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## ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.



1898.

LIST OF INSTITUTIONS ENTITLED TO RECEIVE THE PHILOSOPHICAL TRANSACTIONS OR PROCEEDINGS OF THE ROYAL SOCIETY.

Institutions marked A are entitled to receive Philosophical Transactions, Series A, and Proceedings.

"	"	B	"	"	"	"	Series B, and Proceedings.
"	"	AB	"	"	"	"	Series A and B, and Proceedings.
"	"	P	"	"	"	"	Proceedings only.

**America (Central).**

Mexico.

p. Sociedad Científica "Antonio Alzate."

**America (North).** (See UNITED STATES and CANADA.)

**America (South).**

Buenos Ayres.

AB. Museo Nacional.

Caracas.

B. University Library.

Cordova.

AB. Academia Nacional de Ciencias.

Demerara.

p. Royal Agricultural and Commercial Society, British Guiana.

La Plata.

B. Museo de La Plata.

Rio de Janeiro.

p. Observatorio.

**Australia.**

Adelaide.

p. Royal Society of South Australia.

Brisbane.

p. Royal Society of Queensland.

Melbourne.

p. Observatory.

p. Royal Society of Victoria.

AB. University Library.

Sydney.

p. Australian Museum.

p. Geological Survey.

p. Linnean Society of New South Wales.

AB. Royal Society of New South Wales.

AB. University Library.

**Austria.**

Agram.

p. Jugoslavenska Akademija Znanosti i Umjetnosti.

p. Societas Historico-Naturalis Croatica.

VOL. CXCI.—A.

**Austria (continued).**

Brünn.

AB. Naturforschender Verein.

Gratz.

AB. Naturwissenschaftlicher Verein für Steiermark.

Innsbruck.

AB. Das Ferdinandeum.

p. Naturwissenschaftlich - Medicinischer Verein.

Prague.

AB. Königliche Böhmisches Gesellschaft der Wissenschaften.

Trieste.

B. Museo di Storia Naturale.

p. Società Adriatica di Scienze Naturali.

Vienna.

p. Anthropologische Gesellschaft.

AB. Kaiserliche Akademie der Wissenschaften.

p. K.K. Geographische Gesellschaft.

AB. K.K. Geologische Reichsanstalt.

B. K.K. Naturhistorisches Hof-Museum.

B. K.K. Zoologisch-Botanische Gesellschaft.

p. Oesterreichische Gesellschaft für Meteorologie.

A. Von Kuffner'sche Sternwarte.

**Belgium.**

Brussels.

B. Académie Royale de Médecine.

AB. Académie Royale des Sciences.

B. Musée Royal d'Histoire Naturelle de Belgique.

p. Observatoire Royal.

p. Société Belge de Géologie, de Paléontologie, et d'Hydrologie.

p. Société Malacologique de Belgique.

Ghent.

AB. University.

**Belgium (continued).****Liège.**

- AB. Société des Sciences.
- p. Société Géologique de Belgique.

**Louvain.**

- B. Laboratoire de Microscopie et de Biologie Cellulaire.
- AB. Université.

**Canada.****Hamilton.**

- p. Hamilton Association.

**Montreal.**

- AB. McGill University.
- p. Natural History Society.

**Ottawa.**

- AB. Geological Survey of Canada.
- AB. Royal Society of Canada.

**Toronto.**

- p. Astronomical and Physical Society.
- p. Canadian Institute.
- AB. University.

**Cape of Good Hope.**

- A. Observatory.
- AB. South African Library.

**Ceylon.****Colombo.**

- B. Museum.

**China.****Shanghai.**

- p. China Branch of the Royal Asiatic Society.

**Denmark.****Copenhagen.**

- AB. Kongelige Danske Videnskabernes Selskab.

**Egypt.****Alexandria.**

- AB. Bibliothèque Municipale.

**England and Wales.****Aberystwith.**

- AB. University College.

**Bangor.**

- AB. University College of North Wales.

**Birmingham.**

- AB. Free Central Library.
- AB. Mason College.
- p. Philosophical Society.

**Bolton.**

- p. Public Library.

**Bristol.**

- p. Merchant Venturers' School.
- AB. University College.

**Cambridge.**

- AB. Philosophical Society.
- p. Union Society.

**England and Wales (continued).****Cooper's Hill.**

- AB. Royal Indian Engineering College.

**Dudley.**

- p. Dudley and Midland Geological and Scientific Society.

**Essex.**

- p. Essex Field Club.

**Falmouth.**

- p. Royal Cornwall Polytechnic Society.

**Greenwich.**

- A. Royal Observatory.

**Kew.**

- B. Royal Gardens.

**Leeds.**

- p. Philosophical Society.
- AB. Yorkshire College.

**Liverpool.**

- AB. Free Public Library.
- p. Literary and Philosophical Society.
- A. Observatory.
- AB. University College.

**London.**

- AB. Admiralty.
- p. Anthropological Institute.
- AB. British Museum (Nat. Hist.).
- AB. Chemical Society.
- A. City and Guilds of London Institute.
- p. "Electrician," Editor of the.
- B. Entomological Society.
- AB. Geological Society.
- AB. Geological Survey of Great Britain.
- p. Geologists' Association.
- AB. Guildhall Library.
- A. Institution of Civil Engineers.
- p. Institution of Electrical Engineers.
- A. Institution of Mechanical Engineers.
- A. Institution of Naval Architects.
- p. Iron and Steel Institute.
- AB. King's College.
- B. Linnean Society.
- AB. London Institution.
- p. London Library.
- A. Mathematical Society.
- p. Meteorological Office.
- p. Odontological Society.
- p. Pharmaceutical Society.
- p. Physical Society.
- p. Quekett Microscopical Club.
- p. Royal Agricultural Society.
- A. Royal Astronomical Society.
- B. Royal College of Physicians.
- B. Royal College of Surgeons.



**England and Wales (continued).**

**London (continued).**

- p.* Royal Engineers (for Libraries abroad, six copies).
- AB. Royal Engineers. Head Quarters Library.
- p.* Royal Geographical Society.
- p.* Royal Horticultural Society.
- p.* Royal Institute of British Architects.
- AB. Royal Institution of Great Britain.
- B. Royal Medical and Chirurgical Society.
- p.* Royal Meteorological Society.
- p.* Royal Microscopical Society.
- p.* Royal Statistical Society.
- AB. Royal United Service Institution.
- AB. Society of Arts.
- p.* Society of Biblical Archæology.
- p.* Society of Chemical Industry (London Section).
- p.* Standard Weights and Measures Department.
- AB. The Queen's Library.
- AB. The War Office.
- AB. University College.
- p.* Victoria Institute.
- B. Zoological Society.

**Manchester.**

- AB. Free Library.
- AB. Literary and Philosophical Society.
- p.* Geological Society.
- AB. Owens College.

**Netley.**

- p.* Royal Victoria Hospital.

**Newcastle.**

- AB. Free Library.
- p.* North of England Institute of Mining and Mechanical Engineers.
- p.* Society of Chemical Industry (Newcastle Section).

**Norwich.**

- p.* Norfolk and Norwich Literary Institution.

**Nottingham.**

- AB. Free Public Library.

**Oxford.**

- p.* Ashmolean Society.
- AB. Radcliffe Library.
- A. Radcliffe Observatory.

**Penzance.**

- p.* Geological Society of Cornwall.

**Plymouth.**

- B. Marine Biological Association.
- p.* Plymouth Institution.

**Richmond.**

- A. "Kew" Observatory.

**England and Wales (continued).**

**Salford.**

- p.* Royal Museum and Library.

**Stonyhurst.**

- p.* The College.

**Swansea.**

- AB. Royal Institution.

**Woolwich.**

- AB. Royal Artillery Library.

**Finland.**

**Helsingfors.**

- p.* Societas pro Fauna et Flora Fennica.
- AB. Société des Sciences.

**France.**

**Bordeaux.**

- p.* Académie des Sciences.
- p.* Faculté des Sciences.
- p.* Société de Médecine et de Chirurgie.
- p.* Société des Sciences Physiques et Naturelles.

**Caen.**

- p.* Société Linnéenne de Normandie.

**Cherbourg.**

- p.* Société des Sciences Naturelles.

**Dijon.**

- p.* Académie des Sciences.

**Lille.**

- p.* Faculté des Sciences.

**Lyons.**

- AB. Académie des Sciences, Belles-Lettres et Arts.
- AB. Université.

**Marseilles.**

- AB. Faculté des Sciences.

**Montpellier.**

- AB. Académie des Sciences et Lettres.
- B. Faculté de Médecine.

**Nantes.**

- p.* Société des Sciences Naturelles de l'Ouest de la France.

**Paris.**

- AB. Académie des Sciences de l'Institut.
- p.* Association Française pour l'Avancement des Sciences.
- p.* Bureau des Longitudes.
- A. Bureau International des Poids et Mesures.
- p.* Commission des Annales des Ponts et Chaussées.
- p.* Conservatoire des Arts et Métiers.
- p.* Cosmos (M. L'ABBÉ VALETTE).
- AB. Dépôt de la Marine.
- AB. École des Mines.
- AB. École Normale Supérieure.
- AB. École Polytechnique.
- AB. Faculté des Sciences de la Sorbonne.
- AB. Jardin des Plantes.

**France (continued).**

**Paris (continued).**

- p.* L'Electricien.
- A.* L'Observatoire.
- p.* Revue Scientifique (Mons. H. DE VARIGNY).
- p.* Société de Biologie.
- AB.* Société d'Encouragement pour l'Industrie Nationale.
- AB.* Société de Géographie.
- p.* Société de Physique.
- B.* Société Entomologique.
- AB.* Société Géologique.
- p.* Société Mathématique.
- p.* Société Météorologique de France.

**Toulouse.**

- AB.* Académie des Sciences.
- A.* Faculté des Sciences.

**Germany.**

**Berlin.**

- A.* Deutsche Chemische Gesellschaft.
- A.* Die Sternwarte.
- p.* Gesellschaft für Erdkunde.
- AB.* Königliche Preussische Akademie der Wissenschaften.
- A.* Physikalische Gesellschaft.

**Bonn.**

- AB.* Universität.

**Bremen.**

- p.* Naturwissenschaftlicher Verein.

**Breslau.**

- p.* Schlesische Gesellschaft für Vaterländische Kultur.

**Brunswick.**

- p.* Verein für Naturwissenschaft.

**Carlsruhe. See Karlsruhe.**

**Charlottenburg.**

- A.* Physikalisch-Technische Reichsanstalt.

**Danzig.**

- AB.* Naturforschende Gesellschaft.

**Dresden.**

- p.* Verein für Erdkunde.

**Emden.**

- p.* Naturforschende Gesellschaft.

**Erlangen.**

- AB.* Physikalisch-Medicinische Societät.

**Frankfurt-am-Main.**

- AB.* Senckenbergische Naturforschende Gesellschaft.

- p.* Zoologische Gesellschaft.

**Frankfurt-am-Oder.**

- p.* Naturwissenschaftlicher Verein.

**Freiburg-im-Breisgau.**

- AB.* Universität.

**Giessen.**

- AB.* Grossherzogliche Universität.

**Germany (continued).**

**Görlitz.**

- p.* Naturforschende Gesellschaft.

**Göttingen.**

- AB.* Königliche Gesellschaft der Wissenschaften.

**Halle.**

- AB.* Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.
- p.* Naturwissenschaftlicher Verein für Sachsen und Thüringen.

**Hamburg.**

- p.* Naturhistorisches Museum.
- AB.* Naturwissenschaftlicher Verein.

**Heidelberg.**

- p.* Naturhistorisch-Medizinischer Verein.
- AB.* Universität.

**Jena.**

- AB.* Medicinisch-Naturwissenschaftliche Gesellschaft.

**Karlsruhe.**

- A.* Grossherzogliche Sternwarte.
- p.* Technische Hochschule.

**Kiel.**

- p.* Naturwissenschaftlicher Verein für Schleswig-Holstein.
- A.* Sternwarte.
- AB.* Universität.

**Königsberg.**

- AB.* Königliche Physikalisch - Ökonomische Gesellschaft.

**Leipsic.**

- p.* Annalen der Physik und Chemie.
- AB.* Königliche Sächsische Gesellschaft der Wissenschaften.

**Magdeburg.**

- p.* Naturwissenschaftlicher Verein.

**Marburg.**

- AB.* Universität.

**Munich.**

- AB.* Königliche Bayerische Akademie der Wissenschaften.
- p.* Zeitschrift für Biologie.

**Münster.**

- AB.* Königliche Theologische und Philosophische Akademie.

**Potsdam.**

- A.* Astrophysikalisches Observatorium.

**Rostock.**

- AB.* Universität.

**Strasburg.**

- AB.* Universität.

**Tübingen.**

- AB.* Universität.



**Germany (continued).**

Würzburg.

AB. Physikalisch-Medicinische Gesellschaft.

**Greece.**

Athens.

A. National Observatory.

**Holland. (See NETHERLANDS.)**

**Hungary.**

Buda-pest.

p. Königl. Ungarische Geologische Anstalt.

AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.

Hermannstadt.

p. Siebenbürgischer Verein für die Naturwissenschaften.

Klausenburg.

AB. Az Erdélyi Múzeum. Das Siebenbürgische Museum.

Schemnitz.

p. K. Ungarische Berg- und Forst-Akademie.

**India.**

Bombay.

AB. Elphinstone College.

p. Royal Asiatic Society (Bombay Branch).

Calcutta.

AB. Asiatic Society of Bengal.

AB. Geological Museum.

p. Great Trigonometrical Survey of India.

AB. Indian Museum.

p. The Meteorological Reporter to the Government of India.

Madras.

B. Central Museum.

A. Observatory.

Roorkee.

p. Roorkee College.

**Ireland.**

Armagh.

A. Observatory.

Belfast.

AB. Queen's College.

Cork.

p. Philosophical Society.

AB. Queen's College.

Dublin.

A. Observatory.

AB. National Library of Ireland.

B. Royal College of Surgeons in Ireland.

AB. Royal Dublin Society.

AB. Royal Irish Academy.

Galway.

AB. Queen's College.

**Italy.**

Acireale.

p. Accademia di Scienze, Lettere ed Arti.

Bologna.

AB. Accademia delle Scienze dell' Istituto.

Catania.

AB. Accademia Gioenia di Scienze Naturali.

Florence.

p. Biblioteca Nazionale Centrale.

AB. Museo Botanico.

p. Reale Istituto di Studi Superiori.

Genoa.

p. Società Ligustica di Scienze Naturali e Geografiche.

Milan.

AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.

AB. Società Italiana di Scienze Naturali.

Modena.

p. Le Stazioni Sperimentali Agrarie Italiane.

Naples.

p. Società di Naturalisti.

AB. Società Reale, Accademia delle Scienze.

B. Stazione Zoologica (Dr. DOHRN).

Padua.

p. University.

Palermo.

A. Circolo Matematico.

AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).

A. Reale Osservatorio.

Pisa.

p. Il Nuovo Cimento.

p. Società Toscana di Scienze Naturali.

Rome.

p. Accademia Pontificia de' Nuovi Lincei.

p. Rassegna delle Scienze Geologiche in Italia.

A. Reale Ufficio Centrale di Meteorologia e di Geodinamica, Collegio Romano.

AB. Reale Accademia dei Lincei.

p. R. Comitato Geologico d' Italia.

A. Specola Vaticana.

AB. Società Italiana delle Scienze.

Siena.

p. Reale Accademia dei Fisiocritici.

Turin.

p. Laboratorio di Fisiologia.

AB. Reale Accademia delle Scienze.

Venice.

p. Ateneo Veneto.

AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

**Japan.**

Tokiô.

AB. Imperial University.

p. Asiatic Society of Japan.

**Java.**

Buitenzorg.

p. Jardin Botanique.

**Luxembourg.**

Luxembourg.

p. Société des Sciences Naturelles.

**Malta.**

p. Public Library.

**Mauritius.**

p. Royal Society of Arts and Sciences.

**Netherlands.**

Amsterdam.

AB. Koninklijke Akademie van Wetenschappen.

p. K. Zoologisch Genootschap 'Natura Artis Magistra.'

Delft.

p. École Polytechnique.

Haarlem.

AB. Hollandsche Maatschappij der Wetenschappen.

p. Musée Teyler.

Leyden.

AB. University.

Rotterdam.

AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.

Utrecht.

AB. Provinciaal Genootschap van Kunsten en Wetenschappen.

**New Brunswick.**

St. John.

p. Natural History Society.

**New Zealand.**

Wellington.

AB. New Zealand Institute.

**Norway.**

Bergen.

AB. Bergenske Museum.

Christiania.

AB. Kongelige Norske Frederiks Universitet.

Tromsøe.

p. Museum.

Trondhjem.

AB. Kongelige Norske Videnskabers Selskab.

**Nova Scotia.**

Halifax.

p. Nova Scotian Institute of Science.

Windsor.

p. King's College Library.

**Portugal.**

Coimbra.

AB. Universidade.

Lisbon.

AB. Academia Real das Sciencias.

p. Secção dos Trabalhos Geologicos de Portugal

Oporto.

p. Annaes de Sciencias Naturaes.

**Russia.**

Dorpat.

AB. Université.

Irkutsk.

p. Société Impériale Russe de Géographie  
(Section de la Sibérie Orientale).

Kazan.

AB. Imperatorsky Kazansky Universitet.

p. Société Physico-Mathématique.

Kharkoff.

p. Section Médicale de la Société des Sciences  
Expérimentales, Université de Kharkow.

Kieff.

p. Société des Naturalistes.

Moscow.

AB. Le Musée Public.

B. Société Impériale des Naturalistes.

Odessa.

p. Société des Naturalistes de la Nouvelle-  
Russie.

Pulkowa.

A. Nikolai Haupt-Sternwarte.

St. Petersburg.

AB. Académie Impériale des Sciences.

B. Archives des Sciences Biologiques.

AB. Comité Géologique.

p. Compass Observatory.

A. Observatoire Physique Central.

**Scotland.**

Aberdeen.

AB. University.

Edinburgh.

p. Geological Society.

p. Royal College of Physicians (Research  
Laboratory).

p. Royal Medical Society.

A. Royal Observatory.

p. Royal Physical Society.

p. Royal Scottish Society of Arts

AB. Royal Society.

Glasgow.

AB. Mitchell Free Library.

p. Natural History Society

p. Philosophical Society

**Servia.**

Belgrade.

p. Académie Royale de Serbie.



**Sicily.** (See ITALY.)

**Spain.**

Cadiz.

- A. Instituto y Observatorio de Marina de San Fernando.

Madrid.

- p. Comisión del Mapa Geológico de España.
- AB. Real Academia de Ciencias.

**Sweden.**

Gottenburg.

- AB. Kongl. Vetenskaps och Vitterhets Samhälle.

Lund.

- AB. Universitet.

Stockholm.

- A. Acta Mathematica.
- AB. Kongliga Svenska Vetenskaps-Akademie.
- AB. Sveriges Geologiska Undersökning.

Upsala.

- AB. Universitet.

**Switzerland.**

Basel.

- p. Naturforschende Gesellschaft.

Bern.

- AB. Allg. Schweizerische Gesellschaft.
- p. Naturforschende Gesellschaft.

Geneva.

- AB. Société de Physique et d'Histoire Naturelle.
- AB. Institut National Genevois.

Lausanne.

- p. Société Vaudoise des Sciences Naturelles.

Neuchâtel.

- p. Société des Sciences Naturelles.

Zürich.

- AB. Das Schweizerische Polytechnikum.
- p. Naturforschende Gesellschaft.
- p. Sternwarte.

**Tasmania.**

Hobart.

- p. Royal Society of Tasmania.

**United States.**

Albany.

- AB. New York State Library.

Annapolis.

- AB. Naval Academy.

Austin.

- p. Texas Academy of Sciences.

Baltimore.

- AB. Johns Hopkins University.

Berkeley.

- p. University of California.

**United States (continued).**

Boston.

- AB. American Academy of Sciences.
- B. Boston Society of Natural History.
- A. Technological Institute.

Brooklyn.

- AB. Brooklyn Library.

Cambridge.

- AB. Harvard University.
- B. Museum of Comparative Zoology.

Chapel Hill (N.C.).

- p. Elisha Mitchell Scientific Society.

Charleston.

- p. Elliott Society of Science and Art of South Carolina.

Chicago.

- AB. Academy of Sciences.
- p. Field Columbian Museum.
- p. Journal of Comparative Neurology.

Davenport (Iowa).

- p. Academy of Natural Sciences.

Ithaca (N.Y.).

- p. Physical Review (Cornell University).

Madison.

- p. Wisconsin Academy of Sciences.

Mount Hamilton (California).

- A. Lick Observatory.

New Haven (Conn.).

- AB. American Journal of Science.
- AB. Connecticut Academy of Arts and Sciences.

New York.

- p. American Geographical Society.
- p. American Mathematical Society.
- p. American Museum of Natural History.
- p. New York Academy of Sciences.
- p. New York Medical Journal.
- p. School of Mines, Columbia College.

Philadelphia.

- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- p. Franklin Institute.
- p. Wagner Free Institute of Science.

Rochester (N.Y.).

- p. Academy of Science.

St. Louis.

- p. Academy of Science.

Salem (Mass.).

- p. American Association for the Advancement of Science.

- AB. Essex Institute.

San Francisco.

- AB. California Academy of Sciences.

**United States (continued).**

Washington.

- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- B. United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.

**United States (continued).**

Washington (continued).

- AB. United States Naval Observatory.
- p. United States Department of Agriculture.
- A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)

- AB. United States Military Academy.



# PHILOSOPHICAL TRANSACTIONS.

## I. *Memoir on the Integration of Partial Differential Equations of the Second Order in Three Independent Variables when an Intermediary Integral does not exist in general.*

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THE general feature of most methods for the integration of partial differential equations in two independent variables is, in some form or other, the construction of a set of subsidiary equations in only a single independent variable; and this applies to all orders. In particular, for the first order in any number of variables (not merely in two), the subsidiary system is a set of ordinary equations in a single independent variable, containing as many equations as dependent variables to be determined by that subsidiary system. For equations of the second order which possess an intermediary integral, the best methods (that is, the most effective as giving tests of existence) are those of BOOLE, modified and developed by IMSCHENETSKY, and that of GOURSAT, initially based upon the theory of characteristics, but subsequently brought into the form of Jacobian systems of simultaneous partial equations of the first order. These methods are exceptions to the foregoing general statement. But for equations of the second order or of higher orders, which involve two independent variables and in no case possess an intermediary integral, the most general methods are that of AMPÈRE and that of DARBOUX, with such modifications and reconstruction as have been introduced by other writers; and though in these developments partial differential equations of the first order are introduced, still initially the subsidiary system is in effect a system with one independent variable expressed and the other, suppressed during the integration, playing a parametric part. In other words, the subsidiary system practically has one independent variable fewer than the original equation.

In another paper\* I have given a method for dealing with partial differential equations of the second order in three variables when they possess an intermediary integral; and references will there be found to other writers upon the subject. My aim in the present paper has been to obtain a method for partial differential equations of the second order in three variables when, in general, they possess no intermediary integral. The natural generalisation of the idea in DARBOUX's method has been

\* "Partial Differential Equations of the Second Order, involving Three Independent Variables and possessing an Intermediary Integral," *Camb. Phil. Trans.*, vol. xvi., 1898, pp. 191–218.

adopted, viz., the construction of subsidiary equations in which the number of expressed independent variables is less by unity than the number in the original equation; consequently the number is two. The subsidiary equations thus are a set of simultaneous partial differential equations in two independent variables and a number of dependent variables.

It then appears that such differential equations of the second order having no intermediary integral divide themselves into two classes, discriminated by the distinction that, for the one class, what I have called the characteristic invariant, viz. :

$$p^2 \frac{\partial F}{\partial a} + p q \frac{\partial F}{\partial b} + q^2 \frac{\partial F}{\partial c} - p \frac{\partial F}{\partial g} - q \frac{\partial F}{\partial f} + \frac{\partial F}{\partial c} = 0,$$

can, *quâ* function of  $p$  and  $q$ , be resolved into two linear equations, and, for the other class, the characteristic invariant is irreducible.

The first section of this paper is devoted to the general theory of equations of the second order, so as to construct systems of equations subsidiary to the integration; occasional paragraphs in the other sections develop the general theory in connection with particular types of equations. The second section is devoted to the integration of equations whose characteristic invariant is reducible: a method is devised whereby the integration can be effected in those cases where the integral can be expressed in a finite form without partial quadratures; and various examples are given in elucidation of detailed processes. The third section is devoted to the integration of equations whose characteristic invariant is irreducible; and the method is applied with considerable detail to some of the equations that are important in mathematical physics.

It should be added that the case of three independent variables has been selected for detailed treatment, as being that of complexity next greater than the case of two independent variables, the general theory of which is fairly complete. An inspection of the results, as well as of the processes, will make it manifest that, for many of them, generalisation to the case of  $n$  independent variables is immediate.\*

## SECTION I.

### *General Theory.*

1. Let the number of independent variables be three, and denote them by  $x, y, z$ . Denote the dependent variable by  $v$ , and write

$$\begin{array}{lll} \frac{\partial v}{\partial x} = l, & \frac{\partial v}{\partial y} = m, & \frac{\partial v}{\partial z} = n, \\ \frac{\partial^2 v}{\partial x^2} = a, & \frac{\partial^2 v}{\partial y^2} = b, & \frac{\partial^2 v}{\partial z^2} = c, \\ \frac{\partial^2 v}{\partial y \partial z} = f, & \frac{\partial^2 v}{\partial z \partial x} = g, & \frac{\partial^2 v}{\partial x \partial y} = h. \end{array}$$

\* See a note at the end of the paper.



Let any general differential equation of the second order be taken in the form

$$F(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0.$$

When the proper value of  $v$  is substituted, this becomes an identity, so that, when differentiated with regard to  $x, y, z$ , in succession, the results are identities. Hence, writing

$$\left. \begin{aligned} F_x &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial v} l + a \frac{\partial F}{\partial l} + h \frac{\partial F}{\partial m} + g \frac{\partial F}{\partial n} \\ F_y &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial v} m + h \frac{\partial F}{\partial l} + b \frac{\partial F}{\partial m} + f \frac{\partial F}{\partial n} \\ F_z &= \frac{\partial F}{\partial z} + \frac{\partial F}{\partial v} n + g \frac{\partial F}{\partial l} + f \frac{\partial F}{\partial m} + c \frac{\partial F}{\partial n} \end{aligned} \right\},$$

we have

$$\left. \begin{aligned} F_x + F_a \frac{\partial a}{\partial x} + F_b \frac{\partial b}{\partial x} + \dots + F_h \frac{\partial h}{\partial x} &= 0 \\ F_y + F_a \frac{\partial a}{\partial y} + F_b \frac{\partial b}{\partial y} + \dots + F_h \frac{\partial h}{\partial y} &= 0 \\ F_z + F_a \frac{\partial a}{\partial z} + F_b \frac{\partial b}{\partial z} + \dots + F_h \frac{\partial h}{\partial z} &= 0 \end{aligned} \right\}.$$

By CAUCHY'S theorem, a solution of  $F = 0$  exists, determined by the values of  $v$  and one of its derivatives, assigned for a relation between  $x, y, z$ . This implies that, at all points on the surface represented by the relation, the values of  $v$  and, say,  $\partial v / \partial x$  are given; and consequently the values of  $\partial v / \partial y$  and  $\partial v / \partial z$  are known at all points on the surface.

Taking now  $v, l, m, n$ , as known on the surface, and denoting by  $p, q$ , the derivatives of  $z$  with regard to  $x, y$ , along the surface, we have

$$\begin{aligned} dl &= a dx + h dy + g dz = (a + pg) dx + (h + qg) dy, \\ dm &= h dx + b dy + f dz = (h + pf) dx + (b + qf) dy, \\ dn &= g dx + f dy + c dz = (g + pc) dx + (f + qc) dy, \end{aligned}$$

so that, as  $l, m, n$ , are known everywhere on the surface, the quantities

$$\begin{aligned} a + pg &= \frac{dl}{dx}, & h + qg &= \frac{dl}{dy}, \\ h + pf &= \frac{dm}{dx}, & b + qf &= \frac{dm}{dy}, \\ g + pc &= \frac{dn}{dx}, & f + qc &= \frac{dn}{dy}, \end{aligned}$$

are known along the surface. These equations require the relation

$$\frac{dl}{dy} + p \frac{dn}{dy} = \frac{dm}{dx} + q \frac{dn}{dx},$$

so that they determine five of the quantities  $a, b, c, f, g, h$ , in terms of the remaining one, say

$$\begin{aligned} f &= \frac{dn}{dy} - qc, \\ b &= \frac{dm}{dy} - q \frac{dn}{dy} + q^2c, \\ g &= \frac{dn}{dx} - pc, \\ a &= \frac{dl}{dx} - p \frac{dn}{dx} + p^2c, \\ h &= \left. \begin{aligned} &\frac{dm}{dx} - p \frac{dn}{dy} + pqc \\ &= \frac{dl}{dy} - q \frac{dn}{dx} + pqc \end{aligned} \right\} \end{aligned}$$

We also have

$$F(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

so that, in general, there are six equations to determine the six derivatives of the second order; and if  $F$  is algebraical in  $a, b, c, f, g, h$ , there will be a limited number of sets of values of these quantities, which can therefore be regarded as known along the surface.

In the same way, the derivatives of higher order can be deduced everywhere on the surface, and so, taking any point as an initial point, we have the values of all the derivatives of  $v$  known there; we then have a series in powers of  $x - x_0, y - y_0, z - z_0$ , which, in CAUCHY'S theorem, is proved a converging series when  $x_0, y_0, z_0$  is an ordinary point in space for the equation: and consequently we infer the existence of the solution as established by CAUCHY'S theorem.\*

This conclusion is justified only, however, if the equations do actually determine sets of values of  $a, b, c, f, g, h$ . In the case where sets of values are not determined, so that, *e.g.*, the equation  $F = 0$  becomes evanescent on the substitution of the values

\* The most general form of the theorem may be stated as follows:—

If  $\phi(x, y, z) = 0$  be an ordinary relation for the equation  $F = 0$ , that is, if it is not a solution of the characteristic invariant equation, then a solution  $v$  of the equation  $F = 0$  exists satisfying the conditions:—

(i)  $v$  is equal to a given arbitrary function of  $x, y, z$ , everywhere along the surface  $\phi = 0$ ;  
(ii) one of the derivatives of  $v$  is equal to a given arbitrary function of  $x, y, z$ , everywhere along the same surface;

and a solution satisfying these conditions is uniquely determined by them.

See also a paper, Proc. Lond. Math. Soc., vol. xxix, 1898, pp. 5-13, in particular, p. 12.



of  $a, b, f, g, h$ , the preceding inference is not justified. Manifestly one such condition will be

$$p^2 \frac{\partial F}{\partial a} + pq \frac{\partial F}{\partial h} + q^2 \frac{\partial F}{\partial b} - p \frac{\partial F}{\partial g} - q \frac{\partial F}{\partial f} + \frac{\partial F}{\partial c} = 0;$$

but this is, of course, only one among a number of equations.

The method, practically contained in CAUCHY'S theorem, leads to a result only in an infinite form; moreover, it takes no account of the alternative when conditions are not satisfied. We proceed, accordingly, to give another method, suggested by the corresponding investigation (due to DARBOUX) for equations of the second order in two variables.

2. The principle underlying DARBOUX'S method is, in effect, similar to that which underlies AMPÈRE'S; but as DARBOUX'S method most easily admits of application to equations in which the derivatives of highest order occur linearly, it is customary to form derivatives of a given equation, in order to secure that the equations discussed shall possess this property. If however the equation be already in a linear form, a mere generalisation of AMPÈRE'S method can first be tried, for the number of subsidiary equations is considerably smaller than in the other method, and consequently the integrations (if they can be performed) are correspondingly easier.\* It will be sufficient for the present purpose to consider a particular example, say

$$b - f + g - h = 0.$$

When we substitute for  $b, f, g, h$ , we find

$$\frac{dm}{dy} + (p - q - 1) \frac{dn}{dy} + \frac{dn}{dx} - \frac{dm}{dx} + (q^2 - pq + q - p)c = 0;$$

the equation now must not determine the value of  $c$ , and so we must have

$$q^2 - pq + q - p = 0,$$

$$\frac{dm}{dy} + (p - q - 1) \frac{dn}{dy} + \frac{dn}{dx} - \frac{dm}{dx} = 0;$$

and there is also the identical condition

$$\frac{dm}{dx} - p \frac{dn}{dy} - \frac{dl}{dy} + q \frac{dn}{dx} = 0.$$

From the first of these, it follows that either

$$p - q = 0,$$

or

$$q + 1 = 0.$$

\* A discussion of the subsidiary system, and of the relation of its integral to the solution of the original equation, will be found later, in §§ 27-29.

When  $p - q = 0$ , then  $z$  is a function of  $x + y$ . Also the second equation becomes

$$\frac{d}{dy}(m - n) = \frac{d}{dx}(m - n),$$

so that  $m - n$  is a function of  $x + y$ . We infer, from the general considerations adduced, that we may take

$$m - n = \Phi(x + y, z),$$

where  $\Phi$  is an arbitrary function.

When  $q + 1 = 0$ , then  $y + z$  is a function of  $x$ . Also the second equation is

$$\frac{dm}{dy} - \frac{dm}{dx} + \frac{dn}{dx} + p \frac{dn}{dy} = 0,$$

which, by means of the identical condition, can be transformed to

$$-\frac{dl}{dy} + \frac{dm}{dy} = 0,$$

so that  $l - m$  is a function of  $x$ , and the corresponding inference is that

$$l - m = \Theta(x, y + z),$$

where  $\Theta$  is an arbitrary function.

Each of these is an intermediary integral, and the integration can be completed.\*

3. Passing now to the generalisation of DARBOUX's method, we change the variables from  $x, y, z$ , to  $x, y, u$ , where  $u$  is a function of  $x, y, z$ , as yet undetermined, so that also  $z$  is a function of  $x, y, u$ , not yet determined. For the consequent variations of  $z$  when  $x, y, u$ , vary, write

$$\frac{dz}{dx} = p, \quad \frac{dz}{dy} = q;$$

and when  $x, y, u$ , are the variables, denote the variations of the other quantities by  $d$ . Thus

$$\begin{array}{lll} \frac{dv}{dx} = l + np, & \frac{dv}{dy} = m + nq, & \frac{dv}{du} = n \frac{dz}{du}, \\ \frac{dl}{dx} = a + gp, & \frac{dl}{dy} = h + gq, & \frac{dl}{du} = g \frac{dz}{du}, \\ \frac{dm}{dx} = h + fp, & \frac{dm}{dy} = b + fq, & \frac{dm}{du} = f \frac{dz}{du}, \\ \frac{dn}{dx} = g + cp, & \frac{dn}{dy} = f + cq, & \frac{dn}{du} = c \frac{dz}{du}. \end{array}$$

\* The example is discussed (and the solution completed) in another connection in § 9.

Derivatives of the third order will be required. Write

$$\begin{aligned}\alpha_0 &= \frac{\partial^3 r}{\partial x^3}, & \beta_0 &= \frac{\partial^3 r}{\partial x \partial y^2}, & \gamma_0 &= \frac{\partial^3 v}{\partial x \partial y^2}, & \delta_0 &= \frac{\partial^3 r}{\partial y^3}, \\ \alpha_1 &= \frac{\partial^3 r}{\partial x^2 \partial z}, & \beta_1 &= \frac{\partial^3 r}{\partial x \partial y \partial z}, & \gamma_1 &= \frac{\partial^3 v}{\partial y^2 \partial z}, \\ \alpha_2 &= \frac{\partial^3 r}{\partial x \partial z^2}, & \beta_2 &= \frac{\partial^3 r}{\partial y \partial z^2}, \\ \alpha_3 &= \frac{\partial^3 v}{\partial z^3},\end{aligned}$$

so that

$$\left. \begin{aligned}da &= \alpha_0 dx + \beta_0 dy + \alpha_1 dz \\ dh &= \beta_0 dx + \gamma_0 dy + \beta_1 dz \\ dg &= \alpha_1 dx + \beta_1 dy + \alpha_2 dz \\ db &= \gamma_0 dx + \delta_0 dy + \gamma_1 dz \\ df &= \beta_1 dx + \gamma_1 dy + \beta_2 dz \\ dc &= \alpha_2 dx + \beta_2 dy + \alpha_3 dz\end{aligned} \right\}.$$

Now, from the equation  $F = 0$ , we have, on supposing a proper value of  $r$  substituted, an identity, which thus admits of being differentiated and leading to identities. Thus

$$\frac{dF}{dx} + \frac{\partial F}{\partial a} \alpha_0 + \frac{\partial F}{\partial h} \beta_0 + \frac{\partial F}{\partial g} \alpha_1 + \frac{\partial F}{\partial b} \gamma_0 + \frac{\partial F}{\partial f} \beta_1 + \frac{\partial F}{\partial c} \alpha_2 = 0,$$

or say

and also

$$\left. \begin{aligned}X + A\alpha_0 + H\beta_0 + G\alpha_1 + B\gamma_0 + F\beta_1 + C\alpha_2 &= 0 \\ Y + A\beta_0 + H\gamma_0 + G\beta_1 + B\delta_0 + F\gamma_1 + C\beta_2 &= 0 \\ Z + A\alpha_1 + H\beta_1 + G\alpha_2 + B\gamma_1 + F\beta_2 + C\alpha_3 &= 0\end{aligned} \right\}.$$

Consider the forms taken by these equations when the new variables are used. We have

$$\begin{aligned}\frac{da}{dx} &= \alpha_0 + \alpha_1 p, & \frac{da}{dy} &= \beta_0 + \alpha_1 q, & \frac{da}{du} &= \alpha_1 \frac{dz}{du}, \\ \frac{dh}{dx} &= \beta_0 + \beta_1 p, & \frac{dh}{dy} &= \gamma_0 + \beta_1 q, & \frac{dh}{du} &= \beta_1 \frac{dz}{du}, \\ \frac{dg}{dx} &= \alpha_1 + \alpha_2 p, & \frac{dg}{dy} &= \beta_1 + \alpha_2 q, & \frac{dg}{du} &= \alpha_2 \frac{dz}{du}, \\ \frac{db}{dx} &= \gamma_0 + \gamma_1 p, & \frac{db}{dy} &= \delta_0 + \gamma_1 q, & \frac{db}{du} &= \gamma_1 \frac{dz}{du}, \\ \frac{df}{dx} &= \beta_1 + \beta_2 p, & \frac{df}{dy} &= \gamma_1 + \beta_2 q, & \frac{df}{du} &= \beta_2 \frac{dz}{du}, \\ \frac{dc}{dx} &= \alpha_2 + \alpha_3 p, & \frac{dc}{dy} &= \beta_2 + \alpha_3 q, & \frac{dc}{du} &= \alpha_3 \frac{dz}{du}.\end{aligned}$$



The following relations, free from derivatives of the third order and not involving derivatives with regard to  $u$ , subsist among the derivatives of  $a, b, c, f, g, h$ , viz. :

$$\left. \begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= \beta_1 p - \alpha_1 q = p \frac{dg}{dy} - q \frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= \gamma_1 p - \beta_1 q = p \frac{df}{dy} - q \frac{df}{dx} \\ \frac{df}{dx} - \frac{dg}{dy} &= \beta_2 p - \alpha_2 q = p \frac{dc}{dy} - q \frac{dc}{dx} \end{aligned} \right\}.$$

Further, from those equations, we have

$$\begin{aligned} \alpha_0 &= -\beta_1 \frac{p^2}{q} - \frac{p}{q} \left( \frac{da}{dy} - \frac{dh}{dx} \right) + \frac{da}{dx}, \\ \alpha_1 &= \beta_1 \frac{p}{q} + \frac{1}{q} \left( \frac{da}{dy} - \frac{dh}{dx} \right), \\ \alpha_2 &= -\beta_1 \frac{1}{q} + \frac{1}{q} \frac{dg}{dy}, \\ \alpha_3 &= \beta_1 \frac{1}{pq} + \frac{1}{p} \frac{dc}{dx} - \frac{1}{pq} \frac{dg}{dy}, \\ \beta_0 &= -\beta_1 p + \frac{dh}{dx}, \\ \beta_2 &= -\beta_1 \frac{1}{p} + \frac{1}{p} \frac{df}{dx}, \\ \gamma_0 &= -\beta_1 q + \frac{dh}{dy}, \\ \gamma_1 &= \beta_1 \frac{q}{p} + \frac{1}{p} \left( \frac{db}{dx} - \frac{dh}{dy} \right), \\ \delta_0 &= -\beta_1 \frac{q^2}{p} + \frac{db}{dy} - \frac{q}{p} \left( \frac{db}{dx} - \frac{dh}{dy} \right). \end{aligned}$$

Other expressions are obtainable, but they are equivalent to this set in virtue of the three relations above given. We take the value of  $\beta_1$  to be

$$\frac{dh}{du} \div \frac{dz}{du}.$$

Substituting these values in the three equations, we find

$$\begin{aligned} -\beta_1 \frac{\Delta}{q} + X + A \frac{da}{dx} - A \frac{p}{q} \left( \frac{da}{dy} - \frac{dh}{dx} \right) + H \frac{dh}{dx} + G \frac{1}{q} \left( \frac{da}{dy} - \frac{dh}{dx} \right) \\ + B \frac{dh}{dy} + C \frac{1}{q} \frac{dg}{dy} = 0, \end{aligned}$$

$$\begin{aligned}
& -\beta_1 \frac{\Delta}{p} + Y + H \frac{dh}{dy} + B \frac{db}{dy} - B \frac{q}{p} \left( \frac{db}{dx} - \frac{dh}{dy} \right) + F \frac{1}{p} \left( \frac{db}{dx} - \frac{dh}{dy} \right) \\
& \quad + A \frac{dh}{dx} + C \frac{1}{p} \frac{df}{dx} = 0, \\
& \beta_1 \frac{\Delta}{pq} + Z + A \frac{1}{q} \left( \frac{da}{dy} - \frac{dh}{dx} \right) + G \frac{1}{q} \frac{dy}{dy} + B \frac{1}{p} \left( \frac{db}{dx} - \frac{dh}{dy} \right) + F \frac{1}{p} \frac{df}{dx} \\
& \quad + C \frac{1}{p} \frac{dx}{dx} - C \frac{1}{pq} \frac{dy}{dy} = 0,
\end{aligned}$$

where

$$\Delta = Ap^2 + Hpq + Bq^2 - Gp - Fq + C;$$

and in each of these  $\beta_1$  has the above-mentioned value.

4. There are two considerations, initially distinct, but found in the course of the argument to be concurrent, which enable us to obtain a certain set of subsidiary equations; they correspond to the two modes of obtaining the subsidiary equations in AMPÈRE'S method of solving equations of the second order in two independent variables.

According to the first of them, we note that the new variable  $u$  is as yet limited by no conditions; it has hitherto remained arbitrary. Suppose it chosen so that  $\Delta = 0$ , that is,

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0.$$

Then the term in  $\beta_1$  disappears from the three equations; and these (after some reductions in which  $\Delta = 0$  is used as well as the identical relations affecting the derivatives of  $a, b, c, f, g, h$ ) take the forms

$$\left. \begin{aligned}
\xi &= X + A \left( \frac{du}{dx} - p \frac{dg}{dx} \right) + H \left( \frac{dh}{dx} - p \frac{dy}{dy} \right) + B \left( \frac{dh}{dy} - q \frac{dy}{dy} \right) + G \frac{dy}{dx} + F \frac{dy}{dy} = 0 \\
\eta &= Y + A \left( \frac{dh}{dx} - p \frac{df}{dx} \right) + H \left( \frac{db}{dx} - p \frac{df}{dy} \right) + B \left( \frac{db}{dy} - q \frac{df}{dy} \right) + G \frac{df}{dx} + F \frac{df}{dy} = 0 \\
\zeta &= Z + A \left( \frac{dg}{dx} - p \frac{dc}{dx} \right) + H \left( \frac{df}{dx} - p \frac{dc}{dy} \right) + B \left( \frac{df}{dy} - q \frac{dc}{dy} \right) + G \frac{dc}{dx} + F \frac{dc}{dy} = 0
\end{aligned} \right\}.$$

According to the second of the considerations indicated, we assume that the new variable  $u$ , which has been adopted, is an argument in an arbitrary function that occurs in the solution. Then  $\beta_1$  will, through the term  $dh/du$  in its value  $dh/du \div dz/du$ , introduce a triple differentiation with regard to  $u$  beyond any differentiation that occurs in the integral equations, while no one of the other terms in any of the equations will introduce more than a corresponding double differentiation with regard to  $u$ . Assuming the integral to be of such a form that these differentiations give rise to derivatives of the arbitrary function, it follows\* that  $\beta_1$  will

\* Provided always that the number of derivatives of the arbitrary function in question, as occurring

contain a derivative with regard to  $u$  of the arbitrary function in question, of higher order than any other term in any of the equations. Now the equations must be satisfied identically when the value of  $v$  is substituted in them; hence the term in  $\beta_1$  must disappear in and by itself in each case, that is, we have

$$\Delta = 0,$$

the same conclusion as before. The remaining parts of the equations must also vanish; their forms are already given.

5. The quantities, which have to be determined for the present purpose, are  $a, b, c, f, g, h, l, m, n, v, z$ , viz., eleven in all. They are functions of  $x, y, u$ . Omitting those equations in which derivatives with regard to  $u$  occur, the eleven quantities are to be functions of  $x$  and  $y$ . Constants that arise in the integration are constant because the variation of  $u$  does not appear explicitly; that is to say, the constants are functions of  $u$ .

The equations for the determination of the eleven unknowns are partial differential equations of the first order; their aggregate is constituted as follows:

First, for the equations defining quantities, we have

$$\begin{aligned} \frac{dv}{dx} &= l + np, & \frac{dv}{dy} &= m + nq, \\ \frac{dl}{dx} &= a + gp, & \frac{dl}{dy} &= h + gq, \\ \frac{dm}{dx} &= h + fp, & \frac{dm}{dy} &= b + fq, \\ \frac{dn}{dx} &= g + cp, & \frac{dn}{dy} &= f + cq, \end{aligned}$$

eight in all.

But there are certain relations among derivatives that must be satisfied. We have

$$\frac{d}{dy} \left( \frac{dv}{dx} \right) = \frac{d}{dx} \left( \frac{dv}{dy} \right),$$

that is,

$$\frac{d}{dy} (l + np) = \frac{d}{dx} (m + nq),$$

or, since

$$\frac{dp}{dy} = \frac{d}{dy} \left( \frac{dz}{dx} \right) = \frac{d}{dx} \left( \frac{dz}{dy} \right) = \frac{dq}{dx},$$

we have

$$\frac{dl}{dy} + p \frac{dn}{dy} = \frac{dm}{dx} + q \frac{dn}{dx},$$

in the solution, is finite. If the solution is not expressible in finite terms, the inference is not necessarily justified in the present connection; we should then fall back upon the first of the two arguments. An example will be found in §§ 41-43.



a relation that is satisfied identically in virtue of the defining equations. As it is deduced from two of these equations, and it is satisfied identically in virtue of others, the inference is that the set of equations must consequently be reduced by one in number when only those which are independent are to be retained.

Treating the other three in the same manner, we find

$$\left. \begin{aligned} \frac{dh}{dx} + q \frac{dg}{dx} &= \frac{da}{dy} + p \frac{dg}{dy} \\ \frac{db}{dx} + q \frac{df}{dx} &= \frac{dh}{dy} + p \frac{df}{dy} \\ \frac{df}{dx} + q \frac{dc}{dx} &= \frac{dg}{dy} + p \frac{dc}{dy} \end{aligned} \right\},$$

equivalent to the identities obtained in § 3. As these are deduced from the eight defining equations, they are satisfied in virtue of those eight; they do not constitute any addition to the aggregate. Seven, therefore, is the number in this class.

The remainder are the equations characteristic of  $F = 0$ , viz., they are

$$\Delta = 0, \quad \xi = 0, \quad \eta = 0, \quad \zeta = 0,$$

being four in all. It thus appears that the tale of independent partial differential equations in the system is eleven, being the same as the number of quantities to be determined.

It is to be noted—the verification is simple—that the original equation

$$F = 0$$

is an integral of these eleven simultaneous equations. Hence, for their effective solution, other ten integrals would be required if further considerations cannot be introduced; but it will appear from examples that this can be done, having the effect of appreciably shortening the process of integration.

6. One generalisation is immediately suggested by the results obtained. In solving equations of the second order in two independent variables, the subsidiary system is composed of a set of simultaneous equations involving one independent variable in effect; and the preceding investigation shows that, for equations of the second order in three independent variables, a subsidiary system can be constructed in the form of a set of simultaneous equations involving two independent variables in effect. It is thus suggested—and it is easy to see that the suggestion can be established definitely—that, *for an equation of the second order involving  $n$  independent variables, a subsidiary system can be constructed in the form of a set of simultaneous partial differential equations of the first order involving in effect  $n - 1$  independent variables and a number of dependent variables, this system being subsidiary to the integration of the proposed equation.*

7. The first of the equations,  $\Delta = 0$ , belonging specially to the postulated equation  $F = 0$ , can be expressed in a different form. The quantity  $z$  is regarded as a function of  $x, y, u$ ; and  $p, q$ , denote the values of  $dz/dx, dz/dy$ , respectively, when  $u$  is considered constant. Let the equation connecting  $z, x, y, u$ , be given (or taken) in the form

$$u = u(x, y, z);$$

then we have

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} = 0.$$

Substituting for  $p$  and  $q$ , the equation  $\Delta = 0$  becomes

$$A \left( \frac{\partial u}{\partial x} \right)^2 + H \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + B \left( \frac{\partial u}{\partial y} \right)^2 + G \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + F \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + C \left( \frac{\partial u}{\partial z} \right)^2 = 0,$$

which, after the preceding explanations in § 4, is an equation satisfied by an argument of an arbitrary function in the integral of the differential equation

It is not difficult to prove that this equation is invariantive for all changes of the independent variables. For suppose them changed according to the transformations

$$\begin{aligned} x' &= \xi(x, y, z), \\ y' &= \eta(x, y, z), \\ z' &= \zeta(x, y, z); \end{aligned}$$

and let  $\xi_x, \xi_y, \xi_z, \dots$  denote derivatives of  $\xi, \dots$  while  $l', m', n', a' \dots$  denote derivatives of  $v$  with regard to the new variables. Then

$$\begin{aligned} l, m, n &= (\xi_x, \eta_x, \zeta_x \text{ } \text{ } l', m', n'); \\ &\quad \xi_y, \eta_y, \zeta_y \\ &\quad \xi_z, \eta_z, \zeta_z \end{aligned}$$

and

$$\begin{aligned} a &= (a', b', c', f', g', h' \text{ } \text{ } \xi_x, \eta_x, \zeta_x)^2 + \dots, \\ h &= (a', b', c', f', g', h' \text{ } \text{ } \xi_x, \eta_x, \zeta_x \text{ } \text{ } \xi_y, \eta_y, \zeta_y) + \dots, \\ &\dots \end{aligned}$$

the terms represented by  $+\dots$  being terms involving the derivatives  $l', m', n'$ , of the first order only. If, then, the differential equation

$$F(a, b, c, f, g, h, l, m, n, x, y, z) = 0$$

becomes

$$F'(a', b', \dots) = 0$$

after these substitutions are made, we have

$$\begin{aligned} A' &= A\xi_x^2 + H\xi_x\xi_y + B\xi_y^2 + G\xi_x\xi_z + F\xi_y\xi_z + C\xi_z^2, \\ H' &= 2A\xi_x\eta_x + H(\xi_x\eta_y + \xi_y\eta_x) + 2B\xi_y\eta_y \\ &\quad + G(\xi_x\eta_z + \xi_z\eta_x) + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z, \end{aligned}$$

and so on; so that, if

$$u(x, y, z) = u'(x', y', z'),$$

we have

$$A' \left( \frac{\partial u'}{\partial \xi} \right)^2 + H' \frac{\partial u'}{\partial \xi} \frac{\partial u'}{\partial \eta} + \dots = A \left( \frac{\partial u}{\partial x} \right)^2 + H \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \dots$$

on substitution and collection of terms. The equation may therefore be called the *characteristic invariant* of the original differential equation.

8. A method for integrating partial differential equations of the second order, when they possess an intermediary integral, has been given by me elsewhere; its aim is the actual derivation of the intermediary integral which, being of the first order, can be regarded as soluble. The preceding method makes no assumption as to the existence of an intermediary integral, and indeed is entirely independent of that existence; so that it can be applied not merely to that former class, but also to equations that do not satisfy the preliminary conditions for the possession of an intermediary integral.

## SECTION II.

### *Equations having a Resoluble Characteristic Invariant.*

9. As a first example (which, it will be seen, possesses intermediary integrals), consider the equation

$$b = j - g + h.$$

The characteristic equation is

$$-pq + q^2 - p + q = 0,$$

which can be resolved into the two equations

$$p - q = 0, \quad q + 1 = 0.$$

The other three equations, deduced as in §§ 3, 4, are easily found to be

$$\left. \begin{aligned} -\frac{dh}{dx} + \frac{dh}{dy} + \frac{dg}{dx} + (p - q - 1) \frac{dy}{dy} &= 0 \\ -\frac{db}{dx} + \frac{db}{dy} + \frac{df}{dx} + (p - q - 1) \frac{df}{dy} &= 0 \\ -\frac{df}{dx} + \frac{df}{dy} + \frac{dc}{dx} + (p - q - 1) \frac{dc}{dy} &= 0 \end{aligned} \right\};$$



and we have the relations of identity, viz.,

$$\left. \begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= p \frac{dg}{dy} - q \frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= p \frac{df}{dy} - q \frac{df}{dx} \\ \frac{df}{dx} - \frac{dy}{dy} &= p \frac{dc}{dy} - q \frac{dc}{dx} \end{aligned} \right\}.$$

Take first the form

$$p - q = 0,$$

showing that  $z$  is any function of  $x + y$ . The three deduced equations then become

$$\begin{aligned} \frac{d}{dx} (g - h) &= \frac{d}{dy} (g - h), \\ \frac{d}{dx} (b - f) &= \frac{d}{dy} (b - f), \\ \frac{d}{dx} (c - f) &= \frac{d}{dy} (c - f), \end{aligned}$$

so that  $h - g$ ,  $b - f$ ,  $c - f$ , are functions of  $x + y$ .

Hence, as, by the original equation, we must have

$$h - g = b - f,$$

we take, as integrals of the differential equations of the present type,

$$\begin{aligned} h - g &= F_1 (x + y, z), \\ b - f &= F_1 (x + y, z), \\ c - f &= F_2 (x + y, z), \end{aligned}$$

where  $F_1$  and  $F_2$  are arbitrary. Hence also

$$\begin{aligned} \frac{\partial}{\partial x} (m - n) &= h - g = F_1 (x + y, z), \\ \frac{\partial}{\partial y} (m - n) &= b - f = F_1 (x + y, z), \\ \frac{\partial}{\partial z} (m - n) &= f - c = -F_2 (x + y, z). \end{aligned}$$

It therefore follows that

$$\frac{\partial F_1}{\partial z} = -\frac{\partial F_2}{\partial x} = -\frac{\partial F_2}{\partial y};$$

and consequently there exists a function, say  $\Phi(x + y, z)$ , such that

$$F_1 = \frac{\partial \Phi}{\partial x}, \quad F_1 = \frac{\partial \Phi}{\partial y}, \quad F_2 = -\frac{\partial \Phi}{\partial z},$$

and therefore we have

$$m - n = \Phi(x + y, z),$$

where  $\Phi$  is an arbitrary function. We might proceed from this equation to the primitive.

Next, take the relation

$$q + 1 = 0,$$

deduced from the characteristic; this shows that  $z + y$  is a function of  $x$ . The three deduced equations now are

$$\left. \begin{aligned} -\frac{dh}{dx} + \frac{dh}{dy} + \frac{dg}{dx} + p \frac{dg}{dy} &= 0 \\ -\frac{db}{dx} + \frac{db}{dy} + \frac{df}{dx} + p \frac{df}{dy} &= 0 \\ -\frac{df}{dx} + \frac{df}{dy} + \frac{dc}{dx} + p \frac{dc}{dy} &= 0 \end{aligned} \right\};$$

and inserting the value  $q = -1$  in the three relations of identity, they become

$$\left. \begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= p \frac{dg}{dy} + \frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= p \frac{df}{dy} + \frac{df}{dx} \\ \frac{df}{dx} - \frac{dg}{dy} &= p \frac{dc}{dy} + \frac{dc}{dx} \end{aligned} \right\}.$$

Combining these, we find

$$\frac{dh}{dy} - \frac{da}{dy} = 0, \quad \frac{db}{dy} - \frac{dh}{dy} = 0, \quad \frac{df}{dy} - \frac{dg}{dy} = 0,$$

so that  $h - a$ ,  $b - h$ ,  $f - g$ , are functions of  $x$ . Hence, as we must have

$$f - g = b - h$$

by the original equation we take for an integral equation, as in the former case,

$$\begin{aligned} f - g = b - h &= G_1(x, y + z), \\ a - h &= G_2(x, y + z), \end{aligned}$$

where  $G_1$  and  $G_2$  are arbitrary. But

$$\begin{aligned}\frac{\partial}{\partial x}(m-l) &= h-a = -G_2(x, y+z), \\ \frac{\partial}{\partial y}(m-l) &= b-h = G_1(x, y+z), \\ \frac{\partial}{\partial z}(m-l) &= f-g = G_1(x, y+z),\end{aligned}$$

so that we have

$$\frac{\partial G_1}{\partial x} = -\frac{\partial G_2}{\partial y} = -\frac{\partial G_2}{\partial z};$$

consequently there exists a function, say  $-\Theta(x, y+z)$ , such that

$$G_2 = +\frac{\partial \Theta}{\partial x}, \quad G_1 = -\frac{\partial \Theta}{\partial y} = -\frac{\partial \Theta}{\partial z},$$

and we then have

$$l-m = \Theta(x, y+z).$$

We may proceed from this equation to the primitive.

As two distinct intermediary integrals, viz.:

$$\begin{aligned}l-m &= \Theta(x, y+z), \\ m-n &= \Phi(x+y, z),\end{aligned}$$

have been obtained, it is worth noticing that they can be treated simultaneously, for they verify identically the JACOBI-POISSON condition of coexistence. If we introduce two new functions,  $\theta$  and  $\phi$ , defined by the equations

$$\begin{aligned}\Theta(\xi, \eta) &= \frac{\partial \theta}{\partial \xi} - \frac{\partial \theta}{\partial \eta}, \\ \Phi(\xi, \eta) &= \frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \eta},\end{aligned}$$

so that  $\theta$  and  $\phi$  are arbitrary functions, then the simultaneous integral is easily obtained, say by MAYER'S method,\* in the form

$$v = \theta(x+y, z) + \phi(x, y+z),$$

which is the general primitive.

10. We next proceed to an example in which the given equation does not possess an intermediary integral. It is not difficult to construct differential equations

\* Math. Ann., vol. 5 (1872), pp. 460-466; see also my 'Theory of Differential Equations,' Part I., §§ 41-43.



of the second order in three independent variables, possessing a general primitive involving a couple of arbitrary functions of two arguments, but not possessing an intermediary integral.

Let

$$\begin{aligned}\frac{\partial}{\partial u} \phi(u, v) &= \phi_1, & \frac{\partial}{\partial v} \phi(u, v) &= \phi_2, \\ \frac{\partial^2}{\partial u^2} \phi(u, v) &= \phi_{11}, & \frac{\partial^2}{\partial u \partial v} \phi(u, v) &= \phi_{12}, & \frac{\partial^2}{\partial v^2} \phi(u, v) &= \phi_{22},\end{aligned}$$

and so on for higher derivatives and for other functions. Take two functions

$$\phi = \phi(x + y, z), \quad \theta = \theta(x + z, y);$$

and consider the equation

$$v = \phi + \rho\phi_1 + \sigma\phi_2 + \theta + \lambda\theta_1 + \mu\theta_2,$$

as one from which  $\phi$  and  $\theta$  are to be eliminated by means of derivatives of order not higher than two; the quantities  $\rho, \sigma, \lambda, \mu$ , being defined as

$$\begin{aligned}\rho &= a_1x + b_1y + c_1z, & \lambda &= \alpha x + \beta y + \gamma z, \\ \sigma &= a'_1x + b'_1y + c'_1z, & \mu &= \alpha'x + \beta'y + \gamma'z.\end{aligned}$$

We have

$$\begin{aligned}l &= (1 + a_1)\phi_1 + \alpha'_1\phi_2 + \rho\phi_{11} + \sigma\phi_{12} + (1 + \alpha)\theta_1 + \alpha'\theta_2 + \lambda\theta_{11} + \mu\theta_{12}, \\ m &= (1 + b_1)\phi_1 + b'_1\phi_2 + \rho\phi_{11} + \sigma\phi_{12} + \beta\theta_1 + (1 + \beta')\theta_2 + \lambda\theta_{12} + \mu\theta_{22}, \\ n &= c_1\phi_1 + (1 + c'_1)\phi_2 + \rho\phi_{12} + \sigma\phi_{22} + (1 + \gamma)\theta_1 + \gamma'\theta_2 + \lambda\theta_{11} + \mu\theta_{12}.\end{aligned}$$

For second derivatives, it is unnecessary to form expressions for  $b$  and  $c$ , for the latter gives the only equation which contains  $\phi_{222}$ , and the former the only one which contains  $\theta_{222}$ , so that, when elimination is to be performed, these equations would be ignored. We therefore take

$$\begin{aligned}a &= (1 + 2a_1)\phi_{11} + 2a'_1\phi_{12} + \rho\phi_{111} + \sigma\phi_{112} + (1 + 2\alpha)\theta_{11} + 2\alpha'\theta_{12} + \lambda\theta_{111} + \mu\theta_{112}, \\ h &= (1 + a_1 + b_1)\phi_{11} + (a'_1 + b'_1)\phi_{12} + \rho\phi_{111} + \sigma\phi_{112} \\ &\quad + \beta\theta_{11} + (1 + \alpha + \beta')\theta_{12} + \alpha'\theta_{22} + \lambda\theta_{112} + \mu\theta_{122}, \\ g &= c_1\phi_{11} + (1 + a_1 + c'_1)\phi_{12} + \alpha'_1\phi_{22} + \rho\phi_{112} + \sigma\phi_{123} \\ &\quad + (1 + \alpha + \gamma)\theta_{11} + (\alpha' + \gamma')\theta_{12} + \lambda\theta_{111} + \mu\theta_{112}, \\ f &= c_1\phi_{11} + (1 + b_1 + c'_1)\phi_{12} + b'_1\phi_{22} + \rho\phi_{112} + \sigma\phi_{122} \\ &\quad + \beta\theta_{11} + (1 + \beta' + \gamma)\theta_{12} + \gamma'\theta_{22} + \lambda\theta_{112} + \mu\theta_{122}.\end{aligned}$$

What is required is the elimination of the functional forms from these equations, if this be possible.

Manifestly all the third derivatives of  $\phi$  and  $\theta$  disappear in the combination  $a + f - g - h$ ; in fact,

$$a + f - g - h = (a_1 - b_1) \phi_{11} - (a_1 - b_1 - a'_1 + b'_1) \phi_{12} - (a'_1 - b'_1) \phi_{22} \\ + (\alpha - \gamma) \theta_{11} - (\alpha - \gamma - \alpha' + \gamma') \theta_{12} - (\alpha' - \gamma') \theta_{22}.$$

If, by means of the expressions for  $l, m, n$ , it be possible to eliminate  $\theta$  and  $\phi$ , we must have a relation of the form

$$a + f - g - h + \xi l + \eta m + \zeta n = 0,$$

where  $\xi, \eta, \zeta$ , do not involve  $\theta$  or  $\phi$ .

In order that the terms in  $\phi_1$  and  $\phi_2$  may disappear, we have

$$(1 + a_1) \xi + (1 + b_1) \eta + c_1 \zeta = 0, \quad a'_1 \xi + b'_1 \eta + (1 + c'_1) \zeta = 0;$$

that those in  $\phi_{11}, \phi_{12}, \phi_{22}$ , may disappear, we have

$$a_1 - b_1 + \rho (\xi + \eta) = 0, \quad - (a'_1 - b'_1) + \sigma \zeta = 0, \\ - (a_1 - b_1) + (a'_1 - b'_1) + \sigma (\xi + \eta) + \rho \zeta = 0;$$

that those in  $\theta_1$  and  $\theta_2$  may disappear, we have

$$(1 + \alpha) \xi + \beta \eta + (1 + \gamma) \zeta = 0, \quad \alpha' \xi + (1 + \beta') \eta + \gamma' \zeta = 0;$$

and, finally, that those in  $\theta_{11}, \theta_{12}, \theta_{22}$ , may disappear, we have

$$\alpha - \gamma + \lambda (\xi + \zeta) = 0, \quad - (\alpha' - \gamma') + \mu \eta = 0, \\ - (\alpha - \gamma) + (\alpha' - \gamma') + \mu (\xi + \zeta) + \lambda \eta = 0.$$

These equations are to be satisfied simultaneously.

Using the last set of three, we have, on substituting in the third from the first and second,

$$(\lambda + \mu) (\xi + \eta + \zeta) = 0;$$

and similarly, from the first set of three,

$$(\rho + \sigma) (\xi + \eta + \zeta) = 0.$$

We accordingly take

$$\xi + \eta + \zeta = 0.$$

With this value, the first set gives

$$\frac{a_1 - b_1}{\rho} - \frac{a'_1 - b'_1}{\sigma} = 0,$$

that is,

$$(a'_1 - b'_1)(a_1x + b_1y + c_1z) = (a_1 - b_1)(a'_1x + b'_1y + c'_1z).$$

Since  $x, y, z$ , are independent variables, this can be satisfied only if

$$\frac{a'_1}{a_1} = \frac{b'_1}{b_1} = \frac{c'_1}{c_1} = k, \text{ say,}$$

so that

$$\sigma = k\rho.$$

Similarly, the second set gives

$$\frac{\alpha - \gamma}{\lambda} - \frac{\alpha' - \gamma'}{\mu} = 0,$$

leading to

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\gamma'}{\gamma} = \kappa, \text{ say,}$$

so that

$$\mu = \kappa\lambda.$$

The first set of equations can now be replaced by

$$\begin{aligned}\xi + \eta + \zeta &= 0, \\ \alpha_1 - b_1 - \rho\zeta &= 0, \\ \xi + \eta + \alpha_1\xi + b_1\eta + c_1\zeta &= 0, \\ \zeta + k(\alpha_1\xi + b_1\eta + c_1\zeta) &= 0.\end{aligned}$$

Hence  $k = -1$ , and so  $\sigma = -\rho$ ; also

$$\alpha_1\xi + b_1\eta + (c_1 - 1)\zeta = 0,$$

so that

$$\frac{\xi}{b_1 - c_1 + 1} = \frac{\eta}{c_1 - \alpha_1 - 1} = \frac{\zeta}{\alpha_1 - b_1} = \frac{1}{\alpha_1x + b_1y + c_1z}.$$

Similarly, by the other set, we find  $\kappa = -1$ , and so  $\mu = -\lambda$ ; and

$$\frac{\xi}{1 - \beta + \gamma} = \frac{\eta}{\alpha - \gamma} = \frac{\zeta}{-\alpha + \beta - 1} = \frac{1}{\alpha x + \beta y + \gamma z}.$$

As the values of  $\xi, \eta, \zeta$ , must be the same in the two determinations, and as the variables  $x, y, z$ , are independent, we have



$$\frac{a}{a_1} = \frac{\beta}{b_1} = \frac{\gamma}{c_1} = \frac{1 - \beta + \gamma}{1 + b_1 - c_1} = \frac{\alpha - \gamma}{c_1 - a_1 - 1} = \frac{\beta - \alpha - 1}{a_1 - b_1} = \frac{1}{p}, \text{ say,}$$

whence

$$\rho = p\lambda, \quad \beta = \alpha + \frac{1}{2}, \quad \gamma = \alpha + \frac{1}{2p},$$

so that

$$\lambda = \alpha(x + y + z) + \frac{1}{2}y + \frac{1}{2p}z.$$

It is easy to verify that, when  $p$  is distinct from 0 and  $\infty$ , there is no intermediary integral, that is, no relation between  $l$ ,  $m$ ,  $n$ , involving only one of the arbitrary functions  $\theta$  and  $\phi$ . We have

$$\frac{\xi}{1 + \frac{1}{p}} = \frac{\eta}{-\frac{1}{p}} = \frac{\zeta}{-1} = \frac{1}{2\lambda};$$

and then the differential equation is

$$a + f - g - h + \frac{1}{2\lambda} \left\{ \left( 1 + \frac{1}{p} \right) l - \frac{1}{p} m - n \right\} = 0.$$

It has no intermediary integral. Its primitive is

$$v = \phi + \theta + \lambda \{ p(\phi_1 - \phi_2) + \theta_1 - \theta_2 \},$$

where

$$\phi = \phi(x + y, z), \quad \theta = \theta(x + z, y);$$

and  $\lambda$  has the above value, and  $p$  is neither 0 nor  $\infty$ .

11. Now take a particular case, so as to illustrate the method of integration. Let

$$\alpha = 0, \quad p = 1;$$

then

$$\lambda = \frac{1}{2}(y + z).$$

The differential equation is

$$a + f - g - h + \frac{2l - m - n}{y + z} = 0;$$

and it is required to obtain the primitive

$$v = \phi + \theta + \frac{1}{2}(y + z)(\phi_1 - \phi_2 + \theta_1 - \theta_2),$$

where

$$\phi = \phi(x + y, z), \quad \theta = \theta(x + z, y).$$

For the differential equation thus postulated, the characteristic equation is

$$p^2 - pq + p - q = 0,$$

that is,

$$(p - q)(p + 1) = 0.$$

We therefore have two solutions. The first is

$$p - q = 0,$$

leading to

$$z = \text{function of } x + y;$$

the second is

$$p + 1 = 0,$$

leading to

$$y = \text{function of } x + z.$$

The other three equations, particular to the equation under consideration, are

$$\frac{da}{dx} - p \frac{dg}{dx} - \frac{dh}{dx} + p \frac{dg}{dy} - \frac{dy}{dx} + \frac{dg}{dy} + \frac{2a - h - g}{y + z} = 0,$$

$$\frac{dh}{dx} - p \frac{df}{dx} - \frac{db}{dx} + p \frac{df}{dy} - \frac{df}{dx} + \frac{df}{dy} + \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2} = 0,$$

$$\frac{dg}{dx} - p \frac{dc}{dx} - \frac{df}{dx} + p \frac{dc}{dy} - \frac{dc}{dx} + \frac{dc}{dy} + \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2} = 0,$$

that is,

$$\frac{d}{dx} (a - h) - (p + 1) \left( \frac{dg}{dx} - \frac{dg}{dy} \right) + \frac{2a - h - g}{y + z} = 0,$$

$$\frac{d}{dx} (h - b) - (p + 1) \left( \frac{df}{dx} - \frac{df}{dy} \right) + \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2} = 0,$$

$$\frac{d}{dx} (g - f) - (p + 1) \left( \frac{dc}{dx} - \frac{dc}{dy} \right) + \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2} = 0.$$

Taking first the case

$$p + 1 = 0,$$

we have

$$\frac{d}{dx} (a - h - g + f) + \frac{2a - h - g - 2g + f + c}{y + z} + \frac{2l - m - n}{(y + z)^2} = 0.$$

But

$$2a - h - g - 2g + f + c = \left( \frac{\partial}{\partial x} + p \frac{\partial}{\partial z} \right) (2l - m - n) = \frac{d}{dx} (2l - m - n),$$

and

$$\frac{d}{dx} \frac{1}{y + z} = - \frac{1}{(y + z)^2} p = \frac{1}{(y + z)^2};$$

so that

$$\frac{d}{dx} (a - h - g + f) + \frac{d}{dx} \left( \frac{2l - m - n}{y + z} \right) = 0.$$

We thus recover the differential equation, which is an integral of the system; the

arbitrary function, which would arise through the integration, is definite: we have, in fact,

$$a - h - g + f + \frac{2l - m - n}{y + z} = 0.$$

Using this integral, the second equation becomes (on the elimination of  $2l - m - n$ )

$$\frac{d}{dx}(h - b) + \frac{a + h - b - g}{y + z} = 0.$$

Combining this with the first equation, we have

$$\frac{d}{dx}(a - 2h + b) + \frac{a - 2h + b}{y + z} = 0,$$

so that, as  $p + 1 = 0$ , we have

$$\frac{d}{dx}\left(\frac{a - 2h + b}{y + z}\right) = 0.$$

Since  $p + 1 = 0$  implies that  $z + x$  is a function of  $y$ , we infer that

$$\frac{a - 2h + b}{y + z} = \text{arb. fn. of } y,$$

when

$$z + x = \text{arb. fn. of } y;$$

and consequently an integral that can be associated with the original equation is given by

$$\frac{a - 2h + b}{y + z} = \theta(z + x, y),$$

where  $\theta$  is an arbitrary function.

Taking next the case  $p = q$ , the alternative that arises out of the characteristic equation, we have the three other equations the same as before. It is now necessary to take account of the three equations of identity, which, when the relation  $p = q$  is used, are of the form

$$\begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= p \left( \frac{dg}{dy} - \frac{dg}{dx} \right), \\ \frac{db}{dx} - \frac{dh}{dy} &= p \left( \frac{df}{dy} - \frac{df}{dx} \right), \\ \frac{df}{dx} - \frac{dg}{dy} &= p \left( \frac{dc}{dy} - \frac{dc}{dx} \right); \end{aligned}$$

so that, eliminating the terms in  $p$  from the three equations, we have



$$\begin{aligned}\frac{d}{dx}(a-g) - \frac{d}{dy}(a-g) + \frac{2a-h-g}{y+z} &= 0, \\ \frac{d}{dx}(h-f) - \frac{d}{dy}(h-f) + \frac{2h-b-f}{y+z} - \frac{2l-m-n}{(y+z)^2} &= 0, \\ \frac{d}{dx}(g-c) - \frac{d}{dy}(g-c) + \frac{2g-f-c}{y+z} - \frac{2l-m-n}{(y+z)^2} &= 0.\end{aligned}$$

Eliminating the term in  $2l-m-n$  from the second and the third of these by means of the original differential equation, we obtain the modified equations in the forms

$$\begin{aligned}\frac{d}{dx}(h-f) - \frac{d}{dy}(h-f) + \frac{a+h-b-g}{y+z} &= 0, \\ \frac{d}{dx}(g-c) - \frac{d}{dy}(g-c) + \frac{a-h+g-c}{y+z} &= 0.\end{aligned}$$

By the first of the former and the second of the latter, we find

$$\frac{d}{dx}(a-2g+c) - \frac{d}{dy}(a-2g+c) + \frac{a-2g+c}{y+z} = 0.$$

Now, as  $p = q$ ,  $z$  behaves like a constant under the operation  $d/dx - d/dy$ ; hence we have

$$\left(\frac{d}{dx} - \frac{d}{dy}\right)\left\{\frac{a-2g+c}{y+z}\right\} = 0,$$

when

$$\left(\frac{d}{dx} - \frac{d}{dy}\right)z = 0.$$

Consequently

$$\frac{a-2g+c}{y+z} = \text{arb. fn. of } x+y,$$

when

$$z = \text{arb. fn. of } x+y;$$

and we therefore infer that

$$\frac{a-2g+c}{y+z} = \phi(x+y, z)$$

is an integral that can be associated with the original differential equation,  $\phi$  being an arbitrary function.

In order to proceed to the primitive, we take first

$$a-2g+c = (y+z)\phi(x+y, z),$$

and introduce a new arbitrary function  $f$ , defined by

$$\phi = f_{111} - 3f_{112} + 3f_{122} - f_{222},$$

so that

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)^2 v = (y + z)\{f_{111} - 3f_{112} + 3f_{122} - f_{222}\},$$

where  $f$  is an arbitrary function of  $x + y, z$ . Hence

$$l - n = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial z} = (y + z)(f_{11} - 2f_{12} + f_{22}) + f_1 - f_2 + G(x + z, y),$$

where  $G$  is arbitrary so far as this equation is concerned.

Similarly, introducing a new arbitrary function  $g$ , defined by

$$\theta(x + z, y) = \theta = g_{111} - 3g_{112} + 3g_{122} - g_{222},$$

the equation

$$a - 2h + b = (y + z)\theta(x + z, y)$$

leads to the equation

$$a - 2h + b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 v = (y + z)(g_{111} - 3g_{112} + 3g_{122} - g_{222}).$$

Hence

$$l - m = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = (y + z)(g_{11} - 2g_{12} + g_{22}) + g_1 - g_2 + F(x + y, z),$$

where  $F$  is arbitrary so far as this equation is concerned.

In order that the two equations, giving the values of  $l - n$  and  $l - m$  respectively, may coexist, they must satisfy the POISSON-JACOBI condition  $(U, V) = 0$ , which, when developed, gives

$$f_1 - f_2 + G + g_1 - g_2 + F = 0;$$

so that, taking account of the arbitrary character of the functions, we have

$$F = -(f_1 - f_2), \quad G = -(g_1 - g_2).$$

Thus

$$l - n = (y + z)(f_{11} - 2f_{12} + f_{22}) + f_1 - f_2 - (g_1 - g_2),$$

$$l - m = (y + z)(g_{11} - 2g_{12} + g_{22}) - (f_1 - f_2) + g_1 - g_2.$$

It is easy to verify, not merely that these equations coexist, but also that each of them satisfies the differential equation; but neither is an intermediary integral in the customary sense, for each of them includes two arbitrary functions of two arguments.

The equations are of the first order; it is easy to obtain the primitive in the form

$$v = 2f + 2g + (y + z)(f_1 - f_2 + g_1 - g_2),$$

where  $f_1 = f(x + y, z)$  and  $g = g(x + z, y)$  are arbitrary functions, and  $f_1, f_2, g_1, g_2$  are their respective first derivatives.

12. It will be noticed that in these examples the equation  $\Delta = 0$  (which is of the second degree in  $p$  and  $q$ ) is resolvable, so that it can be replaced by two linear equations, and that the latter have, in turn, been combined with the other equations of the system. Now, these equations are of LAGRANGE'S linear form, and their integral is such that some combination  $\theta$  of variables can be an arbitrary function of some other combination  $\phi$ . Further, it has appeared that the integral of the subsidiary system (other than the original equation) is such as to make some combination  $\psi$  of the variable quantities a functional combination of  $\theta$  or  $\phi$  at the same time that  $\theta$  and  $\phi$  are functionally related, so that, as the functional forms are arbitrary, we infer that

$$\psi = \Psi(\theta, \phi),$$

where  $\Psi$  is arbitrary, is an equation that can coexist with the original equation. Hence it is to be inferred that *when  $\Delta = 0$  is a resolvable equation, that is, can be resolved into two equations linear in  $p$  and  $q$ , arbitrary functions of two arguments occur in the most general integral equivalent of the original equation.*

13. The converse also is true, viz., *if an integral relation involve at least one arbitrary function of a couple of distinct arguments and be equivalent to a partial differential equation of the second order, and not to an equation of order lower than the second freed from arbitrary functional forms, then the characteristic invariant equation can be resolved into two linear equations.* (The number of independent variables is, of course, presumed to be three.)

Let  $\xi$  and  $\eta$  be two independent functions of  $x, y, z$ , so that not more than one of the three quantities

$$\xi_x \eta_y - \eta_x \xi_y, \quad \xi_y \eta_z - \xi_z \eta_y, \quad \xi_z \eta_x - \eta_z \xi_x,$$

can vanish. As regards the arbitrary function of  $\xi$  and  $\eta$ , let it occur in the integral equation in the form

$$v = \Theta \{ \dots, \phi(\xi, \eta), \dots \},$$

where  $\phi$  denotes the derivative of the arbitrary function of highest order occurring in  $\Theta$ . Then we have

$$\left. \begin{aligned} l &= \frac{\partial \Theta}{\partial \phi} (\phi_1 \xi_x + \phi_2 \eta_x) + \dots \\ m &= \frac{\partial \Theta}{\partial \phi} (\phi_1 \xi_y + \phi_2 \eta_y) + \dots \\ n &= \frac{\partial \Theta}{\partial \phi} (\phi_1 \xi_z + \phi_2 \eta_z) + \dots \end{aligned} \right\},$$

$$\alpha = \frac{\partial \Theta}{\partial \phi} \{ \phi_{11} \xi_x^2 + 2\phi_{12} \xi_x \eta_x + \phi_{22} \eta_x^2 \} + \text{derivatives of } \phi \text{ of lower order},$$

$$h = \frac{\partial \Theta}{\partial \phi} \{ \phi_{11} \xi_x \xi_y + \phi_{12} (\xi_x \eta_y + \xi_y \eta_x) + \phi_{22} \eta_x \eta_y \} + \dots,$$



and so for the others. Now the integral equation is, by hypothesis, equivalent to a partial differential equation of the second order, say to

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0;$$

hence when these values are substituted the equation is to be satisfied, and accordingly the terms involving the various combinations of the arbitrary functions must disappear. Thus the highest power of  $\phi_{11}$ —or what in effect is the same thing, the highest power of  $\phi_{11} \frac{\partial \Theta}{\partial \phi}$ —must disappear of itself, and therefore

$$\xi_x^2 \frac{\partial F}{\partial a} + \xi_y^2 \frac{\partial F}{\partial b} + \xi_z^2 \frac{\partial F}{\partial c} + \xi_x \xi_y \frac{\partial F}{\partial f} + \xi_y \xi_z \frac{\partial F}{\partial g} + \xi_z \xi_x \frac{\partial F}{\partial h} = 0,$$

or, with the former notation,

$$A \xi_x^2 + H \xi_x \xi_y + G \xi_x \xi_z + B \xi_y^2 + F \xi_y \xi_z + C \xi_z^2 = 0.$$

But the term involving the highest power of  $\phi_{12} \frac{\partial \Theta}{\partial \phi}$ , which will be of the same degree as the highest power of  $\phi_{11} \frac{\partial \Theta}{\partial \phi}$ , must also disappear of itself; and this gives rise to the equation

$$2A \xi_x \eta_x + H(\xi_x \eta_y + \xi_y \eta_x) + G(\xi_x \eta_z + \xi_z \eta_x) + 2B \xi_y \eta_y + F(\xi_y \eta_z + \xi_z \eta_y) + 2C \xi_z \eta_z = 0;$$

and likewise the term involving the highest power of  $\phi_{22} \frac{\partial \Theta}{\partial \phi}$  must disappear, leading to the equation

$$A \eta_x^2 + H \eta_x \eta_y + G \eta_x \eta_z + B \eta_y^2 + F \eta_y \eta_z + C \eta_z^2 = 0.$$

From these we have

$$\begin{aligned} (A \xi_x + \tfrac{1}{2} H \xi_y + \tfrac{1}{2} G \xi_z)^2 &= (\tfrac{1}{4} H^2 - AB) \xi_y^2 + 2(\tfrac{1}{4} GH - \tfrac{1}{2} AF) \xi_y \xi_z + (\tfrac{1}{4} G^2 - AC) \xi_z^2, \\ (A \eta_x + \tfrac{1}{2} H \eta_y + \tfrac{1}{2} G \eta_z)^2 &= (\tfrac{1}{4} H^2 - AB) \eta_y^2 + 2(\tfrac{1}{4} GH - \tfrac{1}{2} AF) \eta_y \eta_z + (\tfrac{1}{4} G^2 - AC) \eta_z^2, \\ (A \xi_x + \tfrac{1}{2} H \xi_y + \tfrac{1}{2} G \xi_z)(A \eta_x + \tfrac{1}{2} H \eta_y + \tfrac{1}{2} G \eta_z) \\ &= (\tfrac{1}{4} H^2 - AB) \xi_y \eta_y + (\tfrac{1}{4} GH - \tfrac{1}{2} AF) (\xi_y \eta_z + \xi_z \eta_y) + (\tfrac{1}{4} G^2 - AC) \xi_z \eta_z; \end{aligned}$$

so that, squaring the last, subtracting the product of the first two, reducing, and removing the factor A, we have

$$(\text{Disct. of } \Delta) (\xi_y \eta_z - \xi_z \eta_y)^2 = 0.$$

Similarly, by taking modifications of the first equation in the form

$$(\tfrac{1}{2} H \xi_x + B \xi_y + \tfrac{1}{2} F \xi_z)^2 = \dots,$$

or in the form

$$(\tfrac{1}{2} G \xi_x + \tfrac{1}{2} F \xi_y + C \xi_z)^2 = \dots,$$

and removing factors B and C respectively, we find

$$(\text{Disert. of } \Delta) (\xi_z \eta_x - \xi_x \eta_z)^2 = 0,$$

$$(\text{Disert. of } \Delta) (\xi_x \eta_y - \xi_y \eta_x)^2 = 0.$$

Now it has been seen that not more than one of the three quantities

$$\xi_y \eta_z - \xi_z \eta_y, \quad \xi_z \eta_x - \xi_x \eta_z, \quad \xi_x \eta_y - \xi_y \eta_x,$$

can vanish. Consequently

$$\text{the Discriminant of } \Delta = 0,$$

in other words, the equation  $\Delta = 0$ , that is

$$Ap^3 + Hpq + Bq^3 - Gq - Fp + C = 0,$$

can be resolved into two equations linear in  $p$  and  $q$ . This establishes the proposition.

But if  $\phi$ , instead of being a function of two arguments  $\xi$  and  $\eta$ , were a function of only a single argument  $u$ , then instead of three equations we infer only one equation of the type

$$Au_x^2 + Hu_x u_y + Bu_y^2 + Gu_z u_x + Fu_y u_z + Cu_z^2 = 0,$$

and its resolvability cannot be established. (It is, of course, not the case that it is not resolvable in particular cases; it is not resolvable in general.) Hence *when the equation  $\Delta = 0$  cannot be resolved into two equations, linear in  $p$  and  $q$ , we infer that the arbitrary functions which occur in the integral equivalent are functions of only a single argument.*

14. In the case when  $\Delta = 0$  is resolvable into two linear equations, and when the other equations possess integrable combinations, a method can be constructed for obtaining those combinations. Thus take the example considered in § 11, where the deduction of the combinations is fortuitous in the sense that no indication of the kind of combination is given. Let

$$\theta(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination; that is, we must have

$$\frac{d\theta}{dx} = 0, \quad \frac{d\theta}{dy} = 0.$$

Either (i) one of them, or (ii) a linear cross between them, or (iii) both of them, must be satisfied in virtue of the set of equations.

Let us consider, first, the case when

$$p + 1 = 0$$

so that  $q$  remains arbitrary. Evidently

$$\frac{d\theta}{dy} = 0$$

will not be the equation to be satisfied; so that the effective combination must arise in the form

$$\frac{d\theta}{dx} = 0$$

when  $p + 1 = 0$ . In other words, we must have the equation

$$\begin{aligned} \frac{\partial\theta}{\partial x} + a \frac{\partial\theta}{\partial f} + h \frac{\partial\theta}{\partial m} + g \frac{\partial\theta}{\partial n} - \left( \frac{\partial\theta}{\partial z} + g \frac{\partial\theta}{\partial l} + f \frac{\partial\theta}{\partial m} + c \frac{\partial\theta}{\partial n} \right) + \frac{\partial\theta}{\partial x} (l - n) \\ + \frac{\partial\theta}{\partial a} \frac{da}{dx} + \frac{\partial\theta}{\partial b} \frac{db}{dx} + \frac{\partial\theta}{\partial c} \frac{dc}{dx} + \frac{\partial\theta}{\partial f} \frac{df}{dx} + \frac{\partial\theta}{\partial g} \frac{dg}{dx} + \frac{\partial\theta}{\partial h} \frac{dh}{dx} = 0, \end{aligned}$$

(where the value  $p = -1$  has been inserted), satisfied in virtue of the system of subsidiary equations. The relations of identity all involve  $q$ , and therefore cannot be useful for the purpose. Hence the above equation must be a linear combination of

$$\left. \begin{aligned} \frac{da}{dx} - \frac{dh}{dx} + X &= 0 \\ \frac{dh}{dx} - \frac{db}{dx} + Y &= 0 \\ \frac{dg}{dx} - \frac{df}{dx} + Z &= 0 \end{aligned} \right\}.$$

where

$$\begin{aligned} X &= \frac{2a - h - g}{y + z}, \\ Y &= \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2}, \\ Z &= \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2}; \end{aligned}$$

these being the subsidiary equations particular to the present case when the value  $p = -1$  is inserted.

When therefore we substitute

$$\frac{da}{dx} = \frac{dh}{dx} - X, \quad \frac{db}{dx} = \frac{dh}{dx} + Y, \quad \frac{df}{dx} = \frac{dg}{dx} + Z,$$



in the equation derived from  $\theta$ , the latter should become an identity; that is, we should have

$$\begin{aligned} \frac{\partial \theta}{\partial x} + a \frac{\partial \theta}{\partial l} + h \frac{\partial \theta}{\partial m} + g \frac{\partial \theta}{\partial n} - \left( \frac{\partial \theta}{\partial z} + g \frac{\partial \theta}{\partial l} + f \frac{\partial \theta}{\partial m} + c \frac{\partial \theta}{\partial n} \right) + \frac{\partial \theta}{\partial a} \left( \frac{dh}{dx} - X \right) \\ + \frac{\partial \theta}{\partial b} \left( \frac{dh}{dx} + Y \right) + \frac{\partial \theta}{\partial c} \frac{dc}{dx} + (l - n) \frac{\partial \theta}{\partial v} + \frac{\partial \theta}{\partial f} \left( \frac{dy}{dx} + Z \right) + \frac{\partial \theta}{\partial y} \frac{dy}{dx} + \frac{\partial \theta}{\partial h} \frac{dh}{dx} = 0 \end{aligned}$$

satisfied independently of the values of derivatives of  $h, c, g$ , with regard to  $x$ . It therefore follows that, if a function  $\theta$  of the suggested type should exist, it must satisfy the system of equations

$$\frac{\partial \theta}{\partial c} = 0,$$

$$\frac{\partial \theta}{\partial f} + \frac{\partial \theta}{\partial y} = 0,$$

$$\frac{\partial \theta}{\partial a} + \frac{\partial \theta}{\partial h} + \frac{\partial \theta}{\partial b} = 0,$$

and

$$\begin{aligned} \frac{\partial \theta}{\partial x} - \frac{\partial \theta}{\partial z} + (a - g) \frac{\partial \theta}{\partial l} + (h - f) \frac{\partial \theta}{\partial m} + (g - c) \frac{\partial \theta}{\partial n} + \frac{\partial \theta}{\partial v} (l - n) \\ - X \frac{\partial \theta}{\partial a} + Y \frac{\partial \theta}{\partial b} + Z \frac{\partial \theta}{\partial f} = 0; \end{aligned}$$

and the number of functionally independent solutions of these homogeneous simultaneous equations is the number of integrable combinations of the subsidiary system.

This system of partial differential equations of the first order must be rendered complete by associating with it the JACOBI-POISSON conditions. This complete system, obtained by the regular processes, is without difficulty proved to be equivalent to

$$\frac{\partial \theta}{\partial v} = 0, \quad \frac{\partial \theta}{\partial c} = 0,$$

$$- \frac{\partial \theta}{\partial h} = \frac{\partial \theta}{\partial a} + \frac{\partial \theta}{\partial b},$$

$$\frac{1}{2} \frac{\partial \theta}{\partial l} = - \frac{\partial \theta}{\partial m} = - \frac{\partial \theta}{\partial n} = \frac{1}{y+z} \frac{\partial \theta}{\partial f} = - \frac{1}{y+z} \frac{\partial \theta}{\partial y} = \left( \frac{\partial \theta}{\partial a} - \frac{\partial \theta}{\partial b} \right) \frac{1}{y+z},$$

and

$$\frac{\partial \theta}{\partial x} - \frac{\partial \theta}{\partial z} - \frac{2l - m - n}{(y+z)^2} \frac{\partial \theta}{\partial a} - \frac{2a - 3h + b + f - g}{y+z} \frac{\partial \theta}{\partial b} = 0,$$

the latter being the modification of the last of the four initial equations.

The complete system thus contains nine equations: it involves thirteen variables, viz.,  $a, b, c, f, g, h, l, m, n, v, x, y, z$ ; and consequently it possesses four functionally

independent solutions. These can be obtained, by any of the regular methods, in the form

$$y, x + z, \frac{2a - 3h + b + f - g}{y + z} + \frac{2l - m - n}{(y + z)^2},$$

$$a - h - g + f + \frac{2l - m - n}{y + z}.$$

But the last is zero, owing to the original differential equation; and by using this imposed restriction, the second becomes

$$\frac{a - 2h + b}{y + z}.$$

Consequently the most general solution of the system is

$$\Phi \left( \frac{a - 2h + b}{y + z}, x + z, y \right) = 0,$$

where  $\Phi$  is arbitrary; an equivalent of this is

$$\frac{a - 2h + b}{y + z} = \theta(z + x, y),$$

where  $\theta$  is arbitrary.

Similarly a new relation between derivatives of the second order can be deduced by taking the alternative solution  $p - q = 0$  of the characteristic equation.

The rest of the solution proceeds as before when once the system of partial differential equations satisfied by

$$\mathcal{S} = \mathcal{S}(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is obtained. Now, when  $p = q$ , the relations of identity are

$$\frac{dh}{dx} - \frac{da}{dy} = p \left( \frac{dy}{dy} - \frac{dy}{dx} \right),$$

$$\frac{db}{dx} - \frac{dh}{dy} = p \left( \frac{df}{dy} - \frac{df}{dx} \right),$$

$$\frac{dg}{dx} - \frac{df}{dy} = p \left( \frac{dc}{dy} - \frac{dc}{dx} \right).$$

These can be used to eliminate  $p$  from the equations particular to the present case, and the latter then become

$$\left. \begin{aligned} \frac{da}{dx} - \frac{da}{dy} - \left( \frac{dy}{dx} - \frac{dy}{dy} \right) + X &= 0 \\ \frac{dh}{dx} - \frac{dh}{dy} - \left( \frac{df}{dx} - \frac{df}{dy} \right) + Y &= 0 \\ \frac{dg}{dx} - \frac{dg}{dy} - \left( \frac{dc}{dx} - \frac{dc}{dy} \right) + Z &= 0 \end{aligned} \right\},$$

which hold whatever be the value of  $p$ . But from  $\mathfrak{J} = 0$  we have

$$\begin{aligned} \frac{\partial \mathfrak{J}}{\partial x} + \frac{\partial \mathfrak{J}}{\partial v} (l + np) + \frac{\partial \mathfrak{J}}{\partial l} (a + gp) + \frac{\partial \mathfrak{J}}{\partial m} (h + fp) + \frac{\partial \mathfrak{J}}{\partial n} (g + cp) \\ + \frac{\partial \mathfrak{J}}{\partial a} \frac{da}{dx} + \dots + \frac{\partial \mathfrak{J}}{\partial c} \frac{dc}{dx} = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathfrak{J}}{\partial y} + \frac{\partial \mathfrak{J}}{\partial v} (m + nq) + \frac{\partial \mathfrak{J}}{\partial l} (h + gq) + \frac{\partial \mathfrak{J}}{\partial m} (b + fq) + \frac{\partial \mathfrak{J}}{\partial n} (f + cq) \\ + \frac{\partial \mathfrak{J}}{\partial a} \frac{da}{dy} + \dots + \frac{\partial \mathfrak{J}}{\partial c} \frac{dc}{dy} = 0. \end{aligned}$$

For our immediate purpose,  $p = q$ ; hence, subtracting, we have

$$\begin{aligned} \frac{\partial \mathfrak{J}}{\partial x} - \frac{\partial \mathfrak{J}}{\partial y} + \frac{\partial \mathfrak{J}}{\partial v} (l - m) + \frac{\partial \mathfrak{J}}{\partial l} (a - h) + \frac{\partial \mathfrak{J}}{\partial m} (h - b) + \frac{\partial \mathfrak{J}}{\partial n} (g - f) \\ + \Sigma \frac{\partial \mathfrak{J}}{\partial a} \left( \frac{da}{dx} - \frac{da}{dy} \right) = 0, \end{aligned}$$

which, being free from  $p$  and  $q$ , must be satisfied in virtue of the above three equations. This being the case, it must, when we substitute for

$$\frac{da}{dx} - \frac{da}{dy}, \quad \frac{dh}{dx} - \frac{dh}{dy}, \quad \frac{dc}{dx} - \frac{dc}{dy},$$

be satisfied independently of the values of

$$\frac{dh}{dx} - \frac{dh}{dy}, \quad \frac{df}{dx} - \frac{df}{dy}, \quad \frac{dy}{dx} - \frac{dy}{dy}.$$

Assigning the necessary conditions, we find

$$\frac{\partial \mathfrak{J}}{\partial b} = 0,$$

$$\frac{\partial \mathfrak{J}}{\partial f} + \frac{\partial \mathfrak{J}}{\partial h} = 0,$$

$$\frac{\partial \mathfrak{J}}{\partial a} + \frac{\partial \mathfrak{J}}{\partial g} + \frac{\partial \mathfrak{J}}{\partial c} = 0,$$

$$\begin{aligned} \frac{\partial \mathfrak{J}}{\partial x} - \frac{\partial \mathfrak{J}}{\partial y} + \frac{\partial \mathfrak{J}}{\partial v} (l - m) + \frac{\partial \mathfrak{J}}{\partial l} (a - h) + \frac{\partial \mathfrak{J}}{\partial m} (h - p) + \frac{\partial \mathfrak{J}}{\partial n} (g - f) \\ - X \frac{\partial \mathfrak{J}}{\partial a} - Y \frac{\partial \mathfrak{J}}{\partial h} + Z \frac{\partial \mathfrak{J}}{\partial c} = 0. \end{aligned}$$



The integration of these equations can, as already stated, be effected in the same way as for the preceding part of the solution of the original equation; the most general solution of the system is found to be

$$\frac{u - 2g + c}{y + z} = \phi(x + y, z).$$

15. That the method just expounded is not restricted to individual instances of equations, for which  $\Delta = 0$  is resolvable, can be seen as follows.

We consider, more generally, the case when

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0$$

is resolvable into two linear equations. We have

$$(\Lambda p + \frac{1}{2}Hq - \frac{1}{2}G)^2 = (\frac{1}{4}H^2 - AB)q^2 + (AF - \frac{1}{2}GH)q + \frac{1}{4}G^2 - AC.$$

Let

$$\frac{1}{4}H^2 - AB = \theta^2,$$

$$AF - \frac{1}{2}GH = -2\theta^2\phi;$$

then, since

$$(\frac{1}{4}H^2 - AB)(\frac{1}{4}G^2 - AC) = \frac{1}{4}(AF - \frac{1}{2}GH)^2,$$

we have

$$\frac{1}{4}G^2 - AC = \theta^2\phi^2,$$

and the equation is

$$\Lambda p + \frac{1}{2}Hq - \frac{1}{2}G = \pm \theta(q - \phi),$$

that is, we have the two equations

$$\left. \begin{aligned} \Lambda p + (\frac{1}{2}H - \theta)q - (\frac{1}{2}G - \theta\phi) &= 0 \\ \Lambda p + (\frac{1}{2}H + \theta)q - (\frac{1}{2}G + \theta\phi) &= 0 \end{aligned} \right\},$$

where

$$\left. \begin{aligned} \phi &= \frac{\frac{1}{4}GH - \frac{1}{2}AF}{\frac{1}{4}H^2 - AB} \\ \theta^2 &= \frac{1}{4}H^2 - AB \end{aligned} \right\}.$$

Taking the former of the two equations, viz.,

$$\Lambda p + (\frac{1}{2}H - \theta)q - (\frac{1}{2}G - \theta\phi) = 0,$$

we seek to obtain combinations of the three equations of identity with the three equations particular to the present case. The first equations of each of these sets, as given in §§ 4, 5, are

$$\begin{aligned} X + A \frac{da}{dx} + H \frac{dh}{dx} + B \frac{dh}{dy} + G \frac{dg}{dx} + F \frac{dg}{dy} - p \left( A \frac{dg}{dx} + H \frac{dg}{dy} \right) - q B \frac{dg}{dy} &= 0, \\ \frac{da}{dy} - \frac{dh}{dx} + p \frac{dg}{dy} - q \frac{dg}{dx} &= 0. \end{aligned}$$

Multiply the latter by  $\frac{1}{2}H - \theta$ , and add it to the former. In the resulting equation, the coefficient of  $dg/dx$  is

$$G - Ap - (\tfrac{1}{2}H - \theta)q,$$

which is

$$= (\tfrac{1}{2}G + \theta\phi);$$

the coefficient of  $dg/dy$  is

$$\begin{aligned} -Hp - Bq + (\tfrac{1}{2}H - \theta)p + F &= -\{(\tfrac{1}{2}H + \theta)p + Bq\} + F \\ &= -\frac{\tfrac{1}{2}H + \theta}{A}\{Ap + (\tfrac{1}{2}H - \theta)q\} + F \\ &= -\frac{\tfrac{1}{2}H + \theta}{A}(\tfrac{1}{2}G - \theta\phi) + F. \end{aligned}$$

But this coefficient, multiplied by  $A$ ,

$$\begin{aligned} &= AF - \tfrac{1}{4}GH - \tfrac{1}{2}G\theta + \tfrac{1}{2}H\theta\phi + \theta^2\phi \\ &= \tfrac{1}{4}GH - \tfrac{1}{2}G\theta + \tfrac{1}{2}H\theta\phi - \theta^2\phi \\ &= (\tfrac{1}{2}G + \theta\phi)(\tfrac{1}{2}H - \theta); \end{aligned}$$

and therefore the terms, involving derivatives of  $g$ , are

$$(\tfrac{1}{2}G + \theta\phi)\left(\frac{dg}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\frac{dg}{dy}\right)$$

The terms involving derivatives of  $a$  are

$$A\left(\frac{da}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\frac{da}{dy}\right);$$

and those involving derivatives of  $h$  are

$$(\tfrac{1}{2}H + \theta)\frac{dh}{dx} + B\frac{dh}{dy} = (\tfrac{1}{2}H + \theta)\left(\frac{dh}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\frac{dh}{dy}\right).$$

The equation is now in its simplest form. The other pairs may be treated in the same way; and thus, corresponding to the equation

$$Ap + (\tfrac{1}{2}H - \theta)q - (\tfrac{1}{2}G - \theta\phi) = 0,$$

we have a system of three subsidiary equations free from  $p$  and  $q$  in the form

$$\left. \begin{aligned} X + A\delta a + (\tfrac{1}{2}H + \theta)\delta h + (\tfrac{1}{2}G + \theta\phi)\delta g &= 0 \\ Y + A\delta h + (\tfrac{1}{2}H + \theta)\delta b + (\tfrac{1}{2}G + \theta\phi)\delta f &= 0 \\ Z + A\delta g + (\tfrac{1}{2}H + \theta)\delta f + (\tfrac{1}{2}G + \theta\phi)\delta c &= 0, \end{aligned} \right\},$$

where

$$\delta = \frac{d}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\frac{d}{dy}.$$

If

$$u(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination of these equations, then we must have

$$\begin{aligned} \frac{\partial u}{\partial a} \frac{da}{dx} + \dots + \frac{\partial u}{\partial h} \frac{dh}{dx} + \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z} p \\ + \frac{\partial u}{\partial c} (l + np) + \frac{\partial u}{\partial l} (a + gp) + \frac{\partial u}{\partial m} (h + fp) + \frac{\partial u}{\partial n} (g + cp) = 0, \\ \frac{\partial u}{\partial a} \frac{da}{dy} + \dots + \frac{\partial u}{\partial h} \frac{dh}{dy} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q + \frac{\partial u}{\partial c} (m + nq) + \frac{\partial u}{\partial l} (h + gq) \\ + \frac{\partial u}{\partial m} (b + fq) + \frac{\partial u}{\partial n} (f + cq) = 0. \end{aligned}$$

Multiply the latter by  $\frac{\frac{1}{2}H - \theta}{\Lambda}$ , and add to the former; where  $p$  and  $q$  occur in the result, it is in the form

$$p + \frac{\frac{1}{2}H - \theta}{\Lambda} q = \frac{\frac{1}{2}G - \theta\phi}{\Lambda},$$

and so the equation is

$$\begin{aligned} \frac{\partial u}{\partial a} \delta a + \frac{\partial u}{\partial b} \delta b + \frac{\partial u}{\partial c} \delta c + \frac{\partial u}{\partial f} \delta f + \frac{\partial u}{\partial g} \delta g + \frac{\partial u}{\partial h} \delta h + \frac{\partial u}{\partial v} + \frac{\frac{1}{2}H - \theta}{\Lambda} \frac{\partial u}{\partial y} + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} \frac{\partial u}{\partial z} \\ + \frac{\partial u}{\partial c} \left( l + \frac{\frac{1}{2}H - \theta}{\Lambda} m + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} n \right) + \frac{\partial u}{\partial l} \left( a + \frac{\frac{1}{2}H - \theta}{\Lambda} h + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} g \right) \\ + \frac{\partial u}{\partial m} \left( h + \frac{\frac{1}{2}H - \theta}{\Lambda} b + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} f \right) + \frac{\partial u}{\partial n} \left( g + \frac{\frac{1}{2}H - \theta}{\Lambda} f + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} c \right) = 0. \end{aligned}$$

This must be satisfied in virtue of the three preceding equations and independently of the actual values of  $\delta a, \dots, \delta f$ . Substitute for  $\delta a, \delta b, \delta c$ ; then the coefficients of  $\delta f, \delta g, \delta h$ , and the term independent of these must vanish. We thus obtain four linear homogeneous partial differential equations, viz.,

$$\left. \begin{aligned} 0 &= \frac{\partial u}{\partial h} - \frac{\frac{1}{2}H + \theta}{\Lambda} \frac{\partial u}{\partial a} - \frac{\Lambda}{\frac{1}{2}H + \theta} \frac{\partial u}{\partial b} \\ 0 &= \frac{\partial u}{\partial g} - \frac{\frac{1}{2}G + \theta\phi}{\Lambda} \frac{\partial u}{\partial a} - \frac{\Lambda}{\frac{1}{2}G + \theta\phi} \frac{\partial u}{\partial c} \\ 0 &= \frac{\partial u}{\partial f} - \frac{\frac{1}{2}G + \theta\phi}{\frac{1}{2}H + \theta} \frac{\partial u}{\partial b} - \frac{\frac{1}{2}H + \theta}{\frac{1}{2}G + \theta\phi} \frac{\partial u}{\partial c} \\ 0 &= -\frac{X}{\Lambda} - \frac{Y}{\frac{1}{2}H + \theta} - \frac{Z}{\frac{1}{2}G + \theta\phi} + \frac{\partial u}{\partial v} + l \frac{\partial u}{\partial v} + a \frac{\partial u}{\partial l} + h \frac{\partial u}{\partial m} + g \frac{\partial u}{\partial n} \\ &\quad + \frac{\frac{1}{2}H - \theta}{\Lambda} \left( \frac{\partial u}{\partial y} + m \frac{\partial u}{\partial v} + h \frac{\partial u}{\partial l} + b \frac{\partial u}{\partial m} + f \frac{\partial u}{\partial n} \right) \\ &\quad + \frac{\frac{1}{2}G - \theta\phi}{\Lambda} \left( \frac{\partial u}{\partial z} + n \frac{\partial u}{\partial v} + g \frac{\partial u}{\partial l} + f \frac{\partial u}{\partial m} + c \frac{\partial u}{\partial n} \right) \end{aligned} \right\},$$



with

$$0 = Ap + (\tfrac{1}{2}H - \theta)q - (\tfrac{1}{2}G - \theta\phi).$$

This system of four equations must be rendered complete by constructing the additional equations that arise out of the JACOBI-POISSON conditions. If, when complete, the system contains  $n$  equations, then it possesses  $13 - n$  functionally independent solutions. Among these must be included (i) the original differential equation

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0;$$

(ii) the two distinct integrals of

$$\frac{dx}{A} = \frac{dy}{\tfrac{1}{2}H - \theta} = \frac{dz}{\tfrac{1}{2}G - \theta\phi};$$

say these are  $\xi, \eta$ .

Putting these on one side, there are thus  $10 - n$  new functionally independent solutions. A not uncommon case is  $n = 9$ , when there is one new solution, say  $u$ . Then we have

$$u = \psi(\xi, \eta),$$

where  $\psi$  is an arbitrary functional form; and this equation coexists with the original equation

$$F = 0.$$

16. Thus far we have considered only one of the two equations into  $\Delta = 0$  is resolvable. When we consider the other equation, viz.,

$$Ap + (\tfrac{1}{2}H + \theta)q - (\tfrac{1}{2}Q + \theta\phi) = 0,$$

the sole difference in the general analysis is manifestly a change in the sign of  $\theta$ ; and we therefore obtain the corresponding system of linear homogeneous partial differential equations, determining an integral combination (if any), by changing the sign of  $\theta$  in the preceding system. The method of integration is the same as before.

It may happen that neither of these two systems possesses a solution distinct from the differential equation. If, however, either (or both) should possess such a solution, then  $n$  must be less than 10, and certain conditions—viz., those in order that the system when complete should contain not more than nine equations—must be satisfied. These are the conditions in order that one equation—or two equations, if the result hold for both systems—of the second order involving an arbitrary function of two arguments should be associable with the given equation.

And it should be noted that the characteristic invariant of an equation associable with the given equation is satisfied by that linear equation in the characteristic invariant of the given equation which is used to derive the new equation. The result is general, and the proof of the general result is immediate.

17. Now it may happen that the simultaneous system of equations admits of no new common solution in either case; the inference then is that no equation of the second order containing a single arbitrary function can be associated with, or is compatible with, the given differential equation. But it may then be that some new equation of the third order—new, that is, in the sense that it is not one of the immediate derivatives of the given equation—containing an arbitrary function can be associated with the given equation; and this may occur with each of the linear factors of  $\Delta = 0$ . And so on, precisely as in DARBOUX'S method for dealing with partial differential equations in two independent variables; we seek to obtain one equation (or, it may be, two equations) of finite order which are compatible with the given equation, contain one arbitrary function, and are not mere derivatives from that given equation.

We have been proceeding on the supposition that the equation possesses no intermediary integral. If no other equation of finite order is compatible with the given equation,\* then the method ceases to be effective. In that case, the only result generally attainable seems at present to be that which occurs in the establishment of CAUCHY'S existence-theorem; the integral certainly contains two arbitrary functions, but its expression (in the form of a converging series) is not finite.

18. Suppose that the conditions for the existence of a new common solution are satisfied for neither of the systems in §§ 15, 16, so that no new equation of the second order, containing only a single arbitrary function, is compatible with the given equation. We proceed to construct the system of subsidiary equations which determine an equation (if any) of the third order containing only one arbitrary function, and compatible with the given equation

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0.$$

On account of this equation, we have three derived equations of the third order, viz., with the former notation

$$\left. \begin{aligned} X + A\alpha_0 + H\beta_0 + G\alpha_1 + B\gamma_0 + F\beta_1 + C\alpha_2 &= 0 \\ Y + A\beta_0 + H\gamma_0 + G\beta_1 + B\delta_0 + F\gamma_1 + C\beta_2 &= 0 \\ Z + A\alpha_1 + H\beta_1 + G\alpha_2 + B\gamma_1 + F\beta_2 + C\alpha_3 &= 0 \end{aligned} \right\};$$

and the new equation (if any) must be compatible with these.

19. The process is an amplification of that used in § 2. When the proper value of  $v$  is substituted in  $F = 0$ , the latter becomes an identity, so that, when it is

\* A simple instance is given by

$$a - h - g + f + \lambda \frac{2l - m - n}{y + z} = 0,$$

where  $\lambda$  is a positive constant other than an integer.

differentiated with regard to the independent variables, the results are identities. By hypothesis, no new equation is derivable when first derivatives are formed: we therefore form the derivatives of the second order, being six in all; viz., they are

$$\begin{aligned} \frac{d^2 F}{dx^2} &= 0, & \frac{d^2 F}{dy^2} &= 0, & \frac{d^2 F}{dz^2} &= 0, \\ \frac{d^2 F}{dy dz} &= 0, & \frac{d^2 F}{dz dx} &= 0, & \frac{d^2 F}{dx dy} &= 0, \end{aligned}$$

equations which contain derivatives of  $v$  of order 4. Let these fifteen derivatives be denoted by  $s_1, s_2, \dots, s_{15}$ , their definitions being given by the scheme

	$dx +$	$dy +$	$dz$
$d\alpha_0 =$	$r_1$	$r_2$	$r_3$
$d\alpha_1$	$r_3$	$r_5$	$r_6$
$d\alpha_2$	$r_6$	$r_9$	$r_{10}$
$d\alpha_3$	$r_{10}$	$r_{14}$	$r_{15}$
$d\beta_0$	$r_2$	$r_4$	$r_5$
$d\beta_1$	$r_5$	$r_8$	$r_9$
$d\beta_2$	$r_9$	$r_{13}$	$r_{14}$
$d\gamma_0$	$r_4$	$r_7$	$r_8$
$d\gamma_1$	$r_8$	$r_{12}$	$r_{13}$
$d\delta_0$	$r_7$	$r_{11}$	$r_{12}$

Further, let (XX) denote the part of  $\frac{d^2 F}{dx^2}$  which is free from derivatives of the fourth order, (XY) the corresponding part of  $\frac{d^2 F}{dx dy}$ , (XZ) that of  $\frac{d^2 F}{dx dz}$ , and so on. Then the six equations are

$$\left. \begin{aligned} (\text{XX}) + Ar_1 + Hr_2 + Gr_3 + Br_4 + Fr_5 + Cr_6 &= 0 \\ (\text{XY}) + Ar_2 + Hr_4 + Gr_5 + Br_7 + Fr_8 + Cr_9 &= 0 \\ (\text{XZ}) + Ar_3 + Hr_5 + Gr_6 + Br_8 + Fr_9 + Cr_{10} &= 0 \\ (\text{YY}) + Ar_4 + Hr_7 + Gr_8 + Br_{11} + Fr_{12} + Cr_{13} &= 0 \\ (\text{YZ}) + Ar_5 + Hr_8 + Gr_9 + Br_{12} + Fr_{13} + Cr_{14} &= 0 \\ (\text{ZZ}) + Ar_6 + Hr_9 + Gr_{10} + Br_{13} + Fr_{14} + Cr_{15} &= 0 \end{aligned} \right\}.$$

As before, let the variables be changed from  $x, y, z$ , to  $x, y, u$ , where  $u$  is a function of  $x, y, z$ , as yet undetermined, whence also  $z$  is a function of  $x, y, u$ . For the



consequent variations of  $z$  when  $x, y, u$ , vary, we write  $p$  and  $q$  for  $dz/dx, dz/dy$ , respectively; and we adopt the notation of § 3. The new expressions for the variations of  $v, l, m, n, a, b, c, f, g, h$ , are given in § 3; those for the variations of  $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \delta_0$ , are given by the equations

$$\left. \begin{aligned} \frac{d\alpha_0}{dx} &= r_1 + r_3 p, & \frac{d\alpha_0}{dy} &= r_2 + r_3 q, & \frac{d\alpha_0}{du} &= r_3 \frac{dz}{du} \\ \frac{d\alpha_1}{dx} &= r_3 + r_6 p, & \frac{d\alpha_1}{dy} &= r_5 + r_6 q, & \frac{d\alpha_1}{du} &= r_6 \frac{dz}{du} \\ \frac{d\alpha_2}{dx} &= r_6 + r_{10} p, & \frac{d\alpha_2}{dy} &= r_9 + r_{10} q, & \frac{d\alpha_2}{du} &= r_{10} \frac{dz}{du} \\ \frac{d\alpha_3}{dx} &= r_{10} + r_{15} p, & \frac{d\alpha_3}{dy} &= r_{14} + r_{15} q, & \frac{d\alpha_3}{du} &= r_{15} \frac{dz}{du} \\ \frac{d\beta_0}{dx} &= r_2 + r_5 p, & \frac{d\beta_0}{dy} &= r_4 + r_5 q, & \frac{d\beta_0}{du} &= r_5 \frac{dz}{du} \\ \frac{d\beta_1}{dx} &= r_5 + r_9 p, & \frac{d\beta_1}{dy} &= r_8 + r_9 q, & \frac{d\beta_1}{du} &= r_9 \frac{dz}{du} \\ \frac{d\beta_2}{dx} &= r_9 + r_{14} p, & \frac{d\beta_2}{dy} &= r_{13} + r_{14} q, & \frac{d\beta_2}{du} &= r_{14} \frac{dz}{du} \\ \frac{d\gamma_0}{dx} &= r_4 + r_8 p, & \frac{d\gamma_0}{dy} &= r_7 + r_8 q, & \frac{d\gamma_0}{du} &= r_8 \frac{dz}{du} \\ \frac{d\gamma_1}{dx} &= r_8 + r_{13} p, & \frac{d\gamma_1}{dy} &= r_{12} + r_{13} q, & \frac{d\gamma_1}{du} &= r_{13} \frac{dz}{du} \\ \frac{d\delta_0}{dx} &= r_7 + r_{12} p, & \frac{d\delta_0}{dy} &= r_{11} + r_{12} q, & \frac{d\delta_0}{du} &= r_{12} \frac{dz}{du} \end{aligned} \right\}.$$

The following relations subsist among the derivatives of  $\alpha_0, \dots, \delta_0$ , free from derivatives of order 4 and from derivatives with regard to  $u$ , viz.,

$$\left. \begin{aligned} \frac{d\beta_0}{dx} - \frac{d\alpha_0}{dy} &= p \frac{d\alpha_1}{dy} - q \frac{d\alpha_1}{dx} = r_5 p - r_3 q \\ \frac{d\beta_1}{dx} - \frac{d\alpha_1}{dy} &= p \frac{d\alpha_2}{dy} - q \frac{d\alpha_2}{dx} = r_9 p - r_6 q \\ \frac{d\beta_2}{dx} - \frac{d\alpha_2}{dy} &= p \frac{d\alpha_3}{dy} - q \frac{d\alpha_3}{dx} = r_{14} p - r_{10} q \\ \frac{d\gamma_0}{dx} - \frac{d\beta_0}{dy} &= p \frac{d\beta_1}{dy} - q \frac{d\beta_1}{dx} = r_8 p - r_5 q \\ \frac{d\gamma_1}{dx} - \frac{d\beta_1}{dy} &= p \frac{d\beta_2}{dy} - q \frac{d\beta_2}{dx} = r_{13} p - r_9 q \\ \frac{d\delta_0}{dx} - \frac{d\gamma_0}{dy} &= p \frac{d\gamma_1}{dy} - q \frac{d\gamma_1}{dx} = r_{12} p - r_8 q \end{aligned} \right\}.$$

These six identical relations reduce the twenty equations in the two foregoing columns to fourteen independent equations; hence the fifteen derivatives  $r$  can be expressed in terms of one of them, say  $r_5$ , and of derivatives of  $\alpha_0, \dots, \delta_0$ ; and the value of  $r_5$  is  $d\beta_0/du \div dz/du$ . These expressions for the other fourteen, in terms of  $r_5$ , are

$$\begin{aligned}
 r_1 &= -r_5 \frac{p^2}{q} + \frac{p}{q} \left( p \frac{d\alpha_1}{dy} - q \frac{d\alpha_1}{dx} \right) + \frac{d\alpha_0}{dx} \\
 r_2 &= -r_5 p + \frac{d\beta_0}{dx} \\
 r_3 &= r_5 \frac{p}{q} - \frac{1}{q} \left( p \frac{d\alpha_1}{dy} - q \frac{d\alpha_1}{dx} \right) \\
 r_4 &= -r_5 q + \frac{d\beta_0}{dy} \\
 r_6 &= -r_5 \frac{1}{q} + \frac{1}{q} \frac{d\alpha_1}{dy} \\
 r_7 &= -r_5 \frac{q^2}{p} + \frac{q}{p} \left( q \frac{d\beta_1}{dx} - p \frac{d\beta_1}{dy} \right) + \frac{d\gamma_0}{dy} \\
 r_8 &= r_5 \frac{q}{p} - \frac{1}{p} \left( q \frac{d\beta_1}{dx} - p \frac{d\beta_1}{dy} \right) \\
 r_9 &= -r_5 \frac{1}{p} + \frac{1}{p} \frac{d\beta_1}{dx} \\
 r_{10} &= r_5 \frac{1}{pq} - \frac{1}{pq} \frac{d\alpha_1}{dy} + \frac{1}{p} \frac{d\alpha_2}{dx} \\
 r_{11} &= -r_5 \frac{q^3}{p^2} + \frac{q^2}{p^2} \left( q \frac{d\beta_1}{dx} - p \frac{d\beta_1}{dy} \right) + \frac{q}{p} \left( q \frac{d\gamma_1}{dx} - p \frac{d\gamma_1}{dy} \right) + \frac{d\delta_0}{dy} \\
 r_{12} &= r_5 \frac{q^2}{p^2} - \frac{q}{p^2} \left( q \frac{d\beta_1}{dx} - p \frac{d\beta_1}{dy} \right) - \frac{1}{p} \left( q \frac{d\gamma_1}{dx} - p \frac{d\gamma_1}{dy} \right) \\
 r_{13} &= -r_5 \frac{q}{p^2} + \frac{1}{p^2} \left( q \frac{d\beta_1}{dx} - p \frac{d\beta_1}{dy} \right) + \frac{1}{p} \frac{d\gamma_1}{dx} \\
 r_{14} &= r_5 \frac{1}{p^2} - \frac{1}{p^2} \frac{d\beta_1}{dx} + \frac{1}{p} \frac{d\beta_2}{dx} \\
 r_{15} &= -r_5 \frac{1}{qp^2} + \frac{1}{qp^2} \frac{d\beta_1}{dx} - \frac{1}{qp} \frac{d\beta_2}{dx} + \frac{1}{q} \frac{d\alpha_3}{dy}
 \end{aligned}$$

20. When these values are substituted in the six equations, and terms are collected, it appears that the terms in  $r_5$  in the six equations are

$$-r_5 \frac{\Delta}{q}, \quad -r_5 \frac{\Delta}{p}, \quad r_5 \frac{\Delta}{pq}, \quad -r_5 \frac{\Delta q}{p^2}, \quad r_5 \frac{\Delta}{p^2}, \quad -r_5 \frac{\Delta}{p^2 q},$$

respectively. For reasons which, being the same as before, need not be repeated

here, the term in  $r_5$  is made to disappear from each of the equations, and thus we have

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0,$$

the characteristic invariant. (We hence notice that this is a particular illustration of the remark in § 16, viz., if two equations are compatible with one another, their characteristic invariants are either the same or, being resolvable, have at least one factor common.)

The six equations, after some reductions of an easy character, take the form

$$(XX) + A \left( \frac{d\alpha_0}{dx} - p \frac{d\alpha_1}{dx} \right) + H \left( \frac{d\beta_0}{dx} - p \frac{d\beta_1}{dx} \right) + B \left( \frac{d\beta_0}{dy} - q \frac{d\alpha_1}{dy} \right) + G \frac{d\alpha_1}{dx} + F \frac{d\alpha_1}{dy} = 0,$$

$$(XY) + A \left( \frac{d\beta_0}{dx} - p \frac{d\beta_1}{dx} \right) + H \left( \frac{d\gamma_0}{dx} - p \frac{d\beta_1}{dx} \right) + B \left( \frac{d\gamma_0}{dy} - q \frac{d\beta_1}{dy} \right) + G \frac{d\beta_1}{dx} + F \frac{d\beta_1}{dy} = 0,$$

$$(XZ) + A \left( \frac{d\alpha_1}{dx} - p \frac{d\alpha_2}{dx} \right) + H \left( \frac{d\beta_1}{dx} - p \frac{d\alpha_2}{dx} \right) + B \left( \frac{d\beta_1}{dy} - q \frac{d\alpha_2}{dy} \right) + G \frac{d\alpha_2}{dx} + F \frac{d\alpha_2}{dy} = 0,$$

$$(YY) + A \left( \frac{d\gamma_0}{dx} - p \frac{d\gamma_1}{dx} \right) + H \left( \frac{d\delta_0}{dx} - p \frac{d\gamma_1}{dx} \right) + B \left( \frac{d\delta_0}{dy} - q \frac{d\gamma_1}{dy} \right) + G \frac{d\gamma_1}{dx} + F \frac{d\gamma_1}{dy} = 0,$$

$$(YZ) + A \left( \frac{d\beta_1}{dx} - p \frac{d\beta_2}{dx} \right) + H \left( \frac{d\gamma_1}{dx} - p \frac{d\beta_2}{dx} \right) + B \left( \frac{d\gamma_1}{dy} - q \frac{d\beta_2}{dy} \right) + G \frac{d\beta_2}{dx} + F \frac{d\beta_2}{dy} = 0,$$

$$(ZZ) + A \left( \frac{d\alpha_2}{dx} - p \frac{d\alpha_3}{dx} \right) + H \left( \frac{d\beta_2}{dx} - p \frac{d\alpha_3}{dx} \right) + B \left( \frac{d\beta_2}{dy} - q \frac{d\alpha_3}{dy} \right) + G \frac{d\alpha_3}{dx} + F \frac{d\alpha_3}{dy} = 0.$$

For the aggregate of differential equations in the system, we have one pair

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq;$$

three pairs, one of which is

$$\frac{dl}{dx} = a + gp, \quad \frac{dl}{dy} = h + gp;$$

six pairs, one of which is

$$\frac{da}{dx} = \alpha_0 + \alpha_1 p, \quad \frac{da}{dy} = \beta_0 + \alpha_1 q;$$

one characteristic equation,

$$\Delta = 0;$$

and the above six equations.

The three equations of § 18 become three integrable combinations of the above six with the foregoing equations in the six pairs.

The number of quantities to be determined is

10 quantities	. . . .	$\alpha_0, \dots, \delta_0;$
6	„ . . . .	$\alpha, \dots, h;$
3	„ . . . .	$l, m, n;$
1 quantity	. . . .	$v;$
1	„ . . . .	$z;$



or 21 in all. There are certain relations among the differential equations; and further, four integrable combinations of the new system are known to exist, viz., the initial equation  $F = 0$  and the three equations derived from it. What is wanted is, if existing, a new integrable combination.

21. Thus far the analysis applies whether the characteristic invariant is or is not reducible to two linear equations. Suppose now that, as in § 15, it vanishes, in virtue of one or other of the two equations

$$\left. \begin{aligned} Ap + (\tfrac{1}{2}H - \theta)q - (\tfrac{1}{2}G - \theta\phi) &= 0 \\ Ap + (\tfrac{1}{2}H + \theta)q - (\tfrac{1}{2}G + \theta\phi) &= 0 \end{aligned} \right\};$$

and consider these in turn, in the same manner as before.

Then, by combining the equations of identity with the equations particular to  $F = 0$ , we have

$$\left. \begin{aligned} (XX) + A\delta\alpha_0 + (\tfrac{1}{2}H + \theta)\delta\beta_0 + (\tfrac{1}{2}G + \theta\phi)\delta\alpha_1 &= 0 \\ (XY) + A\delta\beta_0 + (\tfrac{1}{2}H + \theta)\delta\gamma_0 + (\tfrac{1}{2}G + \theta\phi)\delta\beta_1 &= 0 \\ (XZ) + A\delta\alpha_1 + (\tfrac{1}{2}H + \theta)\delta\beta_1 + (\tfrac{1}{2}G + \theta\phi)\delta\alpha_2 &= 0 \\ (YY) + A\delta\gamma_0 + (\tfrac{1}{2}H + \theta)\delta\delta_0 + (\tfrac{1}{2}G + \theta\phi)\delta\gamma_1 &= 0 \\ (YZ) + A\delta\beta_1 + (\tfrac{1}{2}H + \theta)\delta\gamma_1 + (\tfrac{1}{2}G + \theta\phi)\delta\beta_2 &= 0 \\ (ZZ) + A\delta\alpha_2 + (\tfrac{1}{2}H + \theta)\delta\beta_2 + (\tfrac{1}{2}G + \theta\phi)\delta\alpha_3 &= 0 \end{aligned} \right\};$$

as a modified form of the equations for

$$Ap + (\tfrac{1}{2}H - \theta)q - (\tfrac{1}{2}G - \theta\phi) = 0; \quad ]$$

and in this form

$$\delta = \frac{d}{dx} + \frac{\frac{1}{2}H - \theta}{A} \frac{d}{dy}.$$

Let now

$$E(\alpha_0, \dots, \delta_0, \dots, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination, so that we must have

$$\begin{aligned} \sum \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dx} + \sum \frac{\partial E}{\partial a} \frac{da}{dx} + \sum \frac{\partial E}{\partial l} \frac{dl}{dx} + \frac{\partial E}{\partial v} (l + n\rho) + \frac{\partial E}{\partial v} + \rho \frac{\partial E}{\partial z} &= 0, \\ \sum \frac{\partial E}{\partial \alpha_0} \frac{d\alpha_0}{dy} + \sum \frac{\partial E}{\partial a} \frac{da}{dy} + \sum \frac{\partial E}{\partial l} \frac{dl}{dy} + \frac{\partial E}{\partial v} (m + nq) + \frac{\partial E}{\partial y} + q \frac{\partial E}{\partial z} &= 0. \end{aligned}$$

Multiply the first by  $A$ , the second by  $\frac{1}{2}H - \theta$ , and add; then using the equation connecting  $p$  and  $q$  only, we find

$$A\sum \frac{\partial E}{\partial \alpha_0} \delta\alpha_0 + A \frac{DE}{Dx} + (\tfrac{1}{2}H - \theta) \frac{DE}{Dy} + (\tfrac{1}{2}G - \theta\phi) \frac{DE}{Dz} = 0,$$

where

$$\begin{aligned}
 \frac{D}{Dx} &= \frac{\partial}{\partial x} + l \frac{\partial}{\partial v} + a \frac{\partial}{\partial l} + h \frac{\partial}{\partial m} + g \frac{\partial}{\partial n} \\
 &\quad + \alpha_0 \frac{\partial}{\partial a} + \beta_0 \frac{\partial}{\partial h} + \alpha_1 \frac{\partial}{\partial g} + \gamma_0 \frac{\partial}{\partial b} + \beta_1 \frac{\partial}{\partial f} + \alpha_2 \frac{\partial}{\partial c} \\
 \frac{D}{Dy} &= \frac{\partial}{\partial y} + m \frac{\partial}{\partial v} + h \frac{\partial}{\partial l} + b \frac{\partial}{\partial m} + f \frac{\partial}{\partial n} \\
 &\quad + \beta_0 \frac{\partial}{\partial a} + \gamma_0 \frac{\partial}{\partial h} + \beta_1 \frac{\partial}{\partial g} + \delta_0 \frac{\partial}{\partial b} + \gamma_1 \frac{\partial}{\partial f} + \beta_2 \frac{\partial}{\partial c} \\
 \frac{D}{Dz} &= \frac{\partial}{\partial z} + n \frac{\partial}{\partial v} + g \frac{\partial}{\partial l} + f \frac{\partial}{\partial m} + c \frac{\partial}{\partial n} \\
 &\quad + \alpha_1 \frac{\partial}{\partial a} + \beta_1 \frac{\partial}{\partial h} + \alpha_2 \frac{\partial}{\partial g} + \gamma_1 \frac{\partial}{\partial b} + \beta_2 \frac{\partial}{\partial f} + \alpha_3 \frac{\partial}{\partial c}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \frac{D}{Dx} \\ \frac{D}{Dy} \\ \frac{D}{Dz} \end{aligned}} \right\}$$

Now this is to be satisfied in virtue of the preceding six equations and independently of the particular values of  $\partial\alpha_0, \dots, \partial\delta_0$ , belonging to any integral of the original equation. Hence the equation must be expressible in the form

$$\begin{aligned}
 &\lambda_1 \{(XX) + \dots\} + \lambda_2 \{(XY) + \dots\} + \lambda_3 \{(XZ) + \dots\} \\
 &+ \lambda_4 \{(YY) + \dots\} + \lambda_5 \{(YZ) + \dots\} + \lambda_6 \{(ZZ) + \dots\},
 \end{aligned}$$

where  $\lambda_1, \dots, \lambda_6$ , are indeterminate multipliers; the conditions sufficient and necessary for this are

$$\begin{aligned}
 (\tfrac{1}{2}H + \theta)^3 \frac{\partial E}{\partial \alpha_0} - A(\tfrac{1}{2}H + \theta)^2 \frac{\partial E}{\partial \beta_0} + A^2(\tfrac{1}{2}H + \theta) \frac{\partial E}{\partial \gamma_0} - A^3 \frac{\partial E}{\partial \delta_0} &= 0, \\
 (\tfrac{1}{2}G + \theta\phi)^3 \frac{\partial E}{\partial \alpha_0} - A(\tfrac{1}{2}G + \theta\phi)^2 \frac{\partial E}{\partial \alpha_1} + A^2(\tfrac{1}{2}G + \theta\phi) \frac{\partial E}{\partial \alpha_2} - A^3 \frac{\partial E}{\partial \alpha_3} &= 0, \\
 (\tfrac{1}{2}G + \theta\phi)^3 \frac{\partial E}{\partial \delta_0} - (\tfrac{1}{2}G + \theta\phi)^2 (\tfrac{1}{2}H + \theta) \frac{\partial E}{\partial \gamma_1} \\
 + (\tfrac{1}{2}G + \theta\phi)(\tfrac{1}{2}H + \theta)^2 \frac{\partial E}{\partial \beta_2} - (\tfrac{1}{2}H + \theta)^3 \frac{\partial E}{\partial \alpha_3} &= 0, \\
 2 \frac{(\tfrac{1}{2}H + \theta)(\tfrac{1}{2}G + \theta\phi)}{A} \frac{\partial E}{\partial \alpha_0} + \frac{A^2(\tfrac{1}{2}H + \theta)}{(\tfrac{1}{2}G + \theta\phi)^2} \frac{\partial E}{\partial \alpha_3} + A \frac{\partial E}{\partial \beta_1} \\
 - (\tfrac{1}{2}G + \theta\phi) \frac{\partial E}{\partial \beta_0} - (\tfrac{1}{2}H + \theta) \frac{\partial E}{\partial \alpha_1} - \frac{A^2}{\tfrac{1}{2}G + \theta\phi} \frac{\partial E}{\partial \beta_2} &= 0,
 \end{aligned}$$

and

$$\begin{aligned}
 (XX) \frac{\partial E}{\partial \alpha_0} + (XY) \left\{ \frac{\partial E}{\partial \beta_0} - \frac{\tfrac{1}{2}H + \theta}{A} \frac{\partial E}{\partial \alpha_0} \right\} + (XZ) \left\{ \frac{\partial E}{\partial \alpha_1} - \frac{\tfrac{1}{2}G + \theta\phi}{A} \frac{\partial E}{\partial \alpha_0} \right\} + (YY) \frac{A}{\tfrac{1}{2}H + \theta} \frac{\partial E}{\partial \delta_0} \\
 + (YZ) \left\{ \frac{A}{\tfrac{1}{2}G + \theta\phi} \frac{\partial E}{\partial \beta_2} - \frac{A(\tfrac{1}{2}H + \theta)}{(\tfrac{1}{2}G + \theta\phi)^2} \frac{\partial E}{\partial \alpha_3} \right\} + (ZZ) \frac{A}{\tfrac{1}{2}G + \theta\phi} \frac{\partial E}{\partial \alpha_3} &= 0.
 \end{aligned}$$

22. This system must be rendered complete by the addition of the JACOBI-POISSON

conditions. If, when complete, the system contains  $N$  equations, then it possesses  $23 - N$  functionally independent solutions. Among these are to be included—

- (i) The original differential equation  $F = 0$ ;
- (ii) The three derivatives of  $F = 0$  with regard to  $x, y, z$ , respectively;
- (iii) The two distinct integrals of

$$\frac{dx}{\Lambda} = \frac{dy}{\frac{1}{2}H - \theta} = \frac{dz}{\frac{1}{2}G - \theta\phi};$$

say these are  $\xi, \eta$ .

Putting these on one side, there are thus  $17 - N$  new functionally independent solutions, so that  $N$  must be not greater than 16 in order that the method may be effective. If, when  $N = 16$ , the solution is  $u$ , then

$$u = \psi(\xi, \eta),$$

where  $\psi$  is arbitrary, is an equation of the third order that can be associated with the given equation.

The same process, with corresponding results when the appropriate conditions are satisfied, is adopted for the alternative linear equation

$$Ap + (\tfrac{1}{2}H + \theta)q - (\tfrac{1}{2}G + \theta\phi)$$

arising out of the reducible characteristic invariant.

23. An example in which no equation of the first order involving only one arbitrary function, or no equation of the second order involving only one arbitrary function, can be associated with a given equation of the second order, is furnished by

$$a - h - g + f + 2 \frac{2l - m - n}{y + z} = 0.$$

The general primitive is

$$v = F + G + \tfrac{1}{2}(y + z) \{F_1 - F_2 + G_1 - G_2\} \\ + \tfrac{1}{12}(y + z)^2 \{F_{11} - 2F_{12} + F_{22} + G_{11} - 2G_{12} + G_{22}\},$$

where

$$F = F(x + y, z), \quad G = G(x + z, y),$$

and the subscripts 1, 2, denote derivation with respect to the first and the second of the arguments in the respective cases. The associable equations are of the third order at lowest; and they are

$$\alpha_0 - 3\beta_0 + 3\gamma_0 - \delta_0 = (y + z)\Phi(x + z, y), \\ \alpha_0 - 3\alpha_1 + 3\alpha_2 - \alpha_3 = (y + z)\Psi(x + y, z),$$

where  $\Phi$  and  $\Psi$  are arbitrary.

In a similar way in part, and by induction in part, it may be proved that the integral of

$$a - h - g + f + I \frac{2l - m - n}{y + z} = 0,$$

where  $I$  is a positive integer, can be expressed in finite terms. To express the integral, let

$$\begin{aligned} F(\alpha, \beta) &= F, & \text{where } \alpha &= x + y, \beta = z, \\ G(\alpha', \beta') &= G, & \text{where } \alpha' &= x + z, \beta' = y; \end{aligned}$$

and denote by  $\Delta$  the operation  $\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}$ , by  $\Delta'$  the operation  $\frac{\partial}{\partial \alpha'} - \frac{\partial}{\partial \beta'}$ , so that

$$\begin{aligned} \Delta F &= \frac{\partial F}{\partial \alpha} - \frac{\partial F}{\partial \beta}, & \Delta^2 F &= \frac{\partial^2 F}{\partial \alpha^2} - 2 \frac{\partial^2 F}{\partial \alpha \partial \beta} + \frac{\partial^2 F}{\partial \beta^2}, \dots \\ \Delta' G &= \frac{\partial G}{\partial \alpha'} - \frac{\partial G}{\partial \beta'}, & \Delta'^2 G &= \frac{\partial^2 G}{\partial \alpha'^2} - 2 \frac{\partial^2 G}{\partial \alpha' \partial \beta'} + \frac{\partial^2 G}{\partial \beta'^2}, \dots \end{aligned}$$

Then the value of  $v$  is

$$\begin{aligned} v &= F + G \\ &\quad + \frac{1}{2} (y + z) (\Delta F + \Delta' G) \\ &\quad + \dots \dots \dots \\ &\quad + \frac{I! (2I - s)!}{(I - s)! s! 2I!} (y + z)^s (\Delta^s F + \Delta'^s G) \\ &\quad + \dots \dots \dots \\ &\quad + \frac{I!}{2I!} (y + z)^I (\Delta^I F + \Delta'^I G). \end{aligned}$$

24. But it is necessary to take account of what has been achieved when one equation or when two equations (say of the second order) have been obtained compatible with the given equation and involving each one arbitrary function. The method adopted in § 11 to pass to the primitive has manifestly no element of generality.

Now the three equations are not sufficient to express  $a, b, c, f, g, h$ , in terms of  $l, m, n$ , and the variables; but they frequently will serve to express groups of combinations of  $a, b, c, f, g, h$ , in terms of those quantities. Thus the three equations in § 11 suggest combinations  $a - h, h - b, g - f$  (which are the derivatives of  $l - m$ ), and  $a - g, h - f, g - c$  (which are the derivatives of  $l - n$ ). This, however, is only a slight modification of the former method; it, again, has no element of generality.

Another plan would be to differentiate the three equations up to any order with the hope of determining all the derivatives of the highest order that occur in terms of derivatives of lower order. If this were possible, substitution in the equations of differential elements such as

$$dl = a dx + h dy + g dz$$

and successive integration would ultimately lead to  $v$ . It appears in general, however, that relations of interdependence among the equations prevents them from



being adequate for the purpose at any stage; the relations are, in fact, satisfied conditions of compatibility. This method is, therefore, ineffective.

An effective method can, however, be obtained as follows. Restricting ourselves for the moment to the equation of the second order with two compatible equations also of that order\*—the restriction is made only to simplify the explanations—we have  $v$  as expressible in terms of two arbitrary functions. Hence each of the quantities  $l, m, n$  (and therefore any combination of  $v, l, m, n$ ), can be expressed in terms of two arbitrary functions. Now in one of the compatible equations we have one arbitrary function which is to be identified with one of the arbitrary functions in  $v$ ; hence it is to be expected that a proper combination of  $v, l, m, n$ , is an intermediary integral of that equation involving a new arbitrary function, which must be identified with the other of the arbitrary functions in  $v$ .

Similarly for the other of the compatible equations, there is an intermediary integral involving the two arbitrary functions. The conditions of coexistence of the two intermediary integrals must be assigned; it will appear that, if the conditions are not satisfied identically, they provide the means of identification of the various arbitrary functions.

It is to be observed that the intermediary integral or integrals thus obtained cannot be regarded as intermediary integrals of the original equation in the ordinary sense of the phrase, for each of them involves two arbitrary functions. But they are intermediary for the respective compatible equations: each of them involves one arbitrary function more than occurs in the compatible equation. The result manifestly does not imply that the original equation possesses any intermediary integral; in fact, the assumption throughout our investigations has been that no proper intermediary integral exists.

25. A method† has been given elsewhere for constructing the intermediary integral. In effect, it amounts to the use of the conditions which must be satisfied in order that the derivatives

$$\left. \begin{aligned} au_l + hu_m + gu_n + u_x &= 0 \\ hu_l + bu_m + fu_n + u_y &= 0 \\ gu_l + fu_m + cu_n + u_z &= 0 \end{aligned} \right\}$$

from the supposed integral

$$u(l, m, n, v, x, y, z) = 0$$

shall cause the compatible equation

$$\Theta(a, \dots, h, l, m, n, v, x, y, z) = 0$$

to be satisfied without regard to the values of the differential coefficients of  $v$  of

\* The explanations will be seen to apply, *mutatis mutandis*, to other cases of the second order, and indeed to cases of any order, when compatible equations are known.

† In the memoir cited in the introductory remarks.

the second order. These conditions are the simultaneous partial differential equations of the first order determining  $u$ .

Thus, dealing with the case of § 11, when the compatible equations of the second order are

$$\begin{aligned} a + f - g - h &= -\frac{2l - m - n}{y + z}, \\ a - 2h + b &= (y + z)\theta(x + z, y), \\ a - 2g + c &= (y + z)\phi(x + y, z), \end{aligned}$$

we know that the first has no intermediary integral. As regards an intermediary for the second, substituting for  $a, b, c$ , from the derivatives of the  $u$ -equation, the result

$$-\left(\frac{u_{xx}}{u_x}h + \frac{u_{xy}}{u_x}g + \frac{u_x}{u_x}\right) - 2h - \left(\frac{u_{xz}}{u_x}h + \frac{u_{yz}}{u_x}f + \frac{u_y}{u_x}\right) = (y + z)\theta$$

must, *quâ* equation in  $f, g, h$ , be evanescent; hence we find

$$(u_{xx} + u_{yy})^2 = 0, \quad u_{xx} = 0, \quad -\frac{u_{xz}}{u_x} - \frac{u_{yz}}{u_x} = (y + z)\theta;$$

that is, the equations for  $u$  are

$$\begin{aligned} \frac{\partial u}{\partial x} &= 0, \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= 0, \\ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + (l - m)\frac{\partial u}{\partial z} - (y + z)\frac{\partial u}{\partial z}\theta &= 0. \end{aligned}$$

The system is a complete system; hence it possesses four functionally independent solutions. Writing

$$\theta = g_{111} - 3g_{112} + 3g_{122} - g_{222},$$

these four solutions can be expressed in the form

$$\begin{aligned} &z, x + y, \\ &l - m - (y + z)(g_{11} - 2g_{12} + g_{22}) - g_1 + g_2, \end{aligned}$$

and another involving  $v$ , which would require either the arbitrary constant or the arbitrary functional form to which the third would be equated. (The fourth is, in fact, a primitive of the compatible equation under discussion, though it is not necessarily the common primitive of the three simultaneous equations.) We thus infer that

$$l - m - (y + z)(g_{11} - 2g_{12} + g_{22}) - g_1 + g_2 = \psi(x + y, z)$$

is an intermediary of the second of the equations.

In a similar manner, by writing

$$\phi = f_{111} - 3f_{112} + 3f_{122} - f_{222},$$

it can be shown that

$$l - n - (y + z) (f_{11} - 2f_{12} + f_{22}) - f_1 + f_2 = \chi(z + x, y)$$

is an intermediary of the third of the equations. When the conditions of coexistence of these two are assigned, they determine the arbitrary functions  $\psi$  and  $\chi$  in the forms

$$\psi = -f_1 + f_2, \quad \chi = -g_1 + g_2;$$

so that we have

$$\left. \begin{aligned} l - m &= (y + z) (g_{11} - 2g_{12} + g_{22}) + g_1 - g_2 - f_1 + f_2 \\ l - n &= (y + z) (f_{11} - 2f_{12} + f_{22}) + f_1 - f_2 - g_1 + g_2 \end{aligned} \right\};$$

and the primitive can be obtained by the customary process, leading to the form

$$v = 2f + 2g + (y + z) (f_1 - f_2 + g_1 - g_2),$$

where  $f = f(x + y, z)$  and  $g = g(x + z, y)$  are arbitrary functions.

26. If one or both of the equations compatible with the original equation were of the third order, we should then seek an equation of the second order involving one arbitrary function more than that equation of the third order; and we should proceed in a manner similar to that of the preceding plan, the conditions of coexistence of the different equations furnishing the means of identification or comparison of the arbitrary functions that occur.

If there be no equation of the third order, we should similarly proceed to obtain possible equations of the fourth order, if any; and so on with the orders in succession. The method is one of general application if equations of any order compatible with the original equation exist.

### SECTION III.

#### *Equations having an irresoluble characteristic invariant.*

27. The investigations contained in the preceding sections of this paper have referred for the most part to those equations

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0,$$

whose characteristic invariant

$$p^2 \frac{\partial F}{\partial a} + pq \frac{\partial F}{\partial b} + q^2 \frac{\partial F}{\partial c} - p \frac{\partial F}{\partial g} - q \frac{\partial F}{\partial f} + \frac{\partial F}{\partial c} = 0$$

is resolvable into equations that are linear in  $p$  and  $q$ . Those contained in the present section refer to equations whose characteristic invariant is irresoluble.

In the first section (§ 2), a generalisation of AMPÈRE's method was dealt with very briefly, partly because that method and DARBOUX's method apply most effectively to equations in which (with few exceptions) the derivatives of the highest order occur linearly; and, of the two, it is DARBOUX's method which can be more effectively applied to other equations. The fact that the characteristic invariant was resolvable proved of material importance in the general theory.

It is to be remarked, however, that some of the equations which occur most frequently in mathematical physics, for example

$$\begin{aligned}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} &= 0, \\ \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} &= \mu \frac{\partial r}{\partial t}, \\ \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} &= c^2 \frac{\partial^2 r}{\partial t^2},\end{aligned}$$

the latter two being, for purposes of application, made to depend upon the equation

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = -\kappa^2 r,$$

belong to the class which have their characteristic invariant not resolvable, and at the same time are linear in the derivatives of the highest orders that occur. Accordingly both AMPÈRE's method and DARBOUX's method generalised can be applied to such equations.

Moreover, the generalisation of AMPÈRE's method can also be applied to equations of the form

$$\begin{aligned}\theta\Theta + A \frac{\partial\Theta}{\partial a} + B \frac{\partial\Theta}{\partial b} + C \frac{\partial\Theta}{\partial c} + F \frac{\partial\Theta}{\partial f} + G \frac{\partial\Theta}{\partial g} + H \frac{\partial\Theta}{\partial h} \\ + A_1 a + B_1 b + C_1 c + F_1 f + G_1 g + H_1 h = U,\end{aligned}$$

where

$$\Theta = \begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix}$$

and the quantities  $\theta, A, \dots, H, A_1, \dots, H_1, U$ , do not involve derivatives of the second order. For, when the equation is transformed by the relations of § 1, it takes the form

$$J + cI = 0,$$

where  $I = 0$  is the characteristic equation; in other words, taking account of  $I = 0$ , we must associate  $J = 0$  with it as an equivalent to the postulated equation.



28. We consequently begin with the generalisation of AMPÈRE's method. Let the variables be changed from  $x, y, z$ , to  $x, y, u$ , where  $u$  is a function of  $x, y, z$ , as yet undetermined, so that  $z$  is a function of  $x, y, u$ , as yet also undetermined. With the notation previously adopted, we have

$$\left. \begin{aligned} \frac{dv}{dx} &= l + np, & \frac{dv}{dy} &= m + nq, & \frac{dv}{du} &= n \frac{dz}{du} \\ \frac{dl}{dx} &= a + gp, & \frac{dl}{dy} &= h + gq, & \frac{dl}{du} &= g \frac{dz}{du} \\ \frac{dm}{dx} &= h + fp, & \frac{dm}{dy} &= b + fq, & \frac{dm}{du} &= f \frac{dz}{du} \\ \frac{dn}{dx} &= g + cp, & \frac{dn}{dy} &= f + cq, & \frac{dn}{du} &= c \frac{dz}{du} \end{aligned} \right\};$$

and therefore, from the equations involving derivatives with regard to  $x$  and  $y$  alone, it follows that

$$\begin{aligned} a &= \frac{dl}{dx} - p \frac{dn}{dx} + p^2 c, & b &= \frac{dm}{dy} - q \frac{dn}{dy} + q^2 c, \\ g &= \frac{dn}{dx} - p c, & f &= \frac{dn}{dy} - q c, \\ h &= \frac{dm}{dx} - p \frac{dn}{dy} + p q c \\ &= \frac{dl}{dy} - q \frac{dn}{dx} + p q c \end{aligned} \left. \vphantom{\begin{aligned} a &= \frac{dl}{dx} - p \frac{dn}{dx} + p^2 c, \\ b &= \frac{dm}{dy} - q \frac{dn}{dy} + q^2 c, \\ g &= \frac{dn}{dx} - p c, \\ f &= \frac{dn}{dy} - q c, \end{aligned}} \right\},$$

so that we have

$$\frac{dm}{dx} - p \frac{dn}{dy} = \frac{dl}{dy} - q \frac{dn}{dx},$$

which is the condition in order that the necessary relation

$$\frac{d}{dy} \left( \frac{dc}{dx} \right) = \frac{d}{dx} \left( \frac{dv}{dy} \right)$$

be satisfied.

When the postulated equation of the second order is such that, on the substitution of the foregoing values for  $a, b, f, g, h$ , it has a linear form in  $c$ , let it be

$$J + cI = 0.$$

Suppose that the variable  $u$  (or  $z$  as a function of  $x, y, u$ ) is determined so that

$$I = 0,$$

which, after the earlier explanations, is the characteristic invariant; then we have also

$$J = 0.$$

The system of equations now is

$$\left. \begin{aligned} I &= 0, & J &= 0 \\ \frac{dm}{dx} - p \frac{dn}{dy} &= \frac{dl}{dy} - q \frac{dn}{dx} \\ \frac{dv}{dx} &= l + np, & \frac{dv}{dy} &= m + nq \end{aligned} \right\},$$

involving the quantities  $l, m, n, v, z$ , as functions of  $x$  and  $y$ . In all the derivatives here contained,  $u$  is parametric; and consequently all the constants that arise in the integration are constants on this supposition: in other words, all of them are functions of  $u$ . Consequently, when the integrals of the system are obtained, one constant (at choice) can be taken to be  $u$ ; all the other constants are then functions of  $u$ , arbitrary so far as the system is concerned; and any arbitrary function of  $x$  and  $y$  that occurs is also (possibly) a function of  $u$ . In order to determine the limitations on the arbitrary functions, the equation

$$\frac{dv}{du} = n \frac{dz}{du},$$

must also be satisfied; this equation will usually give relations among the arbitrary functional forms, or will determine one of them.

29. *The relations thus obtained constitute an integral of the equation.* For suppose that in the expression for  $v$  we consider  $u$  eliminated in favour of  $z$ ; then

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} \left( p dx + q dy + \frac{dz}{du} du \right).$$

But also

$$dv = \frac{dv}{dx} dx + \frac{dv}{dy} dy + \frac{dv}{du} du,$$

whence

$$\begin{aligned} \frac{dv}{du} &= \frac{\partial v}{\partial z} \frac{dz}{du}, \\ \frac{dv}{dx} &= \frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z}, \\ \frac{dv}{dy} &= \frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z}; \end{aligned}$$

and therefore, comparing these with the equations of the system leading to the integral form, it follows that

$$l = \frac{\partial v}{\partial x}, \quad m = \frac{\partial v}{\partial y}, \quad n = \frac{\partial v}{\partial z}.$$

Next, take a quantity  $c$  such that

$$\frac{dn}{du} = c \frac{dz}{du}.$$

Since

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{du} = n \frac{dz}{du}$$

are satisfied by the integral relations, we have

$$\frac{d}{du} (l + np) = \frac{d}{dx} \left( n \frac{dz}{du} \right),$$

and therefore

$$\frac{dl}{du} + p \frac{dn}{du} = \frac{dn}{dx} \frac{dz}{du},$$

the terms in  $\frac{d^2z}{dx du}$  cancelling; consequently

$$\frac{dl}{du} = \left( \frac{dn}{dx} - pc \right) \frac{dz}{du} = g \frac{dz}{du}.$$

Similarly, from

$$\frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n \frac{dz}{du},$$

we find

$$\frac{dm}{du} = f \frac{dz}{du},$$

the quantities  $g$  and  $f$  which occur here being those which formally occur in the derivatives of  $l, m, n$ , with regard to  $x$  and  $y$ .

Again, we have

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} dx + \frac{\partial^2 v}{\partial x \partial y} dy + \frac{\partial^2 v}{\partial x \partial z} (p dx + q dy + \frac{dz}{du} du) \\ = d \frac{\partial v}{\partial x} = dl = \frac{dl}{dx} dx + \frac{dl}{dy} dy + \frac{dl}{du} du, \end{aligned}$$

whence

$$\begin{aligned} \frac{dl}{du} &= \frac{\partial^2 v}{\partial x \partial z} \frac{dz}{du}, \\ \frac{dl}{dx} &= \frac{\partial^2 v}{\partial x^2} + p \frac{\partial^2 v}{\partial x \partial z}, \\ \frac{dl}{dy} &= \frac{\partial^2 v}{\partial x \partial y} + q \frac{\partial^2 v}{\partial x \partial z}; \end{aligned}$$

and therefore, comparing with the former equations, we have

$$a = \frac{\partial^2 v}{\partial x^2}, \quad h = \frac{\partial^2 v}{\partial x \partial y}, \quad g = \frac{\partial^2 v}{\partial x \partial z}.$$

Similarly we find

$$b = \frac{\partial^2 v}{\partial y^2}, \quad f = \frac{\partial^2 v}{\partial y \partial z}, \quad c = \frac{\partial^2 v}{\partial z^2}.$$

Now, when we take the combination

$$J + cI = 0,$$

and eliminate the derivatives of  $l, m, n$ , with regard to  $x$  and  $y$ , we have the original equation

$$F = 0,$$

or the original equation is satisfied in virtue of the integral system. But, from this integral system, the value of  $v$  is such that

$$l, m, n = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) v,$$

$$a, b, c, f, g, h = \left( \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z \partial x}, \frac{\partial^2}{\partial x \partial y} \right) v;$$

and therefore  $v$  is an integral of the partial differential equation. We consequently have the theorem—

*When an equation  $F = 0$  of the second order is transformed into  $J + cI = 0$  by means of the equations*

$$a = \frac{dl}{dx} - p \frac{dn}{dy} + p^2 c, \quad b = \frac{dm}{dy} - q \frac{dn}{dy} + q^2 c,$$

$$g = \frac{dn}{dx} - pc, \quad f = \frac{dn}{dy} - qc,$$

$$\left. \begin{aligned} h &= \frac{dm}{dx} - p \frac{dn}{dy} + pqc \\ &= \frac{dl}{dy} - q \frac{dn}{dx} + pqc \end{aligned} \right\},$$

*and when, in the integral equivalent of the simultaneous system*

$$\left. \begin{aligned} I &= 0, \quad J = 0 \\ \frac{dm}{dx} - p \frac{dn}{dy} &= \frac{dl}{dy} - q \frac{dn}{dx} \\ \frac{dv}{dx} &= l + np, \quad \frac{dv}{dy} = m + nq \end{aligned} \right\},$$

*all the arbitrary constants are made functions of a parameter  $u$ , and the arbitrary functions of  $x$  and  $y$  are also made functions of  $u$ , subject to the equation*

$$\frac{dv}{du} = n \frac{dz}{du},$$

*(which, in fact, will generally determine either an arbitrary function or relations among the arbitrary functions), then the value of  $v$  thus obtained is an integral of the original equation  $F = 0$ .*

30. The integration of the equation  $F = 0$  is thus made to depend upon the integration of a simultaneous system involving fewer independent variables, and



upon the subsequent determination of the arbitrary functions in the integral equivalent of the system. A question therefore arises as to how an integral equivalent can be obtained. At first sight it seems that, as the number of equations (being five) is equal to the number of unknowns ( $l, m, n, v, z$ ) to be determined, HAMBURGER'S method\* might be applied to our special instance, though not when the number of independent variables in the original equation is more than three. But, as a matter of fact, one of the equations of the system is a functional consequence of two others; viz., the equation

$$\frac{dm}{dx} - p \frac{dn}{dy} = \frac{dl}{dy} - q \frac{dn}{dx}$$

is a functional consequence of

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq.$$

It thus follows that there are only four equations independent of one another involving the five variables; consequently HAMBURGER'S method does not apply. On the other hand, the inference is that, as the equations are fewer in number by unity than the number of variables to be determined, one arbitrary element must exist in any general integral equivalent. This arbitrary element and other arbitrary functional forms, by the foregoing theory, are determined by means of the equation

$$\frac{dv}{du} = n \frac{dz}{du},$$

so far as they can be made determinate.

It is therefore necessary to seek for some integral combination of the subsidiary system, apparently without at present having any perfectly general process of constructing such a solution. It may, however, be pointed out that, as there are four independent equations involving five quantities, they can be used to determine four of them in terms of the remaining one or, more symmetrically when this is possible, to express all five of them in terms of some variable. When such expressions have been obtained, they are to be substituted in

$$\frac{dv}{du} = n \frac{dz}{du},$$

the full solution of the resulting form of which equation will then serve to determine the quantities.

We proceed to consider one or two examples in connection with the foregoing theory and explanations, dealing particularly with well-known equations.

\* CRELLE, t. lxxxi. (1876), pp. 243-281; *ib.*, t. xciii. (1882), pp. 188-214, the number of independent variables being two, and the number of equations being equal to the number of dependent variables.

*Application to  $\nabla^2 v = 0$ .*

31. When the method is applied to the potential equation, which is

$$a + b + c = 0$$

with the present notation, the substitution of values (say of  $a$  and  $b$ ) is required to lead to a result evanescent so far as the determination of coefficients of the second order is concerned. The substitution gives

$$\frac{dl}{dx} - p \frac{dn}{dx} + \frac{dm}{dy} - q \frac{dn}{dy} + c (p^2 + q^2 + 1) = 0$$

so that we must have

$$\begin{aligned} p^2 + q^2 + 1 &= 0, \\ \frac{dl}{dx} - p \frac{dn}{dx} + \frac{dm}{dy} - q \frac{dn}{dy} &= 0; \end{aligned}$$

and the differentiations with regard to  $x$  and to  $y$  in these relations are effected on the supposition that the unexpressed variable  $u$  is constant.

The subsidiary simultaneous system thus is

$$\left. \begin{aligned} p^2 + q^2 + 1 &= 0 \\ \frac{dl}{dx} - p \frac{dn}{dx} + \frac{dm}{dy} - q \frac{dn}{dy} &= 0 \\ \frac{dl}{dy} + p \frac{dn}{dy} - \frac{dm}{dx} - q \frac{dn}{dx} &= 0 \\ \frac{dv}{dx} &= l + np \\ \frac{dv}{dy} &= m + nq \end{aligned} \right\}$$

When HAMBURGER'S method, as expounded in the second of his memoirs already quoted (§ 30), is applied to this system, it is found that the algebraical equations for the determination of the subsidiary multipliers are inconsistent with one another unless all the multipliers are zero; there is then a null result. Accordingly integrable combinations must be obtained otherwise.

Now the general solution of the equation

$$p^2 + q^2 + 1 = 0$$

is given by

$$\begin{aligned} p &= \text{constant}, & q &= \text{constant}, \\ z - px - qy &= \text{constant}, \end{aligned}$$

these constants occurring in association with  $u$  constant. We may, therefore, assume

$$z = u + xp(u) + yq(u),$$

where  $p$  and  $q$  are arbitrary functions of  $u$ , subject solely to the condition

$$p^2 + q^2 + 1 = 0.$$

Because the differentiations with regard to  $x$  and to  $y$  are effected on the hypothesis that  $u$  is constant, the other equation can be taken in the form

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) = 0,$$

so that a function  $\xi$  of  $x$  and  $y$  (and possibly also involving  $u$ ) exists such that

$$l - np = \frac{d\xi}{dy}, \quad m - nq = -\frac{d\xi}{dx}.$$

Moreover, we have

$$l + np = \frac{dv}{dx}, \quad m + nq = \frac{dv}{dy};$$

consequently

$$2l = \frac{dv}{dx} + \frac{d\xi}{dy},$$

$$2m = \frac{dv}{dy} - \frac{d\xi}{dx},$$

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy}, \quad 2nq = \frac{dv}{dy} + \frac{d\xi}{dx}.$$

From the last two equations, it follows that

$$q\left(\frac{dv}{dx} - \frac{d\xi}{dy}\right) = p\left(\frac{dv}{dy} + \frac{d\xi}{dx}\right),$$

and thence that

$$\frac{d}{dx}(qv - p\xi) = \frac{d}{dy}(pv + q\xi).$$

Consequently a function  $w$  of  $x$  and  $y$  (and possibly also involving  $u$ ) exists such that

$$pv + q\xi = -\frac{dw}{dx},$$

$$qv - p\xi = -\frac{dw}{dy},$$

whence

$$v = p \frac{dw}{dx} + q \frac{dw}{dy},$$

$$\xi = q \frac{dw}{dx} - p \frac{dw}{dy}.$$

Substituting these values, we have

$$\left. \begin{aligned} 2l &= p \frac{d^2w}{dx^2} + 2q \frac{d^2w}{dx dy} - p \frac{d^2w}{dy^2} \\ 2m &= -q \frac{d^2w}{dx^2} + 2p \frac{d^2w}{dx dy} + q \frac{d^2w}{dy^2} \\ 2n &= \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} \end{aligned} \right\}.$$

Thus far account of variations with regard to  $x$  and to  $y$  has been taken. But, as regards variations of  $u$ , we have

$$\frac{dv}{du} = n \frac{dz}{du},$$

that is,

$$2 \frac{d}{du} \left( p \frac{dw}{dx} + q \frac{dw}{dy} \right) = \Delta \left( \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} \right),$$

where  $\Delta$  denotes  $1 + xp' + yq'$ . This is the equation of limitation upon the form of  $w$ ; if its general integral were known, the general value of  $v$  could be deduced.

2. Before proceeding with the consideration of this result, it is worth noting the relation of the equations

$$l - np = \frac{d\xi}{dy}, \quad m - nq = -\frac{d\xi}{dx},$$

to the original equation. Because

$$z = u + xp + yq,$$

it follows that

$$\frac{1}{p} \frac{\partial u}{\partial x} = \frac{1}{q} \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial z} = -\frac{1}{\Delta},$$

where  $\Delta$  denotes

$$1 + xp' + yq',$$

so that

$$\left. \begin{aligned} l - np &= \frac{\partial \xi}{\partial y} + q \frac{\partial \xi}{\partial z} \\ m - nq &= -\frac{\partial \xi}{\partial x} - p \frac{\partial \xi}{\partial z} \end{aligned} \right\},$$

when in  $\xi$  substitution for  $u$  is made in terms of  $x, y, z$ . From the former we have

$$\begin{aligned} a - gp - np' \frac{\partial u}{\partial x} &= \frac{\partial^2 \xi}{\partial x \partial y} + q \frac{\partial^2 \xi}{\partial x \partial z} + \frac{\partial \xi}{\partial z} q' \frac{\partial u}{\partial x} \\ g - cp - np' \frac{\partial u}{\partial z} &= \frac{\partial^2 \xi}{\partial y \partial z} + q \frac{\partial^2 \xi}{\partial z^2} + \frac{\partial \xi}{\partial z} q' \frac{\partial u}{\partial z}, \end{aligned}$$

and from the latter



$$\begin{aligned} b - fq - nq' \frac{\partial u}{\partial y} &= - \frac{\partial^2 \xi}{\partial x \partial y} - p \frac{\partial^2 \xi}{\partial y \partial z} - \frac{\partial \xi}{\partial z} p' \frac{\partial u}{\partial y}, \\ f - cq - nq' \frac{\partial u}{\partial z} &= - \frac{\partial^2 \xi}{\partial x \partial z} - p \frac{\partial^2 \xi}{\partial z^2} - \frac{\partial \xi}{\partial z} p' \frac{\partial u}{\partial z}. \end{aligned}$$

Multiplying the second of these equations by  $p$ , the fourth by  $q$ , and then adding all four, the function  $\xi$  is eliminated, and we have

$$a + b + c = 0.$$

33. Again, the general theory in a preceding Section (see particularly § 4) immediately suggests that  $u$  occurs as an argument of an arbitrary function. This being so, let the variables be considered as transformed to  $x, y, u$ , where  $u$  now is known, and effect the transformation directly upon the equation

$$a + b + c = 0.$$

We have, for any function  $P$ ,

$$\begin{aligned} \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz &= dP \\ &= \frac{dP}{dx} dx + \frac{dP}{dy} dy + \frac{dP}{du} du \\ &= \frac{dP}{dx} dx + \frac{dP}{dy} dy + \frac{dP}{du} \frac{1}{\Delta} (dz - p dx - q dy), \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{d}{dx} - \frac{p}{\Delta} \frac{d}{du}, \\ \frac{\partial}{\partial y} &= \frac{d}{dy} - \frac{q}{\Delta} \frac{d}{du}, \\ \frac{\partial}{\partial z} &= \frac{1}{\Delta} \frac{d}{du}. \end{aligned}$$

We thus have

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{d^2 v}{dx^2} - 2 \frac{p}{\Delta} \frac{d^2 v}{dx du} + \frac{p^2}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ - \frac{2pp'}{\Delta^2} + \frac{p^2}{\Delta^3} (xp'' + yq'') \right\}, \\ \frac{\partial^2 v}{\partial y^2} &= \frac{d^2 v}{dy^2} - 2 \frac{q}{\Delta} \frac{d^2 v}{dy du} + \frac{q^2}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ - \frac{2qq'}{\Delta^2} + \frac{q^2}{\Delta^3} (xp'' + yq'') \right\}, \\ \frac{\partial^2 v}{\partial z^2} &= \frac{1}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ \frac{1}{\Delta^3} (xp'' + yq'') \right\}, \end{aligned}$$

whence, adding and remembering that  $a + b + c = 0$ , it follows that

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} - \frac{2}{\Delta} \left( p \frac{d^2 v}{dx du} + q \frac{d^2 v}{dy du} \right) = 0.$$

It is immediately obvious that *one solution* of this equation (and therefore of the original equation) is given by equating  $v$  to any arbitrary function of  $u$ , a result that admits of simple verification.\*

34. But it is not at first sight clear how this solution connects itself with the general solution indicated in § 31; the connection can be made as follows.

The general value of  $v$  is

$$v = p \frac{dw}{dx} + q \frac{dw}{dy};$$

if this is to be an arbitrary function of  $u$ , say  $f(u)$ , as for the solution under consideration, we must have

$$p \frac{dw}{dx} + q \frac{dw}{dy} = f(u),$$

and consequently

$$w = \frac{x}{p} f(u) + G(u, p'x + q'y),$$

where, so far as concerns this relation,  $G$  is any arbitrary function of both its arguments. Writing

$$xp' + yq' = \eta,$$

this is

$$w = \frac{x}{p} f(u) + G(u, \eta),$$

so that

$$\frac{d^2w}{dx^2} = p'^2 \frac{\partial^2 G}{\partial \eta^2},$$

$$\frac{d^2w}{dx dy} = p'q' \frac{\partial^2 G}{\partial \eta^2},$$

$$\frac{d^2w}{dy^2} = q'^2 \frac{\partial^2 G}{\partial \eta^2}.$$

Now

$$p^2 + q^2 + 1 = 0,$$

\* This result was published in the 'Messenger of Mathematics,' vol. xxvii. (1898), pp. 99–118, in a short paper entitled "New Solutions of some of the Partial Differential Equations of Mathematical Physics." The form was altered from that in the text, so that it might be symmetric in the variables, and the theorem was given as follows:—

If  $p, q, r$ , be three arbitrary functions of  $u$  such that

$$p^2 + q^2 + r^2 = 0,$$

and if  $u$  be determined as a function of  $x, y, z$ , by the equation

$$au = xp + yq + zr,$$

where  $a$  is any constant, also if  $v$  denote any arbitrary function of  $u$ , then  $v$  satisfies LAPLACE'S equation

$$\nabla^2 v = 0.$$

so that

$$pp' + qq' = 0,$$

say

$$\frac{p'}{q} = \frac{q'}{-p} = \theta;$$

and so

$$\frac{1}{q^2} \frac{d^2 w}{dx^2} = - \frac{1}{pq} \frac{d^2 w}{dx dy} = \frac{1}{p^2} \frac{d^2 w}{dy^2} = \theta^2 \frac{\partial^2 G}{\partial \eta^2}.$$

Consequently

$$2l = p \frac{d^2 w}{dx^2} + 2q \frac{d^2 w}{dx dy} - p \frac{d^2 w}{dy^2} = p\theta^2 \frac{\partial^2 G}{\partial \eta^2},$$

$$2m = q\theta^2 \frac{\partial^2 G}{\partial \eta^2},$$

$$2n = -\theta^2 \frac{\partial^2 G}{\partial \eta^2}.$$

Now we should have

$$\frac{dv}{du} = n \frac{dz}{du} = (1 + \eta) n,$$

that is,

$$f'(u) = -\frac{1}{2} \theta^2 (1 + \eta) \frac{\partial^2 G}{\partial \eta^2},$$

from which the form of  $G$  is given by

$$G = A(u) + \eta B(u) - \frac{2}{\theta^2} f'(u) [(1 + \eta) \{\log(1 + \eta) - 1\}],$$

$A$  and  $B$  being arbitrary functions.

The form of  $G$  is, however, not so important for the present purpose as is the value of  $\partial^2 G / \partial \eta^2$ . We deduce

$$\frac{1}{2} \theta^2 \frac{\partial^2 G}{\partial \eta^2} = - \frac{f'(u)}{1 + \eta},$$

and consequently

$$\frac{l}{p} = \frac{m}{q} = \frac{n}{-1} = \frac{f'(u)}{1 + xp' + yq'},$$

being the proper values of  $l, m, n$ , as given by

$$\left. \begin{aligned} v &= f(u) \\ z &= u + xp + yq \end{aligned} \right\}.$$

The particular solution is thus seen to be included in the general solution defined by the equations of §31.

35. It is worth inquiring whether, in the notation of the preceding articles, there is any solution of the potential equation which is a function of  $u$  and  $\eta$  alone, say

$$v = F(u, \eta),$$

other than the solution  $v = \text{function of } u \text{ alone}$ . When the variables are taken to be  $x, y, u$ , the equation to be satisfied by  $v$  is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} - \frac{2}{\Delta} \left( p \frac{d^2v}{dx du} + q \frac{d^2v}{dy du} \right) = 0,$$

where

$$\Delta = 1 + xp' + yq' = 1 + \eta.$$

Now

$$\left( p \frac{d}{dx} + q \frac{d}{dy} \right) \eta = pp' + qq' = 0,$$

so that, as

$$\frac{dv}{du} = \frac{\partial F}{\partial u} + (xp'' + yq'') \frac{\partial F}{\partial \eta},$$

we have

$$p \frac{d^2v}{dx du} + q \frac{d^2v}{dy du} = (pp'' + qq'') \frac{\partial F}{\partial \eta} = -(p'^2 + q'^2) \frac{\partial F}{\partial \eta} = \theta^2 \frac{\partial F}{\partial \eta};$$

also

$$\begin{aligned} \frac{dv}{dx} &= p' \frac{\partial F}{\partial \eta}, & \frac{d^2v}{dx^2} &= p'^2 \frac{\partial^2 F}{\partial \eta^2}, \\ \frac{dv}{dy} &= q' \frac{\partial F}{\partial \eta}, & \frac{d^2v}{dy^2} &= q'^2 \frac{\partial^2 F}{\partial \eta^2}; \end{aligned}$$

and therefore

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = (p'^2 + q'^2) \frac{\partial^2 F}{\partial \eta^2} = -\theta^2 \frac{\partial^2 F}{\partial \eta^2}.$$

Hence the equation becomes

$$\frac{\partial^2 F}{\partial \eta^2} + \frac{2}{1 + \eta} \frac{\partial F}{\partial \eta} = 0,$$

and therefore

$$F = \frac{\psi(\eta)}{1 + \eta} + \phi(u),$$

where  $\phi$  and  $\psi$  are arbitrary functions. We thus have the theorem\*—

\* This result can also be expressed in the form symmetrical as regards the three variables. When thus modified, we have the theorem—

If  $p, q, r$ , be three arbitrary functions of  $u$  such that

$$p^2 + q^2 + r^2 = 0,$$

and if  $u$  be determined as a function of  $x, y, z$ , by the equation

$$au = xp + yq + zr,$$

where  $a$  is any constant, then, writing

$$v = \frac{G(u)}{a - xp' - yq' - zr'} + F(u),$$

where  $F$  and  $G$  are arbitrary functions,  $v$  satisfies LAPLACE'S equation

$$\nabla^2 v = 0.$$

This theorem also was stated in the paper referred to (§ 33, note).



If  $p(u)$  and  $q(u)$  denote any functions of  $u$  satisfying the equation

$$p^2 + q^2 + 1 = 0,$$

and if  $u$  be defined by the equation

$$z = u + xp(u) + yq(u),$$

then

$$v = \phi(u) + \frac{\psi(u)}{1 + xp'(u) + yq'(u)},$$

where  $\phi$  and  $\psi$  are arbitrary functions, satisfies LAPLACE'S equation

$$\nabla^2 v = 0.$$

36. The solution just obtained can, like the solution of § 33, be connected with the general solution. Owing to the linear form of all the equations and of the expression for  $u$ , it will be sufficient, in the first instance, to take the part

$$v = \frac{\psi(u)}{1 + \eta} = \frac{\psi}{1 + \eta},$$

say ; because the term  $\phi(u)$  in  $v$  is, in effect, identified by the preceding case. We thus must have

$$p \frac{dw}{dx} + q \frac{dw}{dy} = \frac{\psi}{1 + \eta},$$

the most general solution of which is

$$w = \frac{x}{1 + \eta} \frac{\psi}{p} + H(u, \eta),$$

where, so far as concerns this relation,  $H$  is any arbitrary function of both its arguments. We have

$$\begin{aligned} \frac{d^2 w}{dx^2} &= -\frac{2}{(1 + \eta)^2} \frac{p' \psi}{p} + \frac{2x}{(1 + \eta)^3} \frac{p' \psi}{p} + p'^2 \frac{\partial^2 H}{\partial \eta^2}, \\ \frac{d^2 w}{dx dy} &= -\frac{1}{(1 + \eta)^2} \frac{q' \psi}{p} + \frac{2x}{(1 + \eta)^3} \frac{p' q' \psi}{p} + p' q' \frac{\partial^2 H}{\partial \eta^2}, \\ \frac{d^2 w}{dy^2} &= \frac{2x}{(1 + \eta)^3} \frac{q'^2 \psi}{p} + q'^2 \frac{\partial^2 H}{\partial \eta^2}; \end{aligned}$$

and therefore, after some simple reductions on substituting in the expressions of § 31, we have

$$\left. \begin{aligned} l &= \frac{1}{2} \theta^2 p \frac{\partial^2 H}{\partial \eta^2} + \frac{x}{(1 + \eta)^3} \theta^2 \psi \\ m &= \frac{1}{2} \theta^2 q \frac{\partial^2 H}{\partial \eta^2} + \frac{x}{(1 + \eta)^3} \theta^2 \frac{q}{p} \psi - \frac{1}{(1 + \eta)^2} \theta \frac{\psi}{p} \\ n &= -\frac{1}{2} \theta^2 \frac{\partial^2 H}{\partial \eta^2} - \frac{x}{(1 + \eta)^3} \theta^2 \frac{\psi}{p} - \frac{1}{(1 + \eta)^2} \theta \frac{q \psi}{p} \end{aligned} \right\}.$$

But

$$p \frac{dw}{dx} + q \frac{dw}{dy} = \frac{\psi}{1 + \eta},$$

so that

$$\frac{d}{du} \left( p \frac{dw}{dx} + q \frac{dw}{dy} \right) = \frac{\psi'}{1 + \eta} - \frac{\psi}{(1 + \eta)^2} (xp'' + yq'');$$

and then, as

$$\frac{1}{1 + \eta} \frac{d}{du} \left( p \frac{dw}{dx} + q \frac{dw}{dy} \right) = \frac{1}{2} \left( \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right),$$

we have

$$\frac{\psi'}{(1 + \eta)^2} - \frac{\psi}{(1 + \eta)} (xp'' + yq'') = -\frac{1}{2} \theta^2 \frac{\partial^2 H}{\partial \eta^2} - \frac{x}{(1 + \eta)^3} \theta^2 \frac{\psi}{p} - \frac{1}{(1 + \eta)^2} \theta \frac{q\psi}{p},$$

an equation to determine H.

Substituting the value of  $\partial^2 H / \partial \eta^2$  in the above expressions for  $l, m, n$ , we find

$$\left. \begin{aligned} l &= -\frac{1}{(1 + \eta)^2} p\psi' - \frac{1}{(1 + \eta)^2} p'\psi + \frac{1}{(1 + \eta)^3} (xp'' + yq'') p\psi \\ m &= -\frac{1}{(1 + \eta)^2} q\psi' - \frac{1}{(1 + \eta)^2} q'\psi + \frac{1}{(1 + \eta)^3} (xp'' + yq'') q\psi \\ n &= \frac{1}{(1 + \eta)^2} \psi' - \frac{1}{(1 + \eta)^3} (xp'' + yq'') \psi \end{aligned} \right\},$$

which are the proper values of  $l, m, n$ , as given by

$$\left. \begin{aligned} v &= \frac{\psi(u)}{1 + xp' + yq'} \\ z &= u + xp + yq \end{aligned} \right\}.$$

The value of H is not required for the preceding investigation; but it can be actually deduced from the preceding equation. We have

$$xp'' + yq'' = \frac{q''}{q'} (\eta - xp') + xp'' = \frac{q''}{q'} \eta + \frac{x}{q'} (p''q' - q''p'),$$

also

$$\begin{aligned} p''q' - q''p' &= \frac{1}{q} (-p''pp' - q''qp') = -\frac{p'}{q} (pp'' + qq'') \\ &= \frac{p'}{q} (p'^2 + q'^2) = -\theta^2 \frac{p'}{q}, \end{aligned}$$

so that

$$\frac{x}{q'} (p''q' - q''p') = -\theta^2 \frac{p'}{qq'} x = \frac{\theta^2 x}{p}.$$

Thus the equation for H becomes

$$\frac{\psi'}{(1 + \eta)^2} - \left( \frac{q''}{q'} \eta + \theta^2 \frac{x}{p} \right) \frac{\psi}{(1 + \eta)^3} = -\frac{1}{2} \theta^2 \frac{\partial^2 H}{\partial \eta^2} - \frac{x}{(1 + \eta)^3} \theta^2 \frac{\psi}{p} - \frac{1}{(1 + \eta)^2} \theta \frac{p'\psi}{p};$$

the term involving  $x$  explicitly vanishes, as it should, and therefore we have

$$\begin{aligned}\theta^2 \frac{\partial^2 H}{\partial \eta^2} &= -\frac{2}{(1+\eta)^2} \psi' - \frac{2}{(1+\eta)^2} \frac{p'\psi}{p} + \frac{2\eta}{(1+\eta)^3} \frac{q''}{q'} \psi \\ &= -\frac{2}{(1+\eta)^3} \frac{q''}{q'} \psi + \frac{2}{(1+\eta)^2} \left\{ \left( \frac{q''}{q'} - \frac{p'}{p} \right) \psi - \psi' \right\} \\ &= -\frac{2}{(1+\eta)^3} \frac{q''}{q'} \psi + \frac{2}{(1+\eta)^2} \left\{ \frac{\theta'}{\theta} \psi - \psi' \right\}.\end{aligned}$$

Hence

$$H(u, \eta) = \frac{1}{1+\eta} \frac{q''}{q'\theta^2} \psi + \eta B(u) + A(u) + 2 \frac{\theta'\psi - \psi'\theta}{\theta^3} [(1+\eta) \{\log(1+\eta) - 1\}],$$

the explicit value of  $H$ , where  $A$  and  $B$  are arbitrary functions; and thus the value of  $w$  is known. But, as already remarked, the value of  $\partial^2 H / \partial \eta^2$  is sufficient for the identification of the solution.

37. After the preceding investigation, it is natural to inquire whether a solution more general than that which is expressible in terms of  $u$  and  $\eta$ , and the form of which has been suggested by the theory, can be obtained directly from the original differential equation.

When the variables are taken to be  $x, y, u$ , where

$$\left. \begin{aligned}z &= xp + yq + u \\ 0 &= p^2 + q^2 + 1\end{aligned} \right\},$$

$p$  and  $q$  being functions of  $u$ , the equation  $a + b + c = 0$  becomes

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} - \frac{2}{1+\eta} \left( p \frac{d^2 v}{dx du} + q \frac{d^2 v}{dy du} \right) = 0,$$

where

$$\eta = xp' + yq'.$$

Let  $\zeta$  be defined by the equation

$$\zeta = xp'' + yq'',$$

so that  $u, \eta, \zeta$ , are three variables functionally equivalent to  $x, y, z$ ; and let us inquire what solutions of the form

$$v = F(u, \eta, \zeta)$$

are possessed by the foregoing equation. We have

$$\begin{aligned}p' &= q\theta, & q' &= -p\theta, \\ p'' &= q\theta' - p\theta^2, & q'' &= -p\theta' - q\theta^2, \\ p''' &= q\theta'' - 3p\theta\theta' - q\theta^3, & q''' &= -p\theta'' - 3q\theta\theta' + p\theta^3;\end{aligned}$$

so that

$$\begin{aligned}
p'^2 + q'^2 &= -\theta^2, \\
p'p'' + q'q'' &= -\theta\theta', \\
p''^2 + q''^2 &= -\theta'^2 - \theta^4, \\
pp'' + qq'' &= \theta^2, \\
pp''' + qq''' &= 3\theta\theta',
\end{aligned}$$

quantities which occur in the substitution. Now

$$\begin{aligned}
\frac{dv}{dx} &= p' \frac{\partial F}{\partial \eta} + p'' \frac{\partial F}{\partial \zeta}, \\
\frac{d^2v}{dx^2} &= p'^2 \frac{\partial^2 F}{\partial \eta^2} + 2p'p'' \frac{\partial^2 F}{\partial \eta \partial \zeta} + p''^2 \frac{\partial^2 F}{\partial \zeta^2};
\end{aligned}$$

similarly

$$\frac{d^2v}{dy^2} = q'^2 \frac{\partial^2 F}{\partial \eta^2} + 2q'q'' \frac{\partial^2 F}{\partial \eta \partial \zeta} + q''^2 \frac{\partial^2 F}{\partial \zeta^2};$$

and therefore

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = -\theta^2 \frac{\partial^2 F}{\partial \eta^2} - 2\theta\theta' \frac{\partial^2 F}{\partial \eta \partial \zeta} - (\theta'^2 + \theta^4) \frac{\partial^2 F}{\partial \zeta^2}.$$

Again,

$$\begin{aligned}
\left(p \frac{d}{dx} + q \frac{d}{dy}\right) \eta &= 0, \\
\left(p \frac{d}{dx} + q \frac{d}{dy}\right) \zeta &= pp'' + qq'' = \theta^2,
\end{aligned}$$

so that, as

$$\frac{dv}{du} = \frac{\partial F}{\partial u} + \zeta \frac{\partial F}{\partial \eta} + (xp''' + yq''') \frac{\partial F}{\partial \zeta},$$

we have

$$\begin{aligned}
p \frac{d^2v}{dx du} + q \frac{d^2v}{dy du} &= \theta^2 \frac{\partial^2 F}{\partial u \partial \zeta} + \theta^2 \frac{\partial F}{\partial \eta} + \theta^2 \zeta \frac{\partial^2 F}{\partial \eta \partial \zeta} \\
&\quad + (pp''' + qq''') \frac{\partial F}{\partial \zeta} + \theta^2 (xp''' + yq''') \frac{\partial^2 F}{\partial \zeta^2}.
\end{aligned}$$

Taking account of the values of  $\eta$  and  $\zeta$ , it is not difficult to prove that

$$\theta^2 (xp''' + yq''') = \eta (\theta\theta'' - 3\theta'^2 - \theta^2) + \zeta 3\theta\theta';$$

consequently

$$\begin{aligned}
p \frac{d^2v}{dx du} + q \frac{d^2v}{dy du} &= \theta^2 \left\{ \frac{\partial^2 F}{\partial u \partial \zeta} + \zeta \frac{\partial^2 F}{\partial \eta \partial \zeta} + \frac{\partial F}{\partial \eta} \right\} \\
&\quad + 3\theta\theta' \frac{\partial F}{\partial \zeta} + \{\eta (\theta\theta'' - 3\theta'^2 - \theta^2) + 3\theta\theta'\zeta\} \frac{\partial^2 F}{\partial \zeta^2}
\end{aligned}$$

and the equation satisfied by  $F$  becomes

$$\begin{aligned}
\theta^2 \frac{\partial^2 F}{\partial \eta^2} + 2 \frac{\theta^2}{1 + \eta} \left( \frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial u \partial \zeta} \right) + 2 \left( \theta\theta' + \frac{\theta^2}{1 + \eta} \zeta \right) \frac{\partial^2 F}{\partial \eta \partial \zeta} + 6 \frac{\theta\theta'}{1 + \eta} \frac{\partial F}{\partial \zeta} \\
+ \frac{1}{1 + \eta} \{ \theta'^2 + \theta^4 + \eta (2\theta\theta'' - 5\theta'^2 - \theta'') + 6\theta\theta'\zeta \} \frac{\partial^2 F}{\partial \zeta^2} = 0.
\end{aligned}$$



We can at once infer the result of § 35 as to the solutions which, being of the suggested form, explicitly involve  $u$  and  $\eta$  only; but other solutions in a finite form do not suggest themselves.

38. It need only be remarked that, with the slightest change of notation, all the preceding results can be associated also with the equation

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = c^2 \frac{\partial^2 r}{\partial t^2},$$

where  $c$  is a constant.

$$\text{Application to } \nabla^2 v + \kappa^2 v = 0.$$

39. Consider next, but more briefly, the equation

$$\nabla^2 v + \kappa^2 v = 0,$$

where  $\kappa$  is a constant; in our notation, it may be written

$$a + b + c + \kappa^2 v = 0.$$

The characteristic invariant equation is

$$I = p^2 + q^2 + 1 = 0,$$

the same as for  $\nabla^2 v = 0$ . Substituting for  $a$  and  $b$  their values as given in § 1, it is seen that, on account of the characteristic equation, the term in  $c$  disappears from the result; and we have

$$J = \frac{dl}{dx} - p \frac{dn}{dx} + \frac{dm}{dy} - q \frac{dn}{dy} + \kappa^2 v = 0.$$

The subsidiary system thus is

$$\left. \begin{aligned} I &= 0, & J &= 0 \\ \frac{dm}{dx} - p \frac{dn}{dx} &= \frac{dl}{dy} - q \frac{dn}{dy} \\ \frac{dv}{dx} &= l + np, & \frac{dv}{dy} &= m + nq \end{aligned} \right\},$$

the third of which is an analytical consequence of the fourth and the fifth. As in the preceding example, the general solution of  $I = 0$  is

$$z = u + xp(u) + yq(u),$$

where  $p$  and  $q$  are functions subject to the relation

$$\{p(u)\}^2 + \{q(u)\}^2 + 1 = 0;$$

and in the subsidiary system the differentiations with regard to  $x$  and  $y$  are partial

on the supposition that  $u$  is the third variable: that is, in these partial differentiations in the subsidiary system,  $u$  does not vary. Consequently  $J = 0$  is expressible in the form

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) + \kappa^2 v = 0;$$

and therefore

$$\left. \begin{aligned} \frac{d^2}{dx^2}(l - np) + \frac{d^2}{dx dy}(m - nq) &= -\kappa^2 \frac{dv}{dx} = -\kappa^2(l + np) \\ \frac{d^2}{dx dy}(l - np) + \frac{d^2}{dy^2}(m - nq) &= -\kappa^2 \frac{dv}{dy} = -\kappa^2(m + nq) \end{aligned} \right\},$$

so that

$$\left. \begin{aligned} \left(\frac{d^2}{dx^2} + \kappa^2\right)l + \frac{d^2}{dx dy}m &= \left(p \frac{d^2}{dx^2} + q \frac{d^2}{dx dy} - p\kappa^2\right)n \\ \frac{d^2}{dx dy}l + \left(\frac{d^2}{dy^2} + \kappa^2\right)m &= \left(p \frac{d^2}{dx dy} + q \frac{d^2}{dy^2} - q\kappa^2\right)n \end{aligned} \right\}.$$

Operate on the first equation with  $d^2/dy^2 + \kappa^2$ , on the second with  $d^2/dx dy$ , and subtract; then, dividing by  $\kappa^2$ , we have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right)l = \left(p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} - p\kappa^2\right)n.$$

Similarly, operating on the first equation with  $d^2/dx dy$ , on the second with  $d^2/dx^2 + \kappa^2$ , subtracting, and dividing by  $\kappa^2$ , we have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right)m = \left(-q \frac{d^2}{dx^2} + 2p \frac{d^2}{dx dy} + q \frac{d^2}{dy^2} - q\kappa^2\right)m.$$

It therefore follows that some function  $w$  of  $x$ ,  $y$ , and  $u$  exists such that

$$\left. \begin{aligned} l &= p \frac{d^2 w}{dx^2} + 2q \frac{d^2 w}{dx dy} - p \frac{d^2 w}{dy^2} - p\kappa^2 w \\ m &= -q \frac{d^2 w}{dx^2} + 2p \frac{d^2 w}{dx dy} + q \frac{d^2 w}{dy^2} - q\kappa^2 w \\ n &= \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \kappa^2 w \end{aligned} \right\};$$

but, so far as the subsidiary system is concerned,  $w$  is perfectly arbitrary—a result in accordance with the general explanation of § 30, there being only four functionally independent equations for the preliminary determination of five quantities. Also

$$\begin{aligned} l + np &= 2 \left( p \frac{d^2 w}{dx^2} + q \frac{d^2 w}{dx dy} \right) = \frac{dv}{dx}, \\ m + nq &= 2 \left( p \frac{d^2 w}{dx dy} + q \frac{d^2 w}{dy^2} \right) = \frac{dv}{dy}; \end{aligned}$$

hence

$$v = 2 \left( p \frac{dw}{dx} + q \frac{dw}{dy} \right),$$

no arbitrary function of  $u$  needing to be added, for it can be considered as accounted for in the arbitrary function  $w$ .

But now, regarding variations of  $u$ , we must have the equation

$$\frac{dv}{du} = n \frac{dz}{du}$$

satisfied; that is, the function  $w$  must satisfy the equation

$$2 \frac{d}{du} \left( p \frac{dw}{dx} + q \frac{dw}{dy} \right) = \Delta \left( \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \kappa^2 w \right),$$

where  $\Delta$  denotes  $1 + xp' + yq'$ . As in the former example, this equation imposes the limitation upon the arbitrariness of  $w$ ; when its general integral is known, the most general value of  $v$  can be deduced.

40. It is an inference from the general theory that the quantity  $u$ , determined as a function of  $x, y, z$ , by the equations

$$\left. \begin{aligned} 0 &= 1 + \{p(u)\}^2 + \{q(u)\}^2 \\ z &= u + xp(u) + yq(u) \end{aligned} \right\},$$

is an argument of the arbitrary functions that occur in the solution of the equation  $\nabla^2 v + \kappa^2 v = 0$ . We therefore transform the variables from  $x, y, z$ , to  $x, y, u$ , where  $u$  now is known; and the result, obtained by analysis similar to that in § 33, is

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \kappa^2 v - \frac{2}{\Delta} \left( p \frac{d^2 v}{dx du} + q \frac{d^2 v}{dy du} \right) = 0.$$

Manifestly a function of  $u$  alone is not a solution of this equation; but, on the analogy of § 35, we are led to consider what solutions (if any) of the form

$$v = F(u, \eta)$$

are possessed by the equation,  $\eta$  having its former value  $xp' + yq'$ . When this value is substituted, the equation takes the form

$$-\theta^2 \frac{\partial^2 F}{\partial \eta^2} - 2 \frac{\theta^2}{1 + \eta} \frac{\partial F}{\partial \eta} + \kappa^2 F = 0,$$

where

$$-\theta^2 = p'^2 + q'^2.$$

Hence

$$\frac{\partial^2}{\partial \eta^2} \{(1 + \eta)F\} = \frac{\kappa^2}{\theta^2} (1 + \eta) F,$$

so that, as  $\theta$  is a function of  $u$  only, we have

$$(1 + \eta)F = Ae^{\frac{\kappa}{\theta}\eta} + Be^{-\frac{\kappa}{\theta}\eta},$$

where  $A$  and  $B$  are independent of  $\eta$ , that is, are arbitrary functions of  $u$ . We thus have the theorem\*---

*If  $p(u)$  and  $q(u)$  denote any functions of  $u$  satisfying the equation*

$$p^2 + q^2 + 1 = 0,$$

*and if  $u$  be defined as a function of  $x, y, z$ , by the equation*

$$z = u + xp(u) + yq(u),$$

*also if  $v$  denote*

$$\frac{\phi(u) e^{i\kappa(xp' + yq')(p'^2 + q'^2)^{-\frac{1}{2}}} + \psi(u) e^{-i\kappa(xp' + yq')(p'^2 + q'^2)^{-\frac{1}{2}}}}{1 + xp' + yq'},$$

*$\phi$  and  $\psi$  being arbitrary functions of  $u$ , then  $v$  satisfies the equation*

$$\nabla^2 v + \kappa^2 v = 0.$$

The connection between this solution and the general solution indicated in § 39 can be established as in the corresponding case of the potential equation (§§ 34, 36).

$$\text{Application to } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \mu \frac{\partial v}{\partial t}.$$

41. The preceding examples have led, in each case, to a solution which (though not the most general) was expressible in a finite form. We now take one other example, viz.,

$$\gamma l + b + c = 0,$$

which can be regarded as the equation for the variable conduction of heat in two

\* This result can also be expressed (see the paper already quoted, § 33, note) in a form symmetrical as regards the three variables, as follows:—

*If  $p, q, r$ , be three arbitrary functions of  $u$  such that*

$$\{p(u)\}^2 + \{q(u)\}^2 + \{r(u)\}^2 = 0,$$

*and if  $u$  be determined as a function of  $x, y, z$ , by the equation*

$$au = xp(u) + yq(u) + zr(u),$$

*$a$  being any constant, also if  $v$  denote*

$$\frac{\phi(u) e^{i\kappa(xp' + yq' + zr')(p'^2 + q'^2 + r'^2)^{-\frac{1}{2}}} + \psi(u) e^{-i\kappa(xp' + yq' + zr')(p'^2 + q'^2 + r'^2)^{-\frac{1}{2}}}}{a - xp' - yq' - zr'},$$

*where  $\phi$  and  $\psi$  denote arbitrary functions of  $u$ , then  $v$  satisfies the equation*

$$\nabla^2 v + \kappa^2 v = 0.$$



dimensions; it will be seen that the general solution is not expressible in a finite form.

The characteristic equation is

$$q^2 + 1 = 0,$$

say  $q = -i$ ; and then  $u$  is determined by

$$z + iy = xp(u) + u,$$

where  $p(u)$  is arbitrary. The subsidiary system is

$$\left. \begin{aligned} \gamma l + \frac{dm}{dy} + i \frac{dn}{dy} &= 0 \\ \frac{dv}{dx} &= l + np, \quad \frac{dv}{dy} = m - ni \end{aligned} \right\}.$$

We easily find

$$\left. \begin{aligned} \left( \frac{d^2}{dy^2} + \gamma \frac{d}{dx} \right) v + \left( 2i \frac{d}{dy} - \gamma p \right) u &= 0 \\ \left( \frac{d^2}{dy^2} + \gamma \frac{d}{dx} \right) m + \left( i \frac{d^2}{dy^2} - \gamma p \frac{d}{dy} - i \gamma \frac{d}{dx} \right) u &= 0 \end{aligned} \right\};$$

and so we have

$$\left. \begin{aligned} v &= 2i \frac{dw}{dy} - \gamma pw \\ n &= -\frac{d^2 w}{dy^2} - \gamma \frac{dw}{dx} \\ m &= i \frac{d^2 w}{dy^2} - \gamma p \frac{dw}{dy} - i \gamma \frac{dw}{dx} \\ l &= 2i \frac{d^2 w}{dx dy} + p \frac{d^2 w}{dy^2} \end{aligned} \right\},$$

where  $w$  is an arbitrary function of  $x$ ,  $y$ , and  $u$ . But we also must have the equation

$$\frac{dv}{du} = n \frac{dz}{du}.$$

satisfied; that is, we must satisfy the equation

$$2i \frac{d^2 w}{dy du} - \gamma p \frac{dw}{du} - \gamma w p' = -i \{1 + xp'(u)\} \left\{ \frac{d^2 w}{dy^2} + \gamma \frac{dw}{dx} \right\}.$$

If a general value of  $w$  can be deduced, then the general value of  $v$  could be constructed, and conversely.

42. Taking  $z + iy = s$ ,  $z - iy = s'$ , the equation can be written in the form

$$\frac{\partial^2 v}{\partial s \partial s'} + \frac{1}{2} \gamma \frac{\partial v}{\partial x} = 0,$$

a solution of which is given by

$$v = F(s, s') - 4 \frac{x}{\gamma} \frac{\partial^2 F}{\partial s \partial s'} + \frac{16x^2}{\gamma^2} \frac{1}{2!} \frac{\partial^4 F}{\partial s^2 \partial s'^2} - \dots,$$

where  $F$  is any arbitrary function of  $s$  and  $s'$ . Now the solution, which appears to be most frequently useful, of the original equation is obtained by taking

$$v = V e^{-\mu x},$$

where  $\mu$  is constant, and  $V$  is independent of  $x$ . In this case,

$$\frac{\partial v}{\partial x} = -\mu V e^{-\mu x},$$

and the equation for  $V$  is

$$\frac{\partial^2 V}{\partial s \partial s'} = \frac{1}{4} \gamma \mu V.$$

To identify the two forms, we must have

$$\begin{aligned} F(s, s') - 4 \frac{x}{\gamma} \frac{\partial^2 F}{\partial s \partial s'} + \frac{16x^2}{\gamma^2} \left(\frac{1}{2!}\right)^2 \frac{\partial^4 F}{\partial s^2 \partial s'^2} - \dots \\ = V e^{-\mu x} \\ = V - x \mu V + \frac{x^2}{2!} \mu^2 V - \dots; \end{aligned}$$

consequently

$$\left. \begin{aligned} V &= F(s, s') \\ \mu^n V &= \left(\frac{1}{\gamma}\right)^n \frac{\partial^{2n} F}{\partial s^n \partial s'^n} \end{aligned} \right\}$$

for all values of  $n$ . All the conditions are satisfied in virtue of

$$\frac{\partial^2 V}{\partial s \partial s'} = \frac{1}{4} \gamma \mu V;$$

and consequently the more general solution above obtained includes the less general solution customarily used when the arbitrariness of  $F$  is made subject to the equation

$$\frac{\partial^2 F}{\partial s \partial s'} = \frac{1}{4} \gamma \mu F.$$

This equation is of LAPLACE'S linear form with equal invariants, which, moreover, are constants; hence the whole series\* of derived invariants is unlimited in number, and the number of derivatives of an arbitrary function of  $s$  and the number of derivatives of an arbitrary function of  $s'$ , that occur in the most general solution, are unlimited in each case: that is, the solution is not expressible in finite terms.

\* DARBOUX, 'Théorie Générale des Surfaces,' t. ii.

43. To see how this solution is included in the general solution of § 41, it is necessary to determine a function  $w$  such that

$$2i \frac{dw}{dy} - \gamma p w = v = e^{-4 \frac{x}{\gamma} \frac{\delta^2}{\delta s \delta s'}} F(s, s') = \Phi(s, s'),$$

say. Now, for the derivatives with regard to  $y$ ,  $s$  is parametric, being  $xp + u$ ; and  $s' = s - 2iy$ . We thus find on integration

$$e^{\frac{1}{2} i \gamma p y} w = \phi(x, u) + \frac{1}{2i} \int e^{\frac{1}{2} i \gamma p y} \Phi(s, s - 2iy) dy,$$

$\phi$  being arbitrary, and  $s$  parametric in the integral. Evaluating the integral through integration by parts, and dividing by the exponential factor on the left-hand side, we find

$$\begin{aligned} w &= e^{-\frac{1}{2} i \gamma p y} \phi(x, u) - \frac{1}{\gamma p} \left\{ \Phi(s, s') + \sum_{n=1}^{\infty} \left( \frac{4}{\gamma p} \right)^n \frac{d^n}{ds'^n} \Phi(s, s') \right\} \\ &= e^{-\frac{1}{2} i \gamma p y} \phi(x, u) - \frac{1}{\gamma p} \sum_{n=0}^{\infty} \left( \frac{4}{\gamma p} \right)^n \frac{d^n \Phi}{ds'^n} \\ &= w_1 - w_2, \end{aligned}$$

say, so that  $w_1$  is expressed in terms of  $x, y, u$ , and  $w_2$  in terms of  $s, s', x$ .

Denoting derivatives of  $w_2$  with regard to  $s, s', x$ , by  $\delta/\delta s, \delta/\delta s', \delta/\delta x$ , respectively, we have

$$\frac{d}{dx} = p \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) + \frac{\delta}{\delta x} = p' \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) - \frac{4}{\gamma} \frac{\delta^2}{\delta s \delta s'},$$

this last change arising from the fact that every term in  $w_2$  satisfies the transformed equation because  $\Phi$  satisfies it;

$$\frac{d}{dy} = -2i \frac{\delta}{\delta s'}, \quad \frac{d}{du} = (1 + xp') \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right).$$

Consequently

$$\begin{aligned} \frac{dw_2}{dx} &= p \frac{\delta w_2}{\delta s} + p \frac{\delta w_2}{\delta s'} - \frac{4}{\gamma} \frac{\delta^2 w_2}{\delta s \delta s'}, \\ \frac{dw_2}{dy} &= -2i \frac{\delta w_2}{\delta s'}, \\ \frac{d^2 w_2}{dy^2} &= -4 \frac{\delta^2 w_2}{\delta s'^2}, \\ \frac{d^2 w_2}{dx dy} &= -2ip \frac{\delta^2 w_2}{\delta s \delta s'} - 2ip \frac{\delta^2 w_2}{\delta s'^2} + \frac{8i}{\gamma} \frac{\delta^3 w_2}{\delta s \delta s'^2}. \end{aligned}$$

Further,

$$\frac{d^2 w_1}{dy^2} + \gamma \frac{dw_1}{dx} = e^{-\frac{1}{2} i \gamma p y} \left( \gamma \frac{d\phi}{dx} - \frac{1}{4} \gamma^2 p^2 \phi \right),$$

so that

$$\begin{aligned}
-n &= \frac{d^2 w}{dy^2} + \gamma \frac{dw}{dx} \\
&= e^{-\frac{1}{2}i\gamma p y} \left( \gamma \frac{d\phi}{dx} - \frac{1}{4}\gamma^2 p^2 \phi \right) + 4 \frac{\delta^2 w_2}{\delta s'^2} - \gamma p \frac{\delta w_2}{\delta s} - \gamma p \frac{\delta w_2}{\delta s'} + 4 \frac{\delta^2 w_2}{\delta s \delta s'} \\
&= e^{-\frac{1}{2}i\gamma p y} \left( \gamma \frac{d\phi}{dx} - \frac{1}{4}\gamma^2 p^2 \phi \right) - \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) \left( \gamma p w_2 - 4 \frac{\delta w_2}{\delta s'} \right).
\end{aligned}$$

But

$$w_2 - \frac{4}{\gamma p} \frac{\delta w_2}{\delta s'} = \frac{1}{\gamma p} \Phi;$$

and therefore

$$-n = e^{-\frac{1}{2}i\gamma p y} \left( \gamma \frac{d\phi}{dx} - \frac{1}{4}\gamma^2 p^2 \phi \right) - \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) \Phi.$$

Now we must have

$$\frac{dr}{du} = n \frac{dz}{du},$$

that is,

$$(1 + xp') \left( \frac{\delta \Phi}{\delta s} + \frac{\delta \Phi}{\delta s'} \right) = (1 + xp') n,$$

and so

$$n = \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) \Phi.$$

But, because  $r = \Phi$ ,  $s = z + iy$ ,  $s' = z - iy$ , we have

$$\frac{\partial r}{\partial z} = \left( \frac{\delta}{\delta s} + \frac{\delta}{\delta s'} \right) \Phi = n;$$

that is, the foregoing value of  $n$  is the  $z$ -derivative of  $r$ . Further, comparing the two values of  $n$ , we find

$$\gamma \frac{d\phi}{dx} - \frac{1}{4}\gamma^2 p^2 \phi = 0,$$

whence

$$\phi = e^{\frac{1}{2}i\gamma p^2 x} A(u),$$

$A$  being an arbitrary function; hence

$$w = e^{\frac{1}{2}i\gamma p (p^2 x - 2iy)} A(u) - \frac{1}{\gamma p} \sum_{n=0}^{\infty} \left( \frac{4}{\gamma p} \right)^n \frac{d^n \Phi}{ds'^n},$$

the explicit value of  $w$ .

Next, for the value of  $l$ , we have

$$p \frac{d^2 w_1}{dy^2} + 2i \frac{dw_1}{dx dy} = \frac{d}{dy} \left( p \frac{dw_1}{dy} + 2i \frac{dw_1}{dx} \right) = 0$$

from the value of  $w$ , just obtained; and



$$\begin{aligned}
 p \frac{d^2 w_2}{dy^2} + 2i \frac{d^2 w_2}{dx dy} &= -4p \frac{\delta^2 w_2}{\delta s^2} + 4p \frac{\delta^2 w_2}{\delta s \delta s'} + 4p \frac{\delta^2 w_2}{\delta s'^2} - \frac{16}{\gamma} \frac{\delta^3 w_2}{\delta s \delta s'^2} \\
 &= 4 \frac{\delta^2}{\delta s \delta s'} \left( p - \frac{4}{\gamma} \frac{\delta}{\delta s'} \right) w_2 \\
 &= 4 \frac{\delta^2}{\delta s \delta s'} \frac{\Phi}{\gamma} \\
 &= \frac{4}{\gamma} \frac{\delta^2 \Phi}{\delta s \delta s'}.
 \end{aligned}$$

Hence

$$l = p \frac{d^2 w}{dy^2} + 2i \frac{d^2 w}{dx dy} = -\frac{4}{\gamma} \frac{\delta^2 \Phi}{\delta s \delta s'} = \frac{\partial \Phi}{\partial x} = \frac{\partial v}{\partial x},$$

or the value of  $l$  is the  $x$ -derivative of  $v$ .

Similarly for the value of  $m$ . The solution is thus seen to be included in the form given in § 41; moreover, the value of  $w$  obtained in the preceding investigation is a solution of the equation there required to be satisfied.

44. It has been seen (§ 27) that the equations to which the method involving derivatives of  $v, l, m, n$ , alone can be applied—which method has been indicated as the generalisation of AMPÈRE'S for the case of two independent variables—belong to a distinctly limited class. Moreover, even for equations in this class, it may happen that an integrable combination of the subsidiary system cannot be constructed, or that the quasi-general process sketched in § 28 is not effective. Consequently it becomes necessary to have some other method; and this is provided by what has been indicated as the generalisation of DARBOUX'S method for the case of two independent variables.

In the discussion, we shall consider only the case when it proves necessary to construct the first derivatives of an equation  $F = 0$ ; but the explanations apply, *mutatis mutandis*, to other cases when second derivatives of  $F = 0$  and derivatives of higher orders should be constructed. Examples of the latter cases, when the characteristic invariant is resolvable, have already been given in Section II.

45. The original equation is

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0.$$

The characteristic invariant is

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0,$$

and it is supposed to be irreducible. The subsidiary equations are

$$\left. \begin{aligned}
 X + A \left( \frac{da}{dx} - p \frac{dy}{dx} \right) + H \left( \frac{dh}{dx} - p \frac{dy}{dy} \right) + B \left( \frac{dh}{dy} - q \frac{dy}{dy} \right) + G \frac{dy}{dx} + F \frac{dy}{dy} &= 0 \\
 Y + A \left( \frac{db}{dx} - p \frac{df}{dx} \right) + H \left( \frac{db}{dx} - p \frac{df}{dy} \right) + B \left( \frac{db}{dy} - q \frac{df}{dy} \right) + G \frac{df}{dx} + F \frac{df}{dy} &= 0 \\
 Z + A \left( \frac{dg}{dx} - p \frac{dc}{dx} \right) + H \left( \frac{df}{dx} - p \frac{dc}{dy} \right) + B \left( \frac{df}{dy} - q \frac{dc}{dy} \right) + G \frac{dc}{dx} + F \frac{dc}{dy} &= 0
 \end{aligned} \right\};$$

and equations of definition and derivation are

$$\left. \begin{aligned} \frac{dv}{dx} &= l + np, & \frac{dv}{dy} &= m + nq \\ \frac{dl}{dx} &= \alpha + qp, & \frac{dl}{dy} &= h + gq \\ \frac{dm}{dx} &= h + fp, & \frac{dm}{dy} &= b + fq \\ \frac{dn}{dx} &= g + cp, & \frac{dn}{dy} &= f + cq \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{dv}{du} &= n \frac{dz}{du} \\ \frac{dl}{du} &= g \frac{dz}{du} \\ \frac{dm}{du} &= f \frac{dz}{du} \\ \frac{dn}{du} &= c \frac{dz}{du} \end{aligned} \right\}.$$

So far as concerns derivatives with regard to  $x$  and  $y$ , there appear to be twelve equations, viz., the characteristic invariant, the three subsidiary equations deduced by taking

$$\frac{DF}{Dx} = 0, \quad \frac{DF}{Dy} = 0, \quad \frac{DF}{Dz} = 0,$$

and the eight equations of definition and derivation; and the number of quantities to be obtained from the subsidiary system is eleven, viz.,  $a, b, c, f, g, h, l, m, n, v, z$ . On the other hand,  $F = 0$  is an integral of the system, and it is a persistent relation; consequently one of the set of three equations can be regarded as depending on the other two. Again, from the equations

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq,$$

we have

$$\frac{dl}{dy} + p \frac{dn}{dy} = \frac{dm}{dx} + q \frac{dn}{dx},$$

identically satisfied in virtue of the values of  $dl/dy, dm/dx, dn/dx, dn/dy$ , given by the other equations of the set of eight; hence one of these can be regarded as dependent functionally on the rest, and the set of eight are therefore equivalent to seven only.

The whole set of equations in the subsidiary system, involving derivatives with regard to  $x$  and  $y$ , thus contains ten independent equations; and eleven quantities are to be obtained. Hence it may be possible to express ten of these in terms of one of them, or to express the whole eleven in terms of a single new quantity, arbitrary so far as the set is concerned; its arbitrariness will then be limited so that it shall satisfy the subsidiary equations involving derivatives with regard to  $u$ .

46. *The relations thus obtained, as satisfying all the subsidiary equations, constitute an integral of the equation.*

This result is established by an argument similar to that in § 29 for the case when the subsidiary system is simpler; accordingly here it will not be repeated.

*Application to  $\nabla^2 v = 0$ .*

47. The investigations as to the potential equation, given in § 31, did not require the consideration of derivatives of  $a, b, c, f, g, h$ ; but the example can be used to indicate the method of proceeding when such derivatives must be taken into account, as in the preceding theory.

For the equation

$$a + b + c = 0,$$

the special subsidiary equations involving derivatives of  $a, b, c$ , with regard to  $x$  and  $y$ , are

$$\left. \begin{aligned} \frac{da}{dx} - p \frac{dg}{dx} + \frac{dh}{dy} - q \frac{dg}{dy} &= 0 \\ \frac{dh}{dx} - p \frac{df}{dx} + \frac{db}{dy} - q \frac{df}{dy} &= 0 \\ \frac{dg}{dx} - p \frac{dc}{dx} + \frac{df}{dy} - q \frac{dc}{dy} &= 0 \end{aligned} \right\};$$

the characteristic equation is

$$p^2 + q^2 + 1 = 0;$$

and the relations of identity are

$$\left. \begin{aligned} \frac{da}{dy} + p \frac{dg}{dy} - \frac{dh}{dx} - q \frac{dg}{dx} &= 0 \\ \frac{dh}{dy} + p \frac{df}{dy} - \frac{db}{dx} - q \frac{df}{dx} &= 0 \\ \frac{dg}{dy} + p \frac{df}{dy} - \frac{df}{dx} - q \frac{dc}{dx} &= 0 \end{aligned} \right\}.$$

But the latter can be replaced by the equations of definition, viz.,

$$\left. \begin{aligned} \frac{dl}{dx} &= a + gp, & \frac{dl}{dy} &= h + gq, & \frac{dl}{du} &= g \frac{dz}{du} \\ \frac{dm}{dx} &= h + fp, & \frac{dm}{dy} &= b + fq, & \frac{dm}{du} &= f \frac{dz}{du} \\ \frac{dn}{dx} &= g + cp, & \frac{dn}{dy} &= f + cq, & \frac{dn}{du} &= c \frac{dz}{du} \end{aligned} \right\},$$

together with

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq, \quad \frac{dv}{du} = n \frac{dz}{du};$$

it is, indeed, from the first two columns of these equations of definition that the relations of identity are deduced.

In the subsidiary equations and in the relations of identity, the derivatives with regard to  $x$  and  $y$  are framed on the supposition that  $u$  is constant. Now the complete solution of

$$p^2 + q^2 + 1 = 0$$

is given, by CHARPIT'S process, in the form

$$p = \text{constant}, \quad q = \text{constant}, \quad z - px - qy = \text{constant}.$$

But these constants arise when  $u$  is constant; hence we may take

$$z - px - qy = u,$$

where  $p = p(u)$  and  $q = q(u)$  are arbitrary functions of  $u$  subject to the condition

$$p^2 + q^2 + 1 = 0.$$

From the first of the three subsidiary equations, remembering that  $p$  and  $q$  are now constant so far as concerns derivatives with regard to  $x$  and  $y$ , we have

$$\frac{d}{dx}(\alpha - pg) + \frac{d}{dy}(h - qg) = 0,$$

so that some function  $w$  of  $x, y, u$ , exists such that

$$\alpha - pg = \frac{dw}{dy}, \quad h - qg = -\frac{dw}{dx}.$$

Also we have

$$\alpha + pg = \frac{dl}{dx}, \quad h + qg = \frac{dl}{dy},$$

so that

$$\left. \begin{aligned} 2\alpha &= \frac{dw}{dy} + \frac{dl}{dx}, \\ 2h &= -\frac{dw}{dx} + \frac{dl}{dy}, \\ 2pg &= -\frac{dw}{dy} + \frac{dl}{dx} \\ 2qg &= \frac{dw}{dx} + \frac{dl}{dy} \end{aligned} \right\}.$$

From the last we have

$$p\left(\frac{dw}{dx} + \frac{dl}{dy}\right) = q\left(\frac{dw}{dy} + \frac{dl}{dx}\right),$$

that is,

$$\frac{d}{dx}(pw - ql) + \frac{d}{dy}(pl + qw) = 0,$$

so that some function  $\theta$  of  $x, y, u$ , exists such that



$$pw - ql = \frac{d\theta}{dy}, \quad qw + pl = -\frac{d\theta}{dx},$$

or

$$w = -p \frac{d\theta}{dy} + q \frac{d\theta}{dx}, \quad l = q \frac{d\theta}{dy} + p \frac{d\theta}{dx}.$$

Substituting these values above, we find

$$\left. \begin{aligned} 2a &= \left( p, q, -p \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \theta \right) \\ 2h &= \left( -q, p, q \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \theta \right) \\ 2g &= \left( 1, 0, 1 \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \theta \right) \\ l &= \left( p, q \left( \frac{d}{dx}, \frac{d}{dy} \right) \theta \right) \end{aligned} \right\},$$

$\theta$  denoting an arbitrary function of  $x, y, u$ .

The second of the three subsidiary equations similarly gives

$$\left. \begin{aligned} 2h &= \left( p, q, -p \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \chi \right) \\ 2b &= \left( -q, p, q \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \chi \right) \\ 2f &= \left( 1, 0, 1 \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \chi \right) \\ m &= \left( p, q \left( \frac{d}{dx}, \frac{d}{dy} \right) \chi \right) \end{aligned} \right\},$$

$\chi$  denoting an arbitrary function of  $x, y, u$ ; and the third of the subsidiary equations gives

$$\left. \begin{aligned} 2g &= \left( p, q, -p \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \psi \right) \\ 2f &= \left( -q, p, q \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \psi \right) \\ 2c &= \left( 1, 0, 1 \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \psi \right) \\ n &= \left( p, q \left( \frac{d}{dx}, \frac{d}{dy} \right) \psi \right) \end{aligned} \right\},$$

$\psi$  denoting an arbitrary function.

These three arbitrary functions are not independent of one another. Comparing the two values of  $2f, 2g, 2h$ , respectively which have been obtained, we see that some function  $\phi$  of  $x, y, u$ , exists such that

$$\left. \begin{aligned} \theta &= \left( p, q, -p \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \phi \right) \\ \chi &= \left( -q, p, q \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \phi \right) \\ \psi &= \left( 1, 0, 1 \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \phi \right) \end{aligned} \right\} ;$$

manifestly  $\phi$  is an arbitrary function of  $x, y, u$ .

To deduce the value of  $v$  so far as concerns variations of  $x$  and  $y$ , we have

$$\begin{aligned} \frac{dv}{dx} &= l + np \\ &= \left( p, q \left( \frac{d}{dx}, \frac{d}{dy} \right) \right) (\theta + p\psi) \\ &= 2 \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left( p \frac{d^2\phi}{dx^2} + q \frac{d^2\phi}{dx dy} \right), \end{aligned}$$

and

$$\begin{aligned} \frac{dv}{dy} &= m + nq \\ &= \left( p, q \left( \frac{d}{dx}, \frac{d}{dy} \right) \right) (\chi + q\psi) \\ &= 2 \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left( p \frac{d^2\phi}{dx dy} + q \frac{d^2\phi}{dy^2} \right); \end{aligned}$$

therefore

$$v = 2 \left( p^2 \frac{d^2\phi}{dx^2} + 2pq \frac{d^2\phi}{dx dy} + q^2 \frac{d^2\phi}{dy^2} \right) + V(u),$$

where thus far  $V$  is any arbitrary function of  $u$  alone.

48. As yet no account has been taken of derivatives with regard to  $u$ ; but the whole of the equations are subsidiary to the construction of an integral of the original equation.

The part of the value of  $v$  given by  $V(u)$  can be dropped, as it has been considered in the earlier stage, or it can be regarded as absorbed in the other part of the value, all that is necessary for this purpose being that the function  $\phi$  should have a different form.

Assuming this done, the function  $\phi$  must be such as, first, to satisfy the equation

$$\frac{dv}{du} = n \frac{dz}{du} = n\Delta,$$

with the preceding notation. Now

$$\frac{dv}{du} = 2 \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left\{ p \frac{d^2 \phi}{dx du} + q \frac{d^2 \phi}{dy du} + 2 \left( p' \frac{d}{dx} + q' \frac{d}{dy} \right) \phi \right\};$$

also

$$n = \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left( \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right);$$

and, as

$$\Delta = 1 + xp' + yq',$$

we have

$$\left( p \frac{d}{dx} + q \frac{d}{dy} \right) \Delta = 0;$$

that is,  $\Delta$  behaves as a constant with regard to the operator  $p \frac{d}{dx} + q \frac{d}{dy}$ . The required equation will therefore be satisfied if

$$p \frac{d^2 \phi}{dx du} + q \frac{d^2 \phi}{dy du} + 2p' \frac{d\phi}{dx} + 2q' \frac{d\phi}{dy} = \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right) \Delta.$$

But it is necessary that the other three equations

$$\frac{1}{y} \frac{dl}{du} = \frac{1}{r}, \quad \frac{dm}{du} = \frac{1}{c}, \quad \frac{dn}{du} = \frac{d}{du},$$

should also be satisfied; that they lead to no new condition for  $\phi$  can be seen as follows. We have

$$\begin{aligned} & \left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} \right) \left\{ \left( \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right) \Delta \right\} \\ &= \Delta \left( p, q, -p \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \left( \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right) \right. \\ & \quad + 2p \left\{ p' \left( \frac{d^3 \phi}{dx^3} + \frac{d^3 \phi}{dx dy^2} \right) \right\} \\ & \quad + 2q \left\{ q' \left( \frac{d^3 \phi}{dx^3} + \frac{d^3 \phi}{dx dy^2} \right) + p' \left( \frac{d^3 \phi}{dx^2 dy} + \frac{d^3 \phi}{dy^3} \right) \right\} \\ & \quad \left. - 2p \left\{ q' \left( \frac{d^3 \phi}{dx^2 dy} + \frac{d^3 \phi}{dy^3} \right) \right\} \right) \\ &= \Delta 2g + 2(p'q - pq') \left( \frac{d^3 \phi}{dx^2 dy} + \frac{d^3 \phi}{dy^3} \right); \end{aligned}$$

so that

$$\left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} \right) \left\{ \frac{1}{2} \left( \frac{d^2 \phi}{dx^2} + \frac{d^2 \phi}{dy^2} \right) \Delta \right\} = \Delta g + (p'q - pq') \left( \frac{d^3 \phi}{dx^2 dy} + \frac{d^3 \phi}{dy^3} \right).$$

On account of the equation satisfied by  $\phi$ , this is equal to

$$\left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} \right) \left\{ \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \frac{d\phi}{du} + 2 \left( p' \frac{d}{dx} + q' \frac{d}{dy} \right) \phi \right\},$$

which is equal to

$$\begin{aligned} & \frac{d}{du} \left[ \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} \right) \phi \right] \\ & + \left( p' \frac{d}{dx} + q' \frac{d}{dy} \right) \left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} \right) \phi \\ & - \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \left( p' \frac{d^2}{dx^2} + 2q' \frac{d^2}{dx dy} - p' \frac{d^2}{dy^2} \right) \phi \\ & = \frac{dl}{du} + (p'q - pq') \left( \frac{d^3\phi}{dx^2 dy} + \frac{d^3\phi}{dy^3} \right); \end{aligned}$$

and therefore

$$\frac{dl}{du} = \Delta g = g \frac{dz}{du},$$

in virtue of the equation

$$p \frac{d^2\phi}{dx du} + q \frac{d^2\phi}{dx dy} + 2 \left( p' \frac{d\phi}{dx} + q' \frac{d\phi}{dy} \right) = \frac{1}{2} \left( \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} \right) \Delta.$$

Similarly it can be verified that

$$\frac{dm}{du} = f \frac{dz}{du}, \quad \frac{dn}{du} = c \frac{dz}{du},$$

in virtue of this equation; therefore it is the only condition to be imposed upon  $\phi$  in order that all the equations may be satisfied.

49. Solutions of the equation  $a + b + c = 0$  have been given in §§ 31–38; it is not difficult to verify that they are included in the preceding general form. Thus, taking the solution

$$v = f(u)$$

and assuming that  $\phi$  is, as in § 48, the function through which the term  $V(u)$  in  $v$  is absorbed, it is first necessary to determine  $\phi$  so that

$$p^2 \frac{d^2\phi}{dx^2} + 2pq \frac{d^2\phi}{dx dy} + q^2 \frac{d^2\phi}{dy^2} = \frac{1}{2} f(u).$$

This equation is easily solved; and we have

$$\phi = \frac{1}{4} \frac{u^2}{p^2} f(u) + G(u, \eta) + xH(u, \eta),$$

where  $G$  and  $H$  are arbitrary functions, and  $\eta$  denotes  $p'x + q'y$ , as before. We have

$$\begin{aligned} \frac{d^2\phi}{dx^2} &= \frac{1}{2} \frac{1}{p^2} f(u) + p'^2 \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right) + 2p' \frac{\partial H}{\partial \eta}, \\ \frac{d^2\phi}{dx dy} &= p'q' \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right) + q' \frac{\partial H}{\partial \eta}, \\ \frac{d^2\phi}{dy^2} &= q'^2 \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right); \end{aligned}$$



so that

$$\begin{aligned}\psi &= \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = \frac{1}{2} \frac{f(u)}{p^2} - \theta^2 \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right), \\ \chi &= \left( -q, p, q \right) \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \phi = -\frac{1}{2} \frac{qf(u)}{p^2} + q\theta^2 \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right) + 2(pq' - p'q) \frac{\partial H}{\partial \eta}, \\ \theta &= \left( p, q, -p \right) \left( \frac{d}{dx}, \frac{d}{dy} \right)^2 \phi = \frac{1}{2} \frac{f(u)}{p} + p\theta^2 \left( \frac{\partial^2 G}{\partial \eta^2} + x \frac{\partial^2 H}{\partial \eta^2} \right).\end{aligned}$$

Hence

$$\begin{aligned}l &= \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \theta = p^2 \theta^2 \frac{\partial^2 H}{\partial \eta^2}, \\ m &= \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \chi = pq \theta^2 \frac{\partial^2 H}{\partial \eta^2}, \\ n &= \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \psi = -p \theta^2 \frac{\partial^2 H}{\partial \eta^2}.\end{aligned}$$

But  $\phi$  (or, what is the same thing in effect,  $G$  and  $H$ ) must be such as to give

$$\frac{dv}{du} = n \frac{dz}{du},$$

so that

$$n = \frac{f''(u)}{\Delta};$$

consequently we have

$$\frac{l}{-p} = \frac{m}{-q} = \frac{n}{1} = \frac{f''(u)}{\Delta},$$

which are the proper values of  $l, m, n$ , connected with  $v = f(u)$ .

Similarly for the quantities  $a, b, c, f, g, h$ .

And if the value of  $H$  be required, it is determined by the equation

$$-p\theta^2 \frac{\partial^2 H}{\partial \eta^2} = n = \frac{f''(u)}{\Delta} = \frac{f''(u)}{1 + \eta},$$

the integration of which is immediate.

*Application to  $\nabla^2 v + \kappa^2 v = 0$ .*

50. The integrable combinations of the subsidiary system in the case of the potential-equation seem fortuitously obtained. As one other illustration, added chiefly to show that the subsidiary system can be used in the mode indicated in § 39 to express all the variable quantities in terms of a single quantity, we consider the equation  $\nabla^2 v + \kappa^2 v = 0$ , which is

$$a + b + c + \kappa^2 v = 0,$$

in the notation of the present paper.

We have

$$\begin{aligned} A = B = C = 1, & \quad F = G = H = 0, \\ X = \kappa^2 l, & \quad Y = \kappa^2 m, \quad Z = \kappa^2 n. \end{aligned}$$

Thus the characteristic invariant equation is

$$p^2 + q^2 + 1 = 0,$$

from which we have, as before,

$$z = u + xp(u) + yq(u),$$

so that  $p$  and  $q$  are parametric for differentiations with regard to  $x$  and  $y$ . The three subsidiary equations are

$$\left. \begin{aligned} \kappa^2 l + \frac{da}{dx} - p \frac{dg}{dx} + \frac{dh}{dy} - q \frac{dg}{dy} &= 0 \\ \kappa^2 m + \frac{dh}{dx} - p \frac{df}{dx} + \frac{db}{dy} - q \frac{df}{dy} &= 0 \\ \kappa^2 n + \frac{dg}{dx} - p \frac{dc}{dx} + \frac{df}{dy} - q \frac{dc}{dy} &= 0 \end{aligned} \right\};$$

and we have the equations of definition and derivation as before.

51. One simple mode of proceeding is to use the six equations involving derivatives of  $l, m, n$ , with regard to  $x$  and  $y$ —they are equivalent to five independent equations—in order to express  $a, b, f, g, h$ , in terms of  $c$  and those derivatives. These being obtained in the form

$$\begin{aligned} a &= \frac{dl}{dx} - p \frac{dn}{dx} + p^2 c, & g &= \frac{dn}{dx} - pc, \\ b &= \frac{dm}{dy} - q \frac{dm}{dy} + q^2 c, & f &= \frac{dn}{dy} - qc, \\ h &= \frac{dm}{dx} - p \frac{dn}{dy} + pqc = \frac{dl}{dy} - q \frac{dn}{dx} + pqc, \end{aligned}$$

we substitute them in the subsidiary equations. The third of the latter then becomes

$$\left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \right) n = 2 \left( p \frac{d}{dx} + q \frac{d}{dy} \right) c;$$

the second of them, on using the first of the two values for  $h$  and also the relation between  $n$  and  $c$  just deduced, gives

$$\left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \right) m = \left( -q \frac{d^2}{dx^2} + 2p \frac{d^2}{dx dy} + q \frac{d^2}{dy^2} - q\kappa^2 \right) n;$$

and the first of them, on using the second of the two values for  $h$  and also the relation between  $n$  and  $c$ , gives

$$\left( \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \right) l = \left( p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} - p\kappa^2 \right) n.$$

Let the various operators be denoted by  $\delta_1, \delta_2, \delta_3$ , say

$$\left. \begin{aligned} \delta_1 &= p \frac{d^2}{dx^2} + 2q \frac{d^2}{dx dy} - p \frac{d^2}{dy^2} - p\kappa^2 \\ \delta_2 &= -q \frac{d^2}{dx^2} + 2p \frac{d^2}{dx dy} + q \frac{d^2}{dy^2} - q\kappa^2 \\ \delta_3 &= \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \end{aligned} \right\} ;$$

moreover, let

$$\delta = p \frac{d}{dx} + q \frac{d}{dy}$$

be another operator. Then from the relations between  $m$  and  $n$ ,  $l$  and  $n$ , it follows that a function  $\Theta$  exists such that

$$l = \delta_1 \Theta, \quad m = \delta_2 \Theta, \quad n = \delta_3 \Theta ;$$

and then, as

$$\left. \begin{aligned} \frac{dx}{dy} &= l + np = 2 \left( p \frac{d^2}{dx^2} + q \frac{d^2}{dx dy} \right) \Theta \\ \frac{dx}{dy} &= m + nq = 2 \left( p \frac{d^2}{dx dy} + q \frac{d^2}{dy^2} \right) \Theta \end{aligned} \right\} ,$$

we have

$$v = 2 \left( p \frac{d}{dx} + q \frac{d}{dy} \right) \Theta = 2\delta\Theta.$$

From the relation between  $n$  and  $c$ , we have

$$2\delta c = \delta_3 n = \delta_3^2 \Theta,$$

or, if  $\Theta = \delta w$ , then

$$2c = \delta_3^2 w.$$

We thus have, from the foregoing relations, the results

$$\left. \begin{aligned} v &= 2\delta^2 w \\ l &= \delta\delta_1 w \\ m &= \delta\delta_2 w \\ n &= \delta\delta_3 w \\ c &= \frac{1}{2}\delta_3^2 w \end{aligned} \right\} .$$

The values of  $a, b, f, g, h$ , can be obtained by using these values of  $l, m, n, c$ , and we find

$$\left. \begin{aligned} a &= \frac{1}{2} \delta_1^2 w \\ h &= \frac{1}{2} \delta_1 \delta_2 w \\ b &= \frac{1}{2} \delta_2^2 w \\ g &= \frac{1}{2} \delta_1 \delta_3 w \\ f &= \frac{1}{2} \delta_2 \delta_3 w \end{aligned} \right\}.$$

All these forms correspond to the respective expressions obtained in § 47 for the potential-equation.

52. Account must now be taken of derivatives with regard to  $u$ , and the equations to be satisfied are

$$\frac{1}{n} \frac{dv}{du} = \frac{1}{g} \frac{dl}{du} = \frac{1}{f} \frac{dm}{du} = \frac{1}{c} \frac{dn}{du} = \frac{dz}{du}.$$

Taking the first, we have

$$\frac{dz}{du} = 1 + xp' + yq' = \Delta,$$

so that the equation is to be

$$\frac{dv}{du} = n\Delta = \Delta \delta \delta_3 w.$$

Now, denoting

$$p' \frac{d}{dx} + q' \frac{d}{dy} \text{ by } \delta',$$

and so for  $\delta_1'$ ,  $\delta_2'$ —viz.,  $\delta_1'$  is the same as  $\delta_1$  except that  $p'$  and  $q'$  replace  $p$  and  $q$ , and so for  $\delta_2'$ —we have

$$\begin{aligned} \frac{dv}{du} &= \frac{d}{du} (2\delta^2 w) \\ &= 2\delta^2 \frac{dw}{du} + 4\delta \delta' w; \end{aligned}$$

and therefore

$$2\delta^2 \frac{dw}{du} + 4\delta \delta' w = \Delta \delta \delta_3 w.$$

Now

$$\delta \Delta = pp' + qq' = 0,$$

so that

$$\delta (\Delta \delta_3 w) = \Delta \delta \delta_3 w.$$

Hence the equation in  $w$  will be satisfied if

$$2\delta \frac{dw}{du} + 4\delta' w = \Delta \delta_3 w$$

the condition that we may have

$$\frac{dv}{du} = n \frac{dz}{du}.$$



But now, operating on this equation with  $\delta_1$ , we have

$$\delta_1 \left[ 2\delta \frac{dw}{du} + 4\delta' w \right] = \delta_1 (\Delta \delta_3 w).$$

The left-hand side is

$$\begin{aligned} 2\delta\delta_1 \frac{dw}{du} + 4\delta'\delta_1 w &= 2\delta\delta_1 \frac{dw}{du} + 2\delta'\delta_1 w + 2\delta_1'\delta w + 2(\delta'\delta_1 - \delta_1'\delta) w \\ &= \frac{d}{du} \{2\delta\delta_1 w\} + 2(\delta'\delta_1 - \delta_1'\delta) w. \end{aligned}$$

The first term is  $2 \frac{dl}{du}$ ; the second, on reduction, gives

$$2(qp' - pq') \frac{d}{dy} \delta_3 w;$$

and therefore the left-hand side is

$$2 \frac{dl}{du} + 2(qp' - pq') \frac{d}{dy} \delta_3 w.$$

The right-hand side, viz.,  $\delta_1 (\Delta \delta_3 w)$ , becomes on expansion and reduction

$$\Delta \delta_1 \delta_3 w + 2(p'q - pq') \frac{d}{dy} \delta_3 w = \Delta 2g + 2(p'q - pq') \frac{d}{dy} \delta_3 w.$$

Hence, in virtue of the above equation, we have

$$\frac{dl}{du} = g\Delta = g \frac{dz}{du}.$$

Similarly we find

$$\frac{dm}{du} = f \frac{dz}{du}, \quad \frac{dn}{du} = c \frac{dz}{du};$$

and therefore the equation determining  $w$  is

$$\delta \frac{dw}{du} + 2\delta' w = \frac{1}{2} \Delta \delta_3 w,$$

that is,

$$p \frac{d^2 w}{dx du} + q \frac{d^2 w}{dy du} + 2p' \frac{dw}{dx} + 2q' \frac{dw}{dy} = \frac{1}{2} (1 + xp' + yq') \left( \frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} + \kappa^2 w \right).$$

When a solution of this equation is obtained, a solution of the equation for  $v$  can be constructed, and conversely. In particular, the identification of any solution of the  $v$ -equation, as being included in the general solution, can be made as in former cases.

[NOTE.—Added 17th March, 1898. As indicated in the introductory remarks, the theory given in this paper is applicable when the number of independent variables is greater than two, and when the order of the differential equation is higher than the first; its application is not restricted to the case of an equation of the second order in three independent variables.

A brief sketch of the theory for an equation of order  $m$  in  $n$  independent variables is prefixed to a paper\* which contains the integration of some differential equations of types represented by

$$\left( \frac{\partial}{\partial r_1} + x_1 \frac{\partial}{\partial r_2} + x_2 \frac{\partial}{\partial r_3} + \dots + x_{n-1} \frac{\partial}{\partial r_n} \right)^m U = 0,$$

when the iteration of the differential operator is purely symbolical.]

\* "On some Differential Equations in the Theory of Symmetrical Algebra," Camb. Phil. Trans., vol. xvi. (1898), pp. 291-325.

## II. *On the Change of Absorption produced by Fluorescence.*

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### *Introduction.*

ABOUT two years ago, in the spring of 1895, in the course of a conversation with Professor J. H. POYNTING on the nature of the phenomenon of fluorescence, in the study of which I was at the time beginning to engage, the suggestion was thrown out by him that possibly fluorescent bodies absorb differently, according as they are fluorescing or not, the rays which they give out whilst fluorescing, thus that a body A would absorb differently, according as it is fluorescing or not, the rays from a similar body B in a state of fluorescence. Some fluorescent bodies undoubtedly do, others do not, absorb, except to a very small extent, the rays which they emit. A *strong* solution of fluorescein or eosin, for instance, hardly permits its fluorescent light to penetrate even a very small thickness. Glass coloured with oxide of uranium is much more transparent, but sulphate of quinine hardly absorbs these rays at all.

The question was whether during the act of fluorescing any change is produced in the nature of the absorption itself, that is, whether during fluorescence there is an increase or diminution of absorption in that part of the spectrum where the emitted rays lie. For instance, with uranium glass the radiation takes place *chiefly* between the D and E lines, so that the absorption power for rays may be different according as the body is examined in the dark or in daylight in this part of the spectrum. Of the five bright bands of which the radiation consists, three lie between the D and E lines, the other two being of less refrangibility and of less intensity in the red and orange (STOKES, 'Phil. Trans.,' 1852).\* With the spectroscope I have used I have not been able to see the band in the red, but the

\* The series of experiments have been made with uranium glass, as the compounds of uranium exhibit many peculiarities which other bodies do not appear to possess when fluorescing. For instance, the bright bands in the fluorescing spectrum of uranium glass seem to be noticeable in uranium compounds alone. It is also remarkable that crystals of nitrate of uranium have the quality as well as the quantity of their fluorescent light altered by depriving them of part of their water of crystallization.

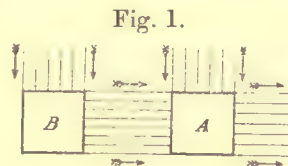
other four were quite distinct. The spectrum seemed to me to be of the nature of maxima and minima.

In the summer of 1895 I made some experiments at Trinity College, Dublin, on the electrical conductivity of fluorescent solutions, and endeavoured to determine whether the action of violet or ultra-violet light upon them gave rise to any alteration in their conductivity.

The many difficulties which the experiment entailed prevented me from arriving at any conclusion upon the matter, and I commenced to look for some such effect as Professor POYNTING had suggested.

The experiments have since been carried out in Professor SCHUSTER's laboratory at Owens College.

1. The leading conception in the following paper may briefly be stated thus: Suppose that a body A is transmitting light from a similar body B, which is fluorescing. It is quite conceivable, as Professor POYNTING remarked, that the amount of light from B transmitted by A should be different according as A is fluorescing or not.



2. There were some reasons which pointed to the conclusion that something of the sort might happen. It seemed quite possible that the action, if any, of the fluorescent light from B might be to strengthen the fluorescence of the body A when already fluorescing; not that the emitted light would be increased merely by scattering, which would be precisely the same in amount whether the body were fluorescing or not, but that the emission of fluorescent radiation would be increased if light of the same frequency were incident on the radiating body.

The experiments which are described in the sequel do not confirm this supposition, but they prove that the amount of fluorescent light apparently transmitted is very different according as the body through which it passes is fluorescing or not, the amount being very much less, only about one-half (0.57), when fluorescence is taking place than when the body is unacted upon by exciting light.

3. It seemed possible that this effect might be due, *indirectly* at least, to an increased absorption of the incident fluorescent light, if it possessed the property of destroying the fluorescence of the body through which it passed. BECQUEREL has shown that certain infra-red rays have the power of destroying phosphorescence in a substance which is exposed to the phosphorescence-exciting rays. It was therefore possible that the result obtained by me was an effect of this kind, but I think that the experiment described on p. 92 sufficiently proves that, though such an explanation would involve an increased absorption of the annulling rays in order that they might be capable of destroying the fluorescence, the effect I have observed is really not essentially of this kind, but is rather due to an increased direct absorption.

It may be useful if, before entering into an account of the experiments, I give,



as briefly as clearness will permit, a short account of the general principles on which the coefficients of absorption may be calculated.

### (a) MODE OF DETERMINING ABSORPTION.

Let us consider two cubes of the same fluorescent substance (fig. 1).

We shall, by comparing the light from B when fluorescing, which is transmitted by A according as the latter is fluorescing or not, obtain

$$\begin{aligned} E_1 &= E_0 (1 + \alpha), \\ E_2 &= E_0 \beta, \end{aligned}$$

$\mathbb{E}_0$  being the intensity of the light emitted by each cube separately and independently of any light transmitted from the other.

The quantities  $\alpha$  and  $\beta$  are clearly the fractions of the incident light transmitted by A according as it is or is not fluorescing. They may be called the coefficients of transmission of the cube under these different circumstances.

Similarly, if we denote the coefficients of absorption of the cube by  $a$  and  $b$ , and that of reflection at the two faces by  $r$ , we have

$$\alpha = 1 - (r + a); \quad \beta = 1 - (r + b).$$

Hence

$$\begin{aligned} E_1 &= E_0[2 - (r + a)]; \\ E_2 &= E_0[1 - (r + b)]. \end{aligned}$$

It has been found more convenient to have to deal with the transmission coefficients rather than with those of absorption, equality in the former of course implying equality in the latter, though, as we have seen, the converse is not necessarily true.

Figs. 2.



Let fig. 2 represent a plan of the plane horizontal surfaces of four similar and equal cubical blocks— $A_1, A_2, A'_1, A'_2$ —of some fluorescent substance such as uranium glass, which is singly refracting. We shall put aside for the present all minor details. Let  $P$  be a photometer furnished with two vertical slits, the breadths of which can be varied, and  $r$  a screen, whose plane is perpendicular to that of the paper, serving to

prevent any light from the vertical surfaces of the cubes  $A_1$  and  $A'_1$  from entering the slits  $s'$  and  $s$  respectively. We shall suppose that the cubes are illuminated by vertical rays incident perpendicularly to the surfaces  $A_1, A_2, A'_1, A'_2$ , i.e., to the plane of the paper. By means of this arrangement we can compare the intensities of the two beams of light coming respectively from the vertical surfaces of the cubes  $A_1$  and  $A'_1$ , opposite to the slits  $s, s'$ . None of the light from  $A_1$  and  $A_2$  is permitted to pass sideways into  $A'_1$  and  $A'_2$ , the two sets of blocks being separated by a screen of black paper. On the other hand, the blocks  $A_1$  and  $A'_1$  are separated from  $A_2$  and  $A'_2$  by a sheet of lead glass, which is very opaque to the fluorescence-exciting rays, so that radiation scattered by  $A_2$  and  $A'_2$  cannot be the source of any part of the fluorescence of  $A_1$  and  $A'_1$ , and *vice versa*. The fluorescence of each block is thus altogether caused by the rays incident perpendicularly to the plane of the paper. If all is right the fluorescent light emitted horizontally from the surfaces  $a_1$  and  $a'_1$  ought to appear the same, and this is tested by the photometer.

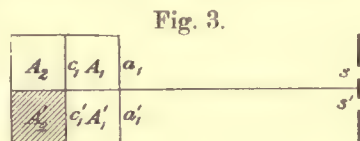
When this condition is fulfilled three experiments are performed: firstly, to determine the coefficient of transmission of  $A_1$  or  $A'_1$  when fluorescing for those rays which the cube itself gives out by fluorescence; secondly, to measure its coefficient of absorption when not fluorescing; and thirdly, to determine independently the ratio of the coefficients, and to compare the value so obtained with that given by the other two results. The last experiment thus serves to verify the first two measurements.

### *The ( $\alpha$ ) Set of Experiments.*

We may call the experiments made to determine the coefficient of transmission during fluorescence the ( $\alpha$ ) set, and similarly the other two the ( $\beta$ ) and  $\beta/(1 + \alpha)$  sets, in all of which a proper combination of the cubes  $A_1, A'_1, A_2, A'_2$ , is chosen, and the fluorescence-exciting light is allowed to fall upon them, whilst the remaining cube or cubes are screened from its action.

Thus, if we wish to determine the transmission during fluorescence of the block  $A_1$ , we screen  $A'_2$  from the incident rays as in fig. 3, these, it must be remembered, are perpendicular to  $A_1, A'_1, A_2, A'_2$ , that is to the plane of the paper; hence the light emitted by the surface  $a_1$  will be that due to two cubes, and that from  $a'_1$  to one only. We can compare relative intensities of these by adjusting the slits  $s, s'$ . The block  $A'_2$  is not removed, as some of the rays emerging from  $c_1$  towards  $A_2$  are reflected back, and these are compensated by a similar reflection from  $A'_2$ . There is no reason to suppose that the reflection would be different according as the reflecting surface is fluorescing or not, because fluorescence is not confined to the surface

A portion of the light emitted by  $A_2$  is reflected by the double air layers separating  $A_2$  from  $A_1$ , but this is included in the coefficient of reflection  $r$ . The light emitted



by the surface  $\alpha_1$  is then made up of two parts, *i.e.*, the fluorescent light of  $A_1$  and the fluorescent light of  $A_2$  transmitted by  $A_1$ .

Hence the intensity of the light given out by  $\alpha_1$  is  $E_0(1 + \alpha)$ . The light from  $A'_1$  consists of its fluorescent light only, the internal reflection being the same as in the other two blocks, and is thus  $= E_0$ .

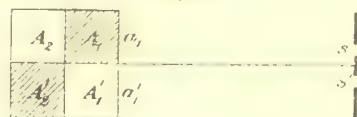
Denoting the widths of the slits by  $s$  and  $s'$ , and remembering that these quantities are inversely proportional to the intensities, we obtain the equation  $1 + \alpha = s'/s$ , from which  $\alpha$  may be determined.

In order to avoid errors arising from want of symmetry, the process is reversed by screening  $A_2$  instead of  $A'_2$ , and similarly  $\alpha$  is given by the equation  $1 + \alpha = \sigma/\sigma'$ ,  $\sigma$  and  $\sigma'$  being now the widths of the slits  $s, s'$ . If there are  $n$  observations with the screen in each of these two positions,

$$\alpha = \frac{1}{2n} \left[ \sum \left( \frac{s'}{s} + \frac{\sigma}{\sigma'} \right) \right] - 1.$$

### *The ( $\beta$ ) Set of Experiments.*

To determine  $\beta$  for  $A_1$ , we screen  $A_1$  and  $A'_2$  from the incident light, as in fig. 4. We can then compare the fraction of the light emitted by  $A_2$  which is transmitted by  $A_1$  (when the latter is not fluorescing) with that emitted by  $A'_1$ , which is the same as that from  $A_2$ ; the constant of reflection being of course involved, as in the previous case.



The light coming from  $\alpha'_1$  is then

$$= E_0,$$

whilst that from  $\alpha_1$

$$= E_0\beta.$$

Hence

$$\beta = s'/s.$$

If the process be reversed,

$$\beta = \sigma/\sigma',$$

and after  $n$  such observations

$$\beta = \frac{1}{2n} \left[ \sum \left( \frac{s'}{s} + \frac{\sigma}{\sigma'} \right) \right].$$

The removal of  $A'_2$  is found to make no appreciable difference. The cube  $A_1$  and the glass screen may be moved before the determination is made, to make sure that  $A_2$  and  $A'_1$  radiate equally.

### *The $\beta/(1 + \alpha)$ Set of Experiments.*

The ratio  $\beta/(1 + \alpha)$  can be obtained independently by the arrangement shown in fig. 5.



It is easy to see, as before, that if  $A'_1$  be screened, the light coming from  $a_1$

$$= E_0(1 + \alpha),$$

whereas that from  $a'_1$

$$= E_0\beta.$$

Hence

$$\beta/(1 + \alpha) = s/s',$$

and reversing,

$$= \sigma'/\sigma,$$

so that

$$\frac{\beta}{1 + \alpha} = \frac{1}{2n} \left[ \Sigma \left( \frac{s}{s'} + \frac{\sigma'}{\sigma} \right) \right].$$

This description of the general method will suffice for the present, and the account of the general methods of carrying out an experiment will be found on p. 96.

*To Determine the Alteration in the Fluorescence on the Supposition that the Fluorescent Light is Capable of Destroying Itself.*

Let us suppose that the intensity of the fluorescent light given out by each of two cubes of uranium glass when not exposed to the radiation from each other—that is, when there is an opaque non-reflecting screen ( $ll$ , fig. 6) separating them—is equal to  $E_0$ , and that when this screen is removed the intensity is equal to  $E'_0$ . The blocks are supposed to be illuminated by rays perpendicular to the plane of the paper, and the radiation, the intensity of which is under investigation, to be in the direction perpendicular to the surfaces,  $a_1, a'_1$ , of the cubes. The method employed for comparing  $E_0$  and  $E'_0$  depends upon obtaining two photographs on one plate, for comparison side by side.

We proceed thus:—

Let the slit  $s'$  be completely closed. Three experiments are made:—

1. The light coming from  $a_1$  is photographed when the cube  $A'_1$  is not fluorescing, or the screen  $ll'$  interposed. We thus obtain the effect due to  $E_0$ .

2. Both cubes are then made to fluoresce, and the screen is removed, in which case we get the effect due to  $E'_0$ .

It can thus be found by adjusting the slit  $s$  when the photographic effects of  $E_0$  and  $E'_0$  are the same, from which the ratio of the two is determined. As the result of these experiments we are not justified in saying that any material difference between  $E_0$  and  $E'_0$  exists other than that possibly due to scattering.

3. The amount of scattering from  $A'_1$  into  $A_1$ , or rather the amount of the  $A'_1$  light scattered by  $A_1$  in the direction outwards from  $a_1$ , may be found by making  $A'_1$  fluoresce alone, and noting the effect of the light then falling on the slit  $s$ . The

Fig. 5.

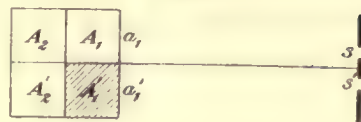
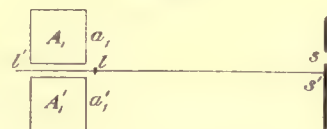


Fig. 6.





difference due to scattering did not appear to be more than 1 part in 8 at the maximum, and was for the purpose of these experiments altogether inappreciable.

#### DESCRIPTION OF APPARATUS.

1. *Fluorescent Cubes.*
2. *Photometer.*

1. The blocks employed are each 1 centim. cube, and are kept together by two indiarubber bands. The two front cubes,  $A_1, A'_1$  (fig. 2), are separated from the two back ones,  $A_2, A'_2$ , as has already been pointed out on p. 89, by a plate of glass of thickness (about 2 millims.) sufficient to stop the fluorescence-exciting rays scattered by one pair of cubes from entering the other, whilst the cubes  $A_1$  and  $A_2$  are divided by a black paper screen from  $A'_1$  and  $A'_2$ , so that no light could pass sideways from  $A_1$  or  $A_2$  to  $A'_1$  or  $A'_2$ .

The horizontal surfaces  $A_1, A_2, A'_1, A'_2$ , are those which are illuminated, whilst the light studied is that emitted by the surfaces  $a_1, a'_1$ .

The surfaces  $b_1, d_1; b'_1, d'_1; b_2, c_2, d_2; b'_2, c'_2, d'_2$ , as well as the lower horizontal surfaces parallel to  $A_1, A'_1; A_2, A'_2$ , are roughened, so that there is no regular reflection from them, and thus are more or less opaque. But the surfaces  $a_1, c_1; a'_1, c'_1; a_2, a'_2$ , are perfectly transparent, as also the upper horizontal surfaces  $A_1, A'_1; A_2, A'_2$ .

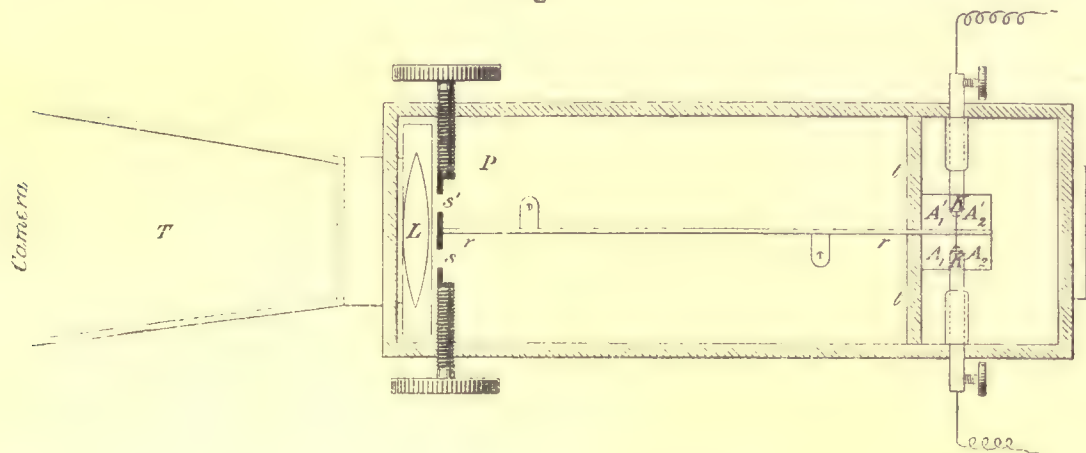
When  $A_1, A'_1, A_2, A'_2$ , are all uniformly illuminated by violet or ultra-violet rays, the fluorescence of each block ought to be the same, if we disregard the very refined correction that the blocks  $A_1, A'_1$ , have each two transparent vertical surfaces, namely  $a_1, c_1$ , and  $a'_1, c'_1$ , whereas the blocks  $A_2$  and  $A'_2$  have each only one,  $a_2$  and  $a'_2$ . At any rate, since the rays enter vertically, and therefore parallel to these surfaces, any difference that may possibly arise in this way in the intensity of the scattered rays would for all practical purposes be quite immaterial.

We have seen (p. 92) that the error, when sought for, was found to be quite inappreciable and lies within the much larger errors of observation. Apart altogether from this consideration, the fluorescent light emitted horizontally from the surfaces  $a_1, a'_1$ , ought to appear the same, if all is right, and this is ascertained by the photometer before an experiment is made.

2. Fig. 7 is a plan of the apparatus showing the position of the fluorescent cubes, &c., in the photographic experiments. The cubes are placed  $A_1$ , &c., and are illuminated by light incident vertically upon them either from the spark in air between cadmium electrodes or the electric arc. The source of illumination was almost invariably the former in the photographic determination, the poles being about 15 millims. apart, and 2 centims. above the fluorescent surface. The spark was obtained by the discharge of a Leyden jar charged by the terminals of the secondary of a 10-inch Apps coil.

The photometer, fluorescent substance, and source of illumination in the photographic experiments were enclosed in a wooden box, shown in the figure. The photometer was at one end, and the fluorescent substance and illuminating arrangement at the other.

Fig. 7.



KK represent the electrodes,  $A_1$ , &c., the fluorescent cubes, and P the photometer, consisting of two vertical slits,  $s$ ,  $s'$ , and a double convex lens, L; while  $tt$  is a cardboard vertical screen, with a horizontal slit, 3 millims. wide, placed immediately in front of the vertical surfaces of the cubes (the slit cannot be shown in the figure). This screen admits of being moved up and down, so that the light at various depths below the illuminated surface may be studied. The usual depth of the upper edge of the slit below the horizontal surface of the fluorescent cubes was about 3 millims. Another vertical screen ( $rr$ ), 6 centims. in length, extends along the whole length of the box from the photometer to the other screen, and divides the horizontal slit into two parts, the light from each part being allowed to pass only through the corresponding vertical slit of the photometer.

The vertical slits  $s$  and  $s'$  were adjustable with micrometer screws.

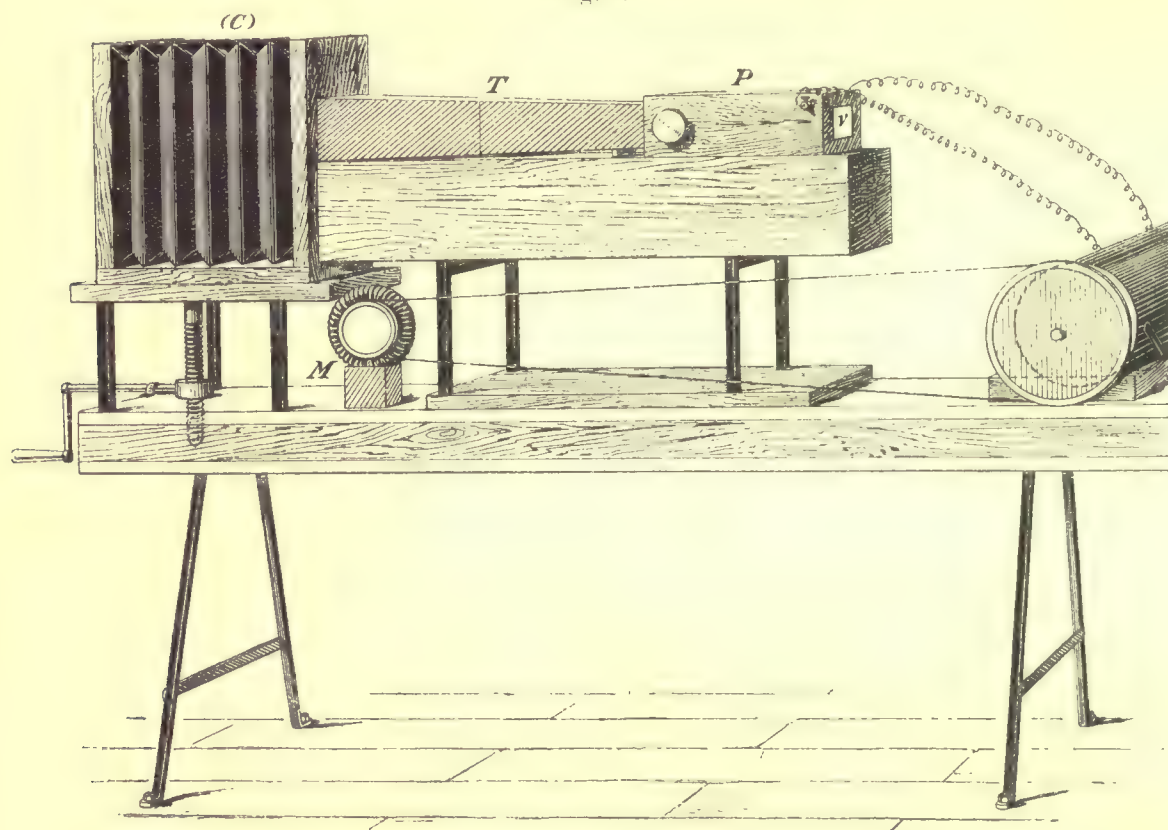
The lens L (of focal length 4.5 centims.) served to form an image of the horizontal slit in the focal plane of the eyepiece, or on the focussing glass of the camera, which admitted of being raised or lowered, so that, when necessary, a number of photographs could be obtained successively on one plate. The photometer contrivance communicated with the camera by means of the conical tube T, leading up to a front slide which remained fixed whilst the camera itself moved up and down (fig. 8). The distance between the lens L and the focal plane of the eyepiece was 18 centims. The whole of the interior of the apparatus was blackened, so that not the slightest trace of reflection could be perceived.

A screen of blue cobalt glass was interposed between the source of illumination and the fluorescent substance so as to cut off the less refrangible rays. When the apparatus is in use a horizontal green band with a dark vertical line or narrow band in

the middle is seen through the eyepiece or on the focussing plate of the camera. The two luminous portions are images of the parts of the surfaces  $a_1, a'_1$ , cut off by the horizontal slit immediately in front of them; the screen  $rr$  and the paper screen which separates  $A_1, A_2$ , from  $A'_1, A'_2$ , divide the image of this slit into two parts, and produce the dark vertical line in the middle.

By this arrangement, which was found to be on the whole the most convenient, we have clearly the opportunity, not merely of comparing the two lights side by side, which is absolutely necessary, but also of illuminating the surfaces  $A_1$ , &c., by the same source simultaneously and under similar conditions.

Fig. 8.



Any slight amount of light, chiefly blue, but partly red, which, having penetrated the cobalt glass, may have been scattered by the fluorescent substance as *false dispersion* (a term used by STOKES to designate the light scattered by irregular reflection from small particles, which is always polarized in the plane of reflection, and of the same wave-length as the incident light), can be eliminated, in consequence of its being polarized, by interposing a NICOL'S prism or a pile of plates between the horizontal and the vertical slits of the photometer. This method is open to the objection that it would weaken the fluorescent light, and it has been thought unnecessary to incur this disadvantage for two reasons. In the first place, the



whole amount of light scattered has been found to be inappreciable (see p. 93), and on this account alone the portion now under consideration may be neglected. In the second place, if the error were appreciable, it would tend to diminish rather than to increase the apparent difference between the absorptions before and during fluorescence respectively. For the scattered light would add to the total light emitted from  $a_1$ ,  $a'_1$ , and would therefore tend to reduce the effect.

The length of the horizontal slit  $tt$  being about 2 centims., and its width 2 millims., the image was 6 centims. by 6 millims., for the focal length was 4.5 centims., and the focal distances 6 centims. and 18 centims. respectively. By using a lens of shorter focal length, however, an image of  $tt$  can be formed on the horizontal slit of a spectroscope placed so that the refraction takes place in a vertical plane, and observations can be made of the absorption in different parts of the spectrum. This, however, is by no means an easy matter to accomplish, and we have confined ourselves to the light taken as a whole.

With reference to the illumination, it was thought at first that one of the electrodes might possibly always be brighter than the other. Accordingly a plan was adopted of continually reversing the current in the primary coil by means of a wheel attached to the commutator of the coil, which was made to rotate by means of a motor. Unfortunately the exposures required in the photographic work were thereby greatly increased. The experiments, however, show that the average illumination obtained from each electrode is, for all purposes with which we are concerned, practically the same.

#### EYE OBSERVATIONS.

The accuracy of these observations has been found to depend, to a very large extent, upon experience on the part of the observer in dealing with the relative intensity of the lights placed side by side. It is, of course, a familiar fact in photometry that it is very difficult to form a judgment as to the equality of two illuminated surfaces when the illumination is either very feeble or very powerful; and accordingly this is a source of considerable inaccuracy in such determinations. It is not desirable in either case to examine the relative intensity attentively for any length of time, as it is obviously possible, by viewing two feebly illuminated surfaces side by side for a sufficiently long time, to imagine that they are almost equal when really they are not so, and this remark appears to be particularly applicable to such cases in which the two colours are of slightly different tints. Instances have arisen in which it has been impossible to arrive at any result whatever as to the relative intensity. In such cases, however, the difference of tint was conspicuously marked.

It is preferable, when the observations are made by eye, to make a large number of measurements quickly, and to take the means of these first impressions, noting in turn the maximum and minimum widths of the slit, and then reversing the system of screens which serve to shelter the combination of the cubes from the exciting rays.



The screen is so arranged as to admit of being worked by the observer, by means of handles, whilst he is otherwise engaged in taking observations through the eyepiece, but the mechanism by which this is effected is not shown in the figure, which is a sketch of the arrangement adapted chiefly to photographic work. In the eye observation an optical bench was used, being a suitable stand for the apparatus.

To test the accuracy attained and also the uniformity of the illumination, or rather its symmetry, we may give the result of thirty observations. These observations were made in the earlier stages of the experiments, when but little experience had been acquired. If the illumination had been uniform, the intensity of the fluorescent light should have appeared the same when the slits were of equal width.

Fifteen readings of the left slit, when the right was fixed at 8, give 7.87 as the mean of the readings, the probable error of a single observation being 0.54, and the probable error from the mean 0.135. Similarly the other fifteen observations, when the left slit was fixed at 8 and the right slit movable, gave the mean of the readings of the right slit to be 7.73, the probable error of a single observation being 0.53, and that from the mean 0.133. It may be remarked that in this particular experiment the readings of each observation were taken with an opening slit, and the apparent equality was obtained before the real equality existed, that is that in both sets of observations the mean of the readings was somewhat less than what it should have been; but the process of reversing eliminates this. Then the readings to the two sets of observations are  $7.87 \pm 0.135$  and  $7.73 \pm 0.133$ . The ratio of the illumination of the cubes on the right to that of those on the left is  $\frac{7.87}{8} \left(1 \pm \frac{0.135}{7.87}\right)$  in the first set of observations, and  $\frac{8}{7.73} \left(1 \pm \frac{0.133}{7.73}\right)$  in the second, if we neglect the square of  $\left(\frac{0.133}{7.73}\right)$ .

The ratios of the illumination of the cubes on the right to that of those on the left, as given by the two sets of observations, are  $(0.984 \pm 0.017)$  and  $(1.035 \pm 0.017)$ . The mean of these two ratios is 1.009. The probable error in the ratio is 0.017 in either case. Hence the probable error from the mean, since the two determinations of the ratio of the two illuminations are quite independent, is  $\frac{0.017}{\sqrt{2}} = 0.012$ . And the probable error of thirty observations is about 1 per cent. from the mean.

In the detection of the change of absorption, an error of 20 per cent. will obliterate the effect.

In tabulating the experiments, the following method has been adopted. The

$A'_2$	$A_2$
$A'_1$	$A_1$

$a'_1$   $a_1$

cubes are indicated by the letters  $A_1$ ,  $A_2$ ,  $A'_1$ ,  $A'_2$ , the arrangement being as shown in the accompanying figure.

The intensities compared are those of the rays proceeding from the faces  $a_1$  and  $a'_1$ .

In each of the experiments one or more of the cubes was shaded from the fluorescence-producing light. The letter in the fifth column indicates which was that protected.

*Experiments ( $\alpha$ ).*

Experiment.	No. of observations.	$s$ .	$s'$ .	Shaded cube.	$\alpha$ .	Difference from mean.
I.	20	5.92	4	$A_2$	0.48	+ 0.03
II.	20	5.45	8	$A_2'$	0.46	+ 0.01
III.	20	8	5.29	$A_2$	0.51	+ 0.06
IV.	20	5.89	8	$A_2'$	0.36	- 0.09
Mean = 0.45						

Similarly taking in all eighty observations for  $\alpha$ ,  $\beta$ , and  $\beta/(1 + \alpha)$ , the following are obtained, the fixed slit being at 8:—

	Mean of eighty observations of movable slit.	Probable error of a single observation.	Probable error from mean.
( $\alpha$ )	5.52	0.296	0.03
( $\beta$ )	6.20	0.514	0.05
$\frac{\beta}{(1 + \alpha)}$	4.17	0.50	0.05

Hence

$$1 + \alpha = \frac{8}{5.52} \left[ 1 \pm \frac{0.03}{5.52} \right] \\ = 1.449 [1 \pm 0.005];$$

therefore

$$\alpha = 0.449 \pm 0.005.$$

Also we easily obtain

$$\beta = 0.787 \pm 0.006$$

and

$$\frac{\beta}{(1 + \alpha)} = 0.521 \pm 0.006.$$

It may be remarked that the probable error in  $\alpha$  is much less than that in  $\beta$  or  $\beta/(1 + \alpha)$ . My attention was called to this when revising the paper. A glance at figs. 3, 4, and 5 will make this clear. The light emanating from  $\alpha_1$  and  $\alpha'_1$  will

be of the same colour in the determination of  $\alpha$ ; but in the determination of  $\beta$  and  $\beta/(1 + \alpha)$  we are comparing the light passing through a non-fluorescing body with that emitted by it directly, and the result shows that there is a difference in tint between the light given out by  $\alpha_1$  and that emanating from  $\alpha'_1$ .

Substituting the values of  $\alpha$  and  $\beta$  thus obtained in  $\beta/(1 + \alpha)$ , we obtain 0.54, which is somewhat larger than that independently determined value 0.52. The difference, though only a third of the probable error of a single observation, is nevertheless three times the probable error from the mean. But this, Professor SCHUSTER suggests, was probably due to slight variations in the symmetry of the illumination. The value of  $\beta/(1 + \alpha)$  determined independently is, however, quite close enough, and may for all practical purposes be considered a very good verification of the accuracy of the values obtained for  $\alpha$  and  $\beta$ .

Taking the observed values of  $\alpha$ ,  $\beta$ , and  $\beta/(1 + \alpha)$  to be 0.450, 0.787, and 0.520 respectively, we obtain by the method of least squares\* the most probable values of  $\alpha$  and  $\beta$  to be  $\alpha = 0.455$  and  $\beta = 0.786$ , which give  $\beta/(1 + \alpha) = 0.54$  as the most probable value of this ratio.

Hence it appears that the observed values of  $\alpha$  and  $\beta$  are more reliable than those of  $\beta/(1 + \alpha)$ .

\* Let

$$\begin{aligned} \alpha_1 &= 1 + \alpha, & X &= \alpha_1 + x, \\ \beta_1 &= \beta, & Y &= \beta_1 + y, \\ \gamma_1 &= \frac{\beta}{1 + \alpha}, & \gamma &= \frac{Y}{X}, \end{aligned}$$

where  $X$  and  $Y$  are the most probable values of  $1 + \alpha$  and  $\beta$ .  $\alpha_1$ ,  $\beta_1$ , and  $\gamma$  being the observed values of  $1 + \alpha$ ,  $\beta$ , and  $\beta/(1 + \alpha)$  respectively.

Then

$$(\alpha_1 - X)^2 + (\beta_1 - Y)^2 + \left(\gamma_1 - \frac{Y}{X}\right)^2 = 0,$$

and differentiating, we get

$$\begin{cases} X^2 + Y^2 - \alpha_1 X - \beta_1 Y = 0; \\ Y - \gamma_1 X - X^2 (\beta_1 - Y) = 0. \end{cases}$$

Substituting for  $X$  and  $Y$  their values  $(\alpha_1 + x)$  and  $(\beta_1 + y)$  and neglecting the powers of  $x$  and  $y$  higher than the first, we obtain

$$\begin{aligned} (1) \quad & \alpha_1 x + \beta_1 y = 0; \\ (2) \quad & y [1 + \alpha_1^2] + \frac{\gamma_1 \beta_1}{\alpha_1} y - \gamma_1 \alpha_1 + \beta_1 = 0. \end{aligned}$$

Substituting for  $\alpha_1$ ,  $\beta_1$ ,  $\gamma_1$ , 1.45, 0.787, and 0.52 respectively, we get

$$\begin{aligned} x &= 0.005; \\ y &= -0.01. \end{aligned}$$

Hence

$$X = 1.455, \quad Y = 0.786,$$

or the most probable values of  $\alpha$  and  $\beta$  are

$$\alpha = 0.455; \quad \beta = 0.786.$$



The coefficients of absorption  $a$  and  $b$  are given by the equations (see p. 89)

$$\begin{aligned}\alpha &= 1 - (\gamma + a); \\ \beta &= 1 - (\gamma + b).\end{aligned}$$

Hence

$$\begin{aligned}r + \alpha &= 0.545, \\ r + b &= 0.214,\end{aligned}$$

and the increase in the absorption, or the difference between  $a$  and  $b$ ,  $= 0.331$ .

#### INFLUENCE OF TEMPERATURE.

It does not appear that temperature can have any effect on the phenomenon except either by weakening the intensity of the fluorescence of the transmitting cube  $A_1$  or  $A'_1$  by heating effects due to the spark, or by a change of absorption due to temperature directly. But, firstly, since all the fluorescing cubes are uniformly illuminated, and as we are merely concerned with the ratio of the intensity of the light emitted from  $a_1$  and  $a'_1$  respectively, the heating being the same for all the illuminated cubes on the one hand and for all the unilluminated ones on the other, the actual intensity  $E_0$  emitted by each cube, which may possibly be diminished by a rise of temperature, does not enter into the results. Secondly, in the eye observations, the various combinations of screening were brought about with considerable rapidity, sufficient to prevent the possibility of any such effect due to changes of temperature arising. No doubt some of the light from  $A_2$  is absorbed by  $A_1$  in the determination of  $\alpha$  and  $\beta$  for  $A_1$ ; any change of temperature, however, by this means would necessarily be slow, and consequently its effect, if any, small.

#### *Photographic Method.*

The following set of experiments was first carried out.

It was afterwards found not to be altogether satisfactory on account of the necessity in these cases of superposing two photographs, and taking the resultant effect to be proportional to the sum of the separate ones. The experiments, however, are of some interest in revealing this fact: that when considerable differences in intensities have to be dealt with the resultant effect of superposing two photographs is not proportional to their sum. (It has not been considered necessary to reproduce these particular photographs here.) Captain ABNEY has previously shown that this is so.

The first photograph taken, which we shall call Photograph A, consisted of five successive images, on the one plate, corresponding respectively to a gradual alteration



in the width of the slits. It was really a blank experiment to test whether the illumination was uniform or not.

All the four cubes were exposed to the exciting illuminations.

(A.)

	$s'$	$s$
1	7.5	8
2	8	8
3	8	7.6
4	8	7.2
5	8	6.8

The table shows the width of the slits for each of the five photographs. The nearest approach to equality was obtained in 2 and 3, and the relative intensity was reversed. The exposures were twelve minutes each. The plate was, however, considerably fogged. The next plate (B) was taken in the same way as before, but the width of the slits was diminished.

(B.)

	$s'$	$s$
1	5.5	5.0
2	5.25	5.0
3	5.0	5.0
4	4.75	5.0
5	4.5	5.0

The equality was here between 2 and 3. The photographs were still too dense, and the fogging too great, to make it possible to say with any degree of accuracy which way the inequality, if any, went; and it was really difficult to say if there was any difference between 3 and 4.

(C) was taken with an experiment of six minutes.

In this case the equality was between 3 and 4.

It was not possible to obtain a greater accuracy than 5 per cent. in any single plate, but it did not seem necessary in the determination of the absorption to attain even to this.

In order to test whether the absorption was the same whether the substance was fluorescing or not, the experiment was arranged in the following way:—

1. A photograph is taken with the screen, as shown in fig. 3. Let  $p$  be the ratio of intensity of the light emitted by  $a_1'$  to that emitted by  $a_1$ ; then

$$\frac{E_0}{E_0(1 + \alpha)} = p_1.$$

2. If  $p_2$  is the ratio of the intensity of the light emitted by  $a_1'$  to that emitted by  $a_1$  in the arrangement fig. 5, then

$$\frac{E_0 \beta}{E_0 (1 + \alpha)} = p_2.$$

Hence

$$\frac{(1 + \beta)}{(1 + \alpha)} = p_1 + p_2,$$

and if  $\alpha = \beta$ ,

$$p_1 + p_2 = 1.$$

The experiment is then performed by first determining  $p_1$ , and putting  $p_2 = 1 - p_1$ , and observing whether the two lights so obtained are of equal intensity.

Again  $p_2$  is first obtained, and  $p_1$  put  $= 1 - p_2$ , and equality again looked for.

In the first case it is found necessary to diminish  $s'$ , or widen  $s$ , to obtain equality, and since the radiation from  $a_1$  and  $a_1'$  are inversely as the width of the slits, it follows that the real value of  $p_2$  is greater than the calculated one, and in the second it is also found necessary to diminish  $s'$  or widen  $s$ , which shows that  $p_1 + p_2 > 1$ , or that  $\frac{1 + \beta}{1 + \alpha} > 1$ , and therefore  $\beta > \alpha$ . Hence the absorption is greater when the substance is fluorescing than when it is not,  $\alpha$  and  $\beta$  being the coefficients of transmission.

To test this photographically, first of all an exposure is given when the screening of the cubes is as in (1), and then, without altering the position of the camera, the screening is arranged as in (2). The slits are such that  $s' = 2s$ , since we want the ratio  $\frac{1 + \beta}{1 + \alpha}$ , not  $\frac{1 + \beta}{2(1 + \alpha)}$ . Thus, if the photographic effect be proportional to the time and the intensity, the effect in the first instance on the photographic plates may be represented thus :

$$\boxed{E_0} \quad \boxed{E_0 \frac{(1 + \alpha)}{2}}$$

and in the second

$$\boxed{E_0 \beta} \quad \boxed{E_0 \frac{(1 + \alpha)}{2}}.$$

Hence the resultant effects, when superposed, are




$$E_0 (1 + \beta); \quad E_0 (1 + \alpha).$$

The camera is then slightly raised,  $s'$  is made equal to  $s$ , all the screens are removed, and a photograph is taken on the same plate with an exposure merely equal to that of each of the other two, the object being to obtain on the one plate a comparison, not merely with reference to the absorption, but also with that of the illumination. The effects on the photographic plate ought to be as follows :—

$$\begin{array}{cc} a \quad \boxed{E_0 (1 + \beta)} & \boxed{E_0 (1 + \alpha)} \quad b \\ c \quad \boxed{E_0 (1 + \alpha)} & \boxed{E_0 (1 + \alpha)} \quad d \end{array}.$$

If there is no change of absorption ( $\alpha$ ) ought to be of the same intensity as ( $b$ ). As a matter of fact, it was always found to be much denser in the negative. The intensities of ( $b$ ) and ( $d$ ) were never exactly equal, but, on the contrary, ( $d$ ) was always darker than ( $b$ ). Now, the width of the slit for ( $d$ ) was double that for ( $b$ ), but the exposure was half, which shows that if we double the intensity of the light to which a photographic plate is exposed, the exposure must be less than half in order that the same effect should be produced. In other words, the photographic effect is not proportional to the intensity and to the time conjointly.

The actual width of the slits and the exposures in one particular instance were as follows :—

		$s'$	$s$	Exposure
Superposed	(1) 	4	2	8 in
	(2) 	4	2	8 "
	(3) 	4	4	8 "

The relative intensity of ( $a$ ) and ( $c$ ) was not constant, which shows that the average illumination may be different at different times, and it was consequently necessary to make comparisons of photographs taken simultaneously, the symmetry of the illumination having first been secured.

Fifteen plates were obtained, which are in the possession of the Royal Society and can be inspected.

Three photographs show the degree of approximation in the uniformity of the illuminations.

Some others exhibit the remarkable difference in the relative intensities under conditions such that if there were no change in the absorptive power the two intensities should be equal.

Three photographs show the superior and inferior limits of the value of  $\alpha$ , the  $A_2$  cube being screened, and the slits are  $s = 4$ ,  $s' = 2.5$ .

The relative intensities as given appear to be the nearest approach to the limits in the width of  $s'$  we should wish to arrive at. Thus we obtain for  $\alpha$  the limits as follows :—

$$1 + \alpha < \frac{4}{2.7}; \text{ therefore } \alpha < 0.48,$$

$$> \frac{4}{.8} \qquad \qquad \qquad > 0.43.$$

The most probable value for  $\alpha$  obtained by the eye observations  $p$ — was 0.455.

Three other photographs give the limits for  $\beta$ ,  $A'_1$  and  $A_2$  being in this case screened.

$s$  lies between 3 and 3.5, so that



$$\begin{aligned} \beta &> \frac{3}{4}, & i.e. & \beta > 0.75, \\ &< \frac{3.5}{4} & & < 0.89, \end{aligned}$$

the value with the eye observations  $p$ — being 0.786.

Assuming the value of  $\alpha$  to be 0.48, and that there is no difference between it and  $\beta$ , we obtain

$$\beta/(1 + \alpha) = 0.32.$$

Consequently if we screen  $A'_1$  and adjust

$$\begin{aligned} s &= 1.3, \\ s' &= 4.0, \end{aligned}$$

we should obtain equality of intensity on a photographic plate. This, however, does not appear to be the case, as shown in a photograph in which two successive images of the horizontal slit are given, one being the result of reversing the screws and screen, so that  $A_1$  was for the second image sheltered from the illuminations, and  $A'_1$  exposed, and

$$\begin{aligned} s &= 4.0, \\ s' &= 1.3. \end{aligned}$$

It shows, moreover, that a reversal in the arrangement also reverses the effect.

Another plate exhibits the effect of superposing two such photographs, showing the absence of want of symmetry in the illumination.

Some other photographs show the equality of illumination and change of absorption from the calculated values, and two others the result of superposing two photographs in the following way.

Firstly,  $A'_1$  is screened, and  $s = s' = 4$ ; and an exposure of 30 minutes gives an effect nearly proportional to  $E_0\beta$  on the left and  $E_0(1 + \alpha)$  on the right. Then  $s$  is closed completely, and the screen altered from  $A'_1$  to  $A'_2$ , and an additional exposure is given without changing the position of the photographic plate, so that an increase proportional to  $E_0$  is produced on the left. Now, since  $\beta$  does not differ so very much from unity, we may take the resultant effect as equal to the sum, though in reality, as shown on p. 103, it is less. The effect on the left side of the plate is then proportional to  $E_0(1 + \beta)$ , and that on the right due to a single exposure to  $E_0(1 + \alpha)$ , and if  $\alpha = \beta$ , these two should have been equal. The plates show that they are not, but give  $\beta > \alpha$ . This is clearly the simplest method of exhibiting the phenomenon. If the photographic effect were proportional to the intensity the difference would clearly be more marked.

I desire to thank Professors ARTHUR SCHUSTER and G. F. FITZGERALD for many useful suggestions and much valuable advice throughout this research, and also Professor J. H. POYNTING, to whom the investigation owes its origin.



### III. *On the Occlusion of Hydrogen and Oxygen by Palladium.*

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#### I. *Introduction.*

SINCE the discovery by GRAHAM ('Phil. Trans.,' 1866, p. 399, also 'GRAHAM'S Researches,' p. 235), that palladium possesses the property of occluding or absorbing large quantities of hydrogen, chemical literature has been enriched by various contributions to our knowledge of this interesting phenomenon. To mention only a few of these, some of which will later on be considered in detail, we have the results of investigations on the change of volume undergone by palladium on the occlusion of hydrogen, the density of the occluded hydrogen, the heat changes which take place, and the pressure of "palladium hydrogen" at different temperatures, &c.

The solution of hydrogen in, or the combination of hydrogen with, palladium, which we may be permitted provisionally to term palladium hydrogen, is now often used in laboratories as a convenient source of pure hydrogen; and finely divided palladium, on account of the facility with which it absorbs hydrogen, is also extensively employed in gas analysis. Hydrogen, occluded by palladium, may conveniently be weighed as palladium hydrogen, and recourse was had by KEISER

(‘Am. Chem. Journ.,’ vol. 10, p. 249) to this method of weighing hydrogen in one of the most delicate of chemical operations, viz., that of atomic weight determination.

Any addition to our knowledge of this substance may therefore be expected to be of interest. It will readily be admitted that the phenomenon of occlusion is one which requires further elucidation, and in submitting the present communication to the Society we hope that the additional work we now bring forward will not only be of interest for its own sake, but that it will finally assist us in arriving at a definite decision with regard to the nature of occlusion.

In previous communications (‘Phil. Trans.,’ A, vol. 186, p. 657, and A, vol. 190, p. 129) we discussed the relations of platinum black to oxygen, hydrogen, and other gases, and on p. 152 of the latter paper we pointed out that until the relation of the palladium to the hydrogen in the comparatively well defined substance palladium hydrogen was ascertained, there was little hope of solving the corresponding problem for the less well defined substance platinum hydrogen.

We therefore turned our attention to the general behaviour of palladium to oxygen and hydrogen, and now give an account of our work on the subject, which includes measurements of the heat evolved on the absorption of these gases, and a comparative study of the occlusion of hydrogen by palladium in different states of aggregation.

## II. Preparation of the Palladium.

The palladium employed in these experiments was in the form of (a) black, (b) sponge, and (c) foil.

*Palladium black.*—Some strips of metallic palladium, which were said to have been prepared by WOLLASTON, were purified in the following manner:—After dissolving in *aqua regia* and evaporating several times successively with water and hydrochloric acid, the solution was treated with excess of ammonia, and filtered. From the filtrate pure palladosamine chloride was precipitated by leading a stream of hydrochloric acid gas through it. This was filtered off, washed with strong hydrochloric acid, dried, and ignited. The residue, consisting of palladium sponge, was then again brought into solution by digesting with aqueous hydrochloric acid through which a current of chlorine gas was led. After evaporating, to get rid of the excess of hydrochloric acid and chlorine, the solution of palladium was neutralized with sodium carbonate, and slowly poured into a boiling, very dilute solution containing excess of sodium formate. The palladium black, precipitated in this way, was then repeatedly boiled out with distilled water until free from alkalis and alkaline chlorides, and dried at 100° C.

*Palladium sponge.*—The palladium sponge employed in the course of these experiments was prepared by fully charging the necessary quantity of palladium black with hydrogen, and then igniting at a red heat *in vacuo*, since simple ignition of

palladium black in the air or *in vacuo*, unless conducted at an extremely high temperature, is insufficient to remove the oxygen initially contained in it.

*Palladium foil.*—A portion of thin foil was procured from Messrs. JOHNSON and MATTHEY, and was guaranteed to contain over 99 per cent. of palladium. The special treatment to which this was subjected will be described in Section VI.

### III. *Density of Palladium.*

It has always been customary, in stating the quantity of hydrogen occluded by palladium, to express the result as so many volumes of hydrogen occluded by unit volume of palladium.

This makes a knowledge of the density of palladium necessary.

The density of the foil employed was found to be 12.1, whilst a direct determination of the density of palladium black, dried at 100° C. in a specially constructed pyknometer, gave the value 10.6. This refers to palladium black prepared as described. An analysis of the black soon after it was dried showed that it contained 0.72 per cent. of water and 1.65 per cent. of oxygen, existing probably as oxide. If we make an allowance for this amount of water contained in the black the density rises to 11.4.

The palladium black used for the density determination was the remainder (over 5 grams) contained in a bottle which had been frequently opened during about four months, and which in all probability had absorbed more water. If we assume that the percentage of water had gone up from 0.72 per cent. to 1 per cent. during the four months, then the density of dry palladium black would become 11.8. No allowance, however, has been made for the fact that the palladium black contained oxygen, which would also tend to lower the density.

Since the knowledge of the density was only essential for the purpose of translating the actual quantity of hydrogen occluded into the number of volumes of hydrogen occluded by unit volume of palladium, we have thought it better, in view of the above circumstances and of the fact that the determination of the density of a fine powder like palladium black is not at all easy, to adopt, for comparative purposes, the value 12.0 as the density of palladium in its three states of aggregation, viz., black, sponge, and foil. As the actual measurements are always given in addition, this assumption ought not to give rise to any error, and is to a certain extent justified by the fact that, under proper conditions, the number of volumes of hydrogen occluded by unit volume of the different varieties of palladium remains practically the same.

### IV. *Analysis of Palladium Black.*

Before beginning any extended experiments with palladium black a preliminary examination was made. From the fact that platinum black prepared in the same



way invariably contains oxygen, it was thought that palladium black would also contain oxygen. On igniting 1.5650 gram palladium black in a hard glass tube *in vacuo*, however, only 0.33 cub. centim. oxygen and 0.56 cub. centim. carbon dioxide were extracted. This proves that the oxygen, if any be present, is more firmly retained by palladium than by platinum.

Another portion of palladium black, weighing 1.9795 gram, was placed in a hard glass bulb tube, one end of which was attached to a weighed U-tube containing  $P_2O_5$ , and thence to the pump, whilst the other end was connected with an apparatus supplying pure dry hydrogen. On ignition *in vacuo* 0.0142 gram  $H_2O$  was collected in the U-tube. This corresponds to the presence of 0.72 per cent. of water. No oxygen was given off, but on passing hydrogen over the sponge remaining in the hard glass tube 0.0326 gram  $H_2O$  was obtained. The palladium black therefore contained 1.65 per cent. of oxygen, corresponding to 23.69 per cent.  $Pd_2O$ , or 12.69 per cent.  $PdO$ . If we assume that the oxygen exists as  $PdO$ , and if we estimate the pure palladium by difference, we get for the analysis of palladium black:—

Palladium . . . . .	86.59	per cent. ;	
Palladium oxide ( $PdO$ ) . .	12.69	„	= 1.65 per cent. $O_2$ ;
Water . . . . .	0.72	„	

From the weight of water formed, it follows that the palladium black contained 22.82 cub. centims., or 138 volumes of oxygen. The oxygen initially present in palladium black was also determined in another way.

Palladium black was fully charged with hydrogen, a portion of which was really occluded, whilst the remainder formed water with the oxygen pre-existing in the black. The occluded hydrogen is easily determined by extracting at a red heat *in vacuo*, and half the difference between the volume of hydrogen used and that pumped off represents the volume of oxygen initially present. In this way it was found (see table, p. 125) that the palladium black contained 140, 141, and 143 volumes of oxygen, whilst the gravimetric determination indicated the presence of 138 volumes.

Pure palladium sponge is almost white, whilst that obtained by the direct ignition of palladium black *in vacuo* is much darker in colour owing to the presence of oxygen or oxide in it.

#### V. Absorption of Oxygen by Palladium.

We have already shown ('Phil. Trans.' A, 1895, vol. 186, p. 682) that when platinum black is heated in an atmosphere of oxygen, kept approximately at ordinary pressure, absorption of oxygen takes place until the temperature rises to about  $360^\circ C.$ , when the gas is again expelled. If this is really a process of oxidation, then, since palladium is usually regarded as a more easily oxidisable



metal, it is to be inferred that palladium black will behave in the same way, although perhaps to a greater extent.

In the first experiment 3.519 grams of palladium black were heated in an atmosphere of oxygen in an apparatus similar to that employed for the corresponding experiments with platinum. The following results show that oxygen was steadily absorbed :—

Heated for	Oxygen absorbed.	
	cub. centims.	volumes.
2 hours at 132° . . . . .	3.29	11.2
6 „ 184° . . . . .	18.18	62.0
6 „ 237° . . . . .	33.73	115.1
6 „ 280° . . . . .	46.18	157.6
Total . . . . .	101.38	345.9

Up to 280° C. the total quantity of oxygen absorbed, viz., 101.38 cub. centims. = 0.1450 gram, is more than half the quantity of oxygen (0.2656 gram) necessary to form the compound  $\text{Pd}_2\text{O}$ . Owing to an accident, which prevented the experiment being carried further, a new portion of palladium black was taken, and the heating in oxygen started at 280° C. The following results were obtained :—

#### OXYGEN Absorbed by Palladium at High Temperatures.

Palladium black used, 2.8100 grams = 0.234 cub. centim.			
Heated for	Oxygen absorbed.		
	cub. centims.	volumes.	gram.
12 hours at 280° C. . . . .	87.59	374	0.1253
5 „ 280° C. . . . .	6.42	27	0.0092
1½ „ 360° C. . . . .	16.28	70	0.0233
3 „ 444° C. . . . .	18.21	78	0.0260
3 „ 444° C. . . . .	17.79	76	0.0254
2 „ 444° C. . . . .	10.68	47	0.0153
1½ „ 600° C. . . . .	9.17	39	0.0131
3 „ a red heat (naked flame)	60.34	258	0.0863
Total . . . . .	226.48	969	0.3239

The quantity of oxygen theoretically necessary for the formation of the oxide  $\text{Pd}_2\text{O}$  is 0.2121 gram, and it will be observed that the oxygen actually absorbed exceeds this quantity. There is no reason to suppose that the absorption of oxygen had ceased after the experimental tube had been heated for three hours in the naked flame. The experiment was stopped at this stage, however, in order to see whether the

excess of oxygen over and above that required for the formation of  $\text{Pd}_2\text{O}$  could be removed *in vacuo* at any intermediate temperature.

After being placed in communication with the pump the experimental tube was then exposed to gradually increasing temperatures, and finally to as high a temperature as the hard glass tube would stand without collapsing.

Only a few bubbles of gas (about 1 cub. centim.) could be extracted, and hence it appears that if either or both of the oxides  $\text{Pd}_2\text{O}$  and  $\text{PdO}$  are formed, they must be stable *in vacuo* at a dull red heat. The substance formed during the absorption of oxygen has a dark brown colour, has lost all the appearance of palladium sponge, and is, without doubt, an oxide or mixture of oxides.

According to WILM ('Berichte,' 1882, p. 2225), the sub-oxide of palladium,  $\text{Pd}_2\text{O}$ , may be prepared by heating the sponge in a current of air until it attains a constant weight. If this is so, it is curious that we have not been able to observe any discontinuity in the absorption of oxygen, and we are inclined to think that palladium sponge, if heated for a sufficiently long time in a current of air, should yield not the sub-oxide, but palladium oxide,  $\text{PdO}$ .

We have already seen that palladium black, prepared in the way described, contains about 140 volumes, or 1.65 per cent. of oxygen, and the fact that the substance was dried at  $100^\circ\text{C}$ . probably accounts for the presence of a certain quantity of this oxygen. In connection with the calorimetric experiments to be described later on, it was of interest to us to know how much oxygen palladium black, rendered free from oxygen, would absorb directly at  $0^\circ$  or at the ordinary temperature. We found it impossible to remove the oxygen without converting the black into sponge, but the required information was obtained in the following way:—

The sample of palladium black was fully charged up with hydrogen, whereby all the oxygen was removed as water. As will be seen presently, the bulk (over 90 per cent.) of the occluded hydrogen was next extracted by exhausting at  $100^\circ\text{C}$ . To the palladium black containing only a comparatively small quantity of hydrogen, oxygen was then admitted, in the first instance very slowly, whilst the experimental tube was kept cold by immersion in melting ice, and in the second rapidly, whilst the tube was simply exposed to the atmosphere. Of course a portion of the oxygen formed water with the residual hydrogen, the rest being absorbed, in the first case without appreciable rise of temperature, and in the second with considerable rise of temperature. The absorbed oxygen was then determined in both specimens by charging up fully with hydrogen and exhausting at a red heat. It was found that about 40 volumes and 120 volumes respectively of oxygen were absorbed.

At  $0^\circ\text{C}$ ., therefore, the absorptive power of palladium black for oxygen is considerably lessened.

It will readily be admitted that the absorption of oxygen by palladium black is really a process of oxidation, and although palladium and platinum belong to different sub-divisions of the platinum group of metals, we think the absorption of oxygen by

platinum black may also best be explained as a superficial oxidation, the chief difference being that the oxide of platinum decomposes or dissociates at a lower temperature than the corresponding oxide of palladium.

#### VI. *Observations on the Occlusion of Hydrogen by Palladium Black, Sponge, and Foil.*

A series of experiments was next undertaken with the object of comparing the relative occlusive power of palladium in the form of (*a*) black, (*b*) sponge, and (*c*) foil for hydrogen, and investigating the behaviour of the substances produced. This was especially necessary in the case of palladium foil or wire, since the statements of different observers are at considerable variance with each other. Whilst most are agreed that palladium in the compact state readily occludes the maximum quantity of hydrogen when charged electrolytically, there are many cases on record in which the compact metal only occludes a relatively small quantity of hydrogen when it is simply exposed or ignited in the gas.

*a. Palladium Black.*—1.619 gram = 0.135 cub. centim. palladium black was charged with pure dry hydrogen in an apparatus similar to that employed for the corresponding experiments with platinum black ('Phil. Trans.,' A, 1895, vol. 186, pp. 668 and 687). Altogether 152.76 cub. centims. (0° and 760 millims.) = 1131 volumes of hydrogen were absorbed, the greater portion of which was taken up almost instantaneously.

From the experiments already described, it is to be expected that a certain fraction of the total hydrogen absorbed formed water with the oxygen pre-existing in the palladium black. In order to determine the quantity of hydrogen which was really occluded and how much was given off *in vacuo* at different temperatures, the experimental tube was exhausted first at the ordinary temperature, then at higher temperatures, and finally at a red heat. The results are given in the following table:—

Palladium black used, 1.619 gram = 0.135 cub. centim. Total H <sub>2</sub> absorbed, 152.76 cub. centims. = 1131 volumes.		
Temperature.	Occluded hydrogen pumped off.	
° C.	cub. centims.	volumes.
20	105.72	783.2
100	2.41	17.8
184	2.27	16.8
280	3.29	24.4
444	1.32	9.8
700 (?)	0.21	1.6
Total . . . .	115.22	853.6

The difference between the total hydrogen absorbed and that extracted at a dull



red heat *in vacuo* corresponds to the presence of 18.77 cub. centims., or 139.1 volumes, of oxygen in the palladium black. Since this volumetric estimation of the amount of absorbed oxygen is in good agreement with the direct gravimetric determination, viz., 138 volumes, we are justified in concluding that 115.22 cub. centims., or 853.6 volumes, is the quantity of hydrogen really occluded.

It will be observed that practically the whole of the occluded hydrogen was extracted at 444°, and that about 92 per cent. of it can be pumped off at the ordinary temperature. The last traces of the hydrogen given off at the ordinary temperature come off very slowly, so that the pumping requires to be continued for a day, or longer.

It was noticed that the palladium black contracts and passes into sponge above 184° C.

This result, viz., that palladium black occludes about 854\* volumes of hydrogen, was confirmed by additional experiments (*v.* Table, p. 125), in which 860\*, 868\*, and 868\* volumes were occluded.

b. *Palladium Sponge*.—The pure palladium sponge remaining behind in the experimental tube after the preceding experiment was completed was charged with hydrogen. 112.4 cub. centims. = 833\* volumes were occluded. Of this 111.5 cub. centims., or 826.1 volumes, representing about 99 per cent. of the whole, were pumped off at the ordinary temperature, whilst on heating to *redness* 1.4 cub. centim., or 10.4 volumes, were extracted. It would appear, therefore, from this experiment that more hydrogen can be removed from palladium sponge than from palladium black at the ordinary temperature.

This does not agree with GRAHAM'S observation ('Researches,' p. 287) that "in the pulverulent spongy state palladium took up 655 volumes of hydrogen, and so charged, it gave off no gas *in vacuo* at the ordinary temperature, nor till its temperature was raised to nearly 100°."

c. *Palladium Foil*.—The palladium foil employed weighed 1.333 gram, and was about 0.025 millim. thick. It had been prepared by rolling out sponge, and was said to contain over 99 per cent. of palladium.

Before being introduced into the experimental tube it was boiled in caustic potash solution, washed, and dried between filter paper.

On ignition *in vacuo* a small quantity of gas, which was not examined, was given off. After cooling down to the ordinary temperature again, hydrogen was admitted, but apparently none was occluded. The palladium was therefore ignited in the hydrogen and allowed to cool down slowly. Next day it was found that only 29 volumes had been occluded. Practically none of this could be extracted *in vacuo* at the ordinary temperature, but on ignition 28 volumes were pumped out.

The result of this experiment, therefore, goes to show that the occlusion of hydrogen by new compact palladium foil is thus either (1) very small, or (2) very slow.

\* A slight correction must be applied to these numbers, the nature of which is explained on p. 120, and the corrected values will be found in the Table on p. 121.



When palladium foil is charged electrolytically with hydrogen it undergoes a considerable increase in volume, as has been pointed out by GRAHAM, DEWAR, THOMA, and others, and when discharged it again contracts, and it is said that its volume becomes less than the original volume. In order to see whether the occlusive power of the foil was altered by successive expansion and contraction, it was charged successively and alternately with hydrogen and oxygen in a voltaic cell. After exhaustion at a red heat *in vacuo*, hydrogen was admitted at the ordinary temperature. The following results were obtained :—

On first admission of hydrogen	. 0.3 cub. centim.	= 2.7 volumes were occluded;
After gently heating for two hours	3.6 „	= 32.2 „ „ „
After eighteen hours at ordinary	} 0.3 „	= 2.7 „ „ „
temperature . . . . .		

Altogether, therefore, 4.2 cub. centims., or 37.6 volumes, of hydrogen were occluded. The occlusive power is thus not much greater than before. If the compactness of the metal has anything to do with the rate of occlusion, this result is what might have been anticipated, since on charging electrolytically with hydrogen and then discharging the volume of the metal becomes less, and presumably the degree of compactness greater.

There can be little doubt that the phenomenon of occlusion consists first of the *solution* or *combination* of the gas in the outer skin of the metal, and then of *diffusion* inwards towards the centre.

In the case of compact foil, produced by the welding together of particles of sponge, the hydrogen is possibly condensed, dissolved, or combined in the two chief outer surfaces of the foil, and then diffuses inwards in both directions through half its thickness. In the case of palladium sponge or black, however, the surface itself is very much greater, and the diameter or radius of the individual particles probably very much less than the thickness of the foil; and consequently it would seem that with palladium sponge we have to deal chiefly with the rate of solution, and that we are less concerned with the factor of diffusion or rate of diffusion.

Hence, if we could by any possible treatment so change new compact palladium foil that it becomes spongy in texture, it is to be expected that it would behave more nearly like palladium sponge as regards the amount of hydrogen occluded and the rate at which occlusion takes place.

It is well known that when copper is oxidised and then reduced it becomes spongy. Now, a favourite device for cleaning platinum and palladium goods is by ignition in the blowpipe flame. Palladium, however, differs from platinum inasmuch as it is more easily oxidised. At a still higher temperature (higher than can conveniently be maintained in an evacuated hard glass tube), it is again reduced to metal. By repeatedly igniting palladium foil in this way it probably acquires a

more or less spongy nature, due to the transition through the state of oxide, and consequently we should expect it to behave more like the sponge.

Most of the palladium employed by other observers has probably been subjected to this treatment, which possibly explains the readiness with which in most instances it takes up hydrogen. There is also evidence, from other sources, that new palladium foil possesses a lower absorption power for hydrogen. For example, a portion of new palladium foil, examined by GRAHAM ('Researches,' p. 268), and believed to be from *fused* metal, occluded only 68 volumes of hydrogen. This is ascribed by GRAHAM to the fact that the metal had been fused. "An inferior absorbing power for hydrogen appears to be connected in both platinum and palladium with the fusion of the metal." GRAHAM's explanation is quite consistent with the view here put forward, for the fusion of the metal would simply produce (only, perhaps, in a more perfect way) the same effect as a thorough welding or rolling.

The foil examined by us had not been fused, but it persistently refused to occlude hydrogen in any quantity.

After it had been submitted to the preceding operations it was ignited several times in the blowpipe flame, rolled up, reignited, and introduced hot into the experimental tube.

On now admitting hydrogen at the ordinary temperature about 33 volumes were immediately absorbed. When the experimental tube was warmed by a naked BUNSEN flame an additional quantity of hydrogen, amounting to over 100 volumes, was suddenly occluded. On heating more strongly this gas was expelled and again reabsorbed on cooling. The experimental tube was now placed in a water bath. When the temperature reached  $88^{\circ}$ – $90^{\circ}$  C. absorption again began and continued slowly at  $100^{\circ}$  C., beginning at the rate of about 1 cub. centim., or 9 volumes, per minute and gradually diminishing. In an hour and a half at  $100^{\circ}$  C. about 500 additional volumes had been occluded, and on standing overnight at the ordinary temperature only a very little, if any, hydrogen was absorbed.

Next day an oil bath was substituted for the water bath, and the temperature was gradually raised above  $100^{\circ}$  C. Absorption again went on until  $130^{\circ}$  was reached, when the occluded gas began to be expelled. The temperature was therefore lowered to  $120^{\circ}$ . Absorption continued for two hours, and then appeared to diminish. By allowing the temperature of the bath to fall to  $100^{\circ}$  absorption again took place, and continued for another hour. The apparatus was now allowed to remain for 44 hours at the ordinary temperature, when we were surprised to find that a further absorption of about 100 volumes had taken place.

It would thus appear that after the occlusion of hydrogen has made a fair start, or when the palladium has been largely converted into palladium hydrogen, a further absorption is able to go on at the ordinary temperature. This may be due to the change in texture of the palladium caused by ruptures between the particles of the



metal produced by the expansion during the first part of the charging. GRAHAM (*loc. cit.*, p. 267) remarks, "The foil was much crumpled and rather friable after repeated use."

The fact that increase of temperature is necessary during the first part of the absorption is to be ascribed not to an increase in the solubility of the hydrogen, but to a more rapid diffusion, throughout the mass of the metal, of such hydrogen as can be retained at that particular temperature. The meaning of this will be rendered clearer by reference to DEVILLE and TROOST's well-known experiment on the passage of hydrogen through a red-hot platinum tube, or to the similar passage of hydrogen through a palladium tube above  $237^{\circ}$  C. (*cf.* RAMSAY, 'Phil. Mag.', August, 1894, p. 206). Now platinum and palladium at this high temperature can only retain under atmospheric pressure\* a very minute quantity of hydrogen, but, minute as it is, it can *diffuse* with great readiness through the hot metal, its place being immediately taken by fresh hydrogen. At moderately high temperatures therefore, when palladium can still absorb a considerable quantity of hydrogen, we should expect a more rapid diffusion through, or permeation of, the foil, and consequently the production of a spongy or fissured mass, caused by expansion on the occlusion of just as much hydrogen as can be retained at this temperature, so that on cooling down to the ordinary temperature the foil ought to behave more nearly like palladium sponge.

At this stage an exact measurement of the total quantity of hydrogen absorbed was made, and it was found that 94.50 cub. centims., or 851 volumes, had been taken up by the palladium foil.

That the absorption at  $100^{\circ}$  was complete was shown by the fact that on raising the temperature to  $100^{\circ}$  C. about 100 volumes of gas were expelled, and of this about three-quarters was reabsorbed on cooling down to the ordinary temperature, whilst the remainder, along with seven additional volumes, was occluded on standing for eighteen hours. The final measurement showed that 95.18 cub. centims., or 858 volumes, of hydrogen had been absorbed.

Just as palladium foil which has been rendered more or less spongy in texture requires a considerable time for the absorption of its quantum of hydrogen, so also it gives off its occluded gas extremely slowly *in vacuo* at the ordinary temperature. On applying the pump a practically complete vacuum was produced in a few minutes, when it was found that only 0.57 cub. centim., or five volumes, had been removed. Hydrogen was, however, given off very slowly, for in another hour 2.14 cub. centims., or nineteen volumes, were extracted. The results are contained in the following table, which shows that nearly the whole of the hydrogen is given off easily and rapidly at  $100^{\circ}$  C. *in vacuo*. There is a difference of twelve volumes between the total hydrogen absorbed and the total hydrogen extracted or occluded. This may be due to (1) experimental error, (2) all the hydrogen may not be extracted at a red

\* The most recent work on the subject by Professor DEWAR ('Proc. Chem. Soc.', 1897, No. 183, 192) shows that a rod of palladium is still capable of occluding about 300 volumes of hydrogen, under a pressure of 100–120 atmospheres, at a temperature of  $500^{\circ}$  C.

heat, or (3) the presence of a film of oxide on the foil. Of these (2) is unlikely, so that the deficit may be regarded as due to (1) and (3), and probably chiefly to (3).

Palladium foil used, 1.333 gram = 0.111 cub. centim. Total H <sub>2</sub> absorbed, 95.18 cub. centims. = 858 volumes.		
Temperature.	Occluded hydrogen pumped off.	
	cub. centims.	volumes.
20° (rapidly) . . . . .	0.57	5
20° (in one hour) . . . . .	2.14	19
100° . . . . .	90.77	818
700° (red heat) . . . . .	0.39	4
Total . . . . .	93.87	846

After all the hydrogen had been extracted from the foil, it was again charged with hydrogen. A slow absorption went on, and this was promoted at the beginning by warming. On opening the tube and bringing the foil, partially charged with hydrogen, into contact with the air, it became distinctly warm to the touch, and after standing for a short time drops of water were deposited on the walls of the tube.

Under proper conditions, therefore, as above set forth, a sample of new palladium foil which initially would only occlude a few volumes of hydrogen may be made to occlude 846, or roughly 850, volumes, that is approximately the same quantity of hydrogen as palladium black or sponge.

New palladium foil, or even fused palladium, if left for a sufficiently long time (possibly months or years) in an atmosphere of hydrogen at the ordinary temperature, would probably absorb the full quantity of hydrogen, the effect of gentle warming being to accelerate (as above) the diffusion of the hydrogen from the fully charged outer skin of the metal towards the centre.

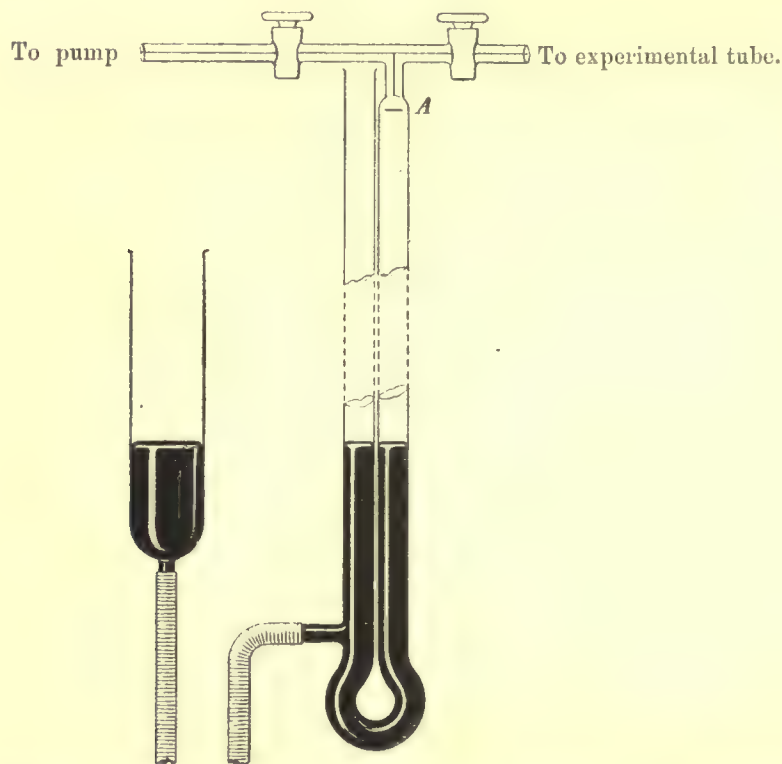
A few additional experiments on the behaviour of palladium to other gases may conveniently be recorded here. Some palladium black was fully charged with hydrogen, and from this as much hydrogen as possible was extracted at 100°. On now admitting sulphur dioxide only about 90 volumes were taken up, in place of about 800 volumes of hydrogen removed. Palladium sponge when treated with sulphur dioxide absorbed only two volumes. About seventeen volumes of carbon monoxide were occluded by palladium sponge.

Palladium black and sponge, like platinum sponge, cause a jet of hydrogen to ignite. When hydrogen was led over palladium black which had absorbed more oxygen at a high temperature, it glowed brightly. A specimen of ordinary palladium black, however, did not glow on passing hydrogen over it, although it became very hot. Apparently it was not itself sufficiently rich in oxygen. Palladium black or sponge charged with hydrogen glows like pyrophoric iron on being shaken out into the air.



VII. *The Heat of Occlusion of Hydrogen by Palladium Black.*

The determination of the heat of occlusion of hydrogen by palladium black was made in the ice calorimeter already described by us ('Phil. Trans.,' A, 1897, vol. 190, p. 131), and it may be interesting to note that during the course of these investigations the temperature of the room in which the calorimeter was placed varied between  $20^{\circ}$ – $24^{\circ}$  C. The experiments were made in a manner precisely similar to the corresponding series with platinum black, the only essential difference being that a constant volume manometer was attached to the experimental tube.



A constant volume manometer was constructed, as shown in the accompanying figure. This form was chosen for two reasons. In the first place, it obviated the necessity of calibration. The capacity of the capillary tubing between the two taps and the mark at the point A was determined once for all and added to that of the experimental tube. During the course of the experiments the mercury in the right limb was always kept *near* the mark A, and during an actual measurement of pressure exactly *at* the mark. The second and chief reason for selecting this form of manometer, however, was that the main bulk of the gas in the experimental tube was at  $0^{\circ}$  C., whilst the projecting portion was at atmospheric temperature, and in the case of an ordinary manometer it would have been very difficult to estimate the average temperature of the projecting volume of gas, which would have been considerable, since capillary manometers are unreliable.

Before being used the manometer was exhausted and found to give readings practically identical with those of the barometer at the time.

A quantity of palladium black was introduced into the experimental tube, the capacity of which was determined. It was then fully charged with hydrogen and exhausted as completely as possible at  $100^{\circ}$  C. before being placed in the calorimeter, when it was sealed on to the burette, furnishing pure hydrogen, the pump, and the manometer.

After the experimental tube had remained several days in the calorimeter it was found, by making connection with the manometer, that the internal pressure was practically zero. The reduced readings of the barometer and manometer were 757.5 millims. and 757.7 millims. respectively. It was estimated that the quantity of palladium black in the calorimeter was now in a position to occlude over 110 cub. centims. of hydrogen, which was admitted in four separate fractions.

In estimating the amount of hydrogen actually occluded in each case, account was taken of the unabsorbed hydrogen existing in the experimental tube before and after the admission of the gas from the burette. The quantity of unabsorbed hydrogen was ascertained from the known capacity of the apparatus (6.30 cub. centims.), its temperature, viz.,  $0^{\circ}$  C., and the pressure indicated by the manometer.

The deflection of the mercury meniscus in the capillary tube of the calorimeter was corrected, when necessary, for the slight errors due to the fact that the bore of the tube was not quite uniform. The results are given below in tabular form.

#### HEAT of Occlusion of Hydrogen by Palladium Black.

Palladium black used, 1.6658 gram.						
Experiment.	Hydrogen occluded.		Pressure of palladium hydrogen.	Deflection.	Heat produced, K.=100 g. cal.	Heat produced per gram of hydrogen occluded.
	cub. centims.	gram.	millims.	millims.		K.
I.	37.26	0.003353	8.1	147.5	+0.1553	+46.32
II.	35.16	0.003164	13.1	139.0	0.1464	46.27
III.	32.40	0.002916	106.6	129.4	0.1362	46.72
IV.	6.79	0.000611	757.6	27.6	0.0291	47.56
V.	-22.72	-0.002045	4.5	- 88.4	-0.0931	-45.52

Altogether 111.61 cub. centims., or 803 volumes, of hydrogen were occluded in addition to that which was already present in the palladium, and which could not be removed *in vacuo* at  $100^{\circ}$ . From previous experiments (see table, p. 111) this probably amounted to a little over 50 volumes.

It is evident from the last column of the table that the heat evolved per gram of hydrogen occluded remains the same for the different fractions.

The mean of the four experiments is 46.72 K., and if we exclude the last, in which experimental errors have a greater influence owing to the smallness of the quantity of hydrogen occluded, the mean of the three principal results is + 46.44 K., and the deviation from this mean in the case of the fourth result is only 0.6 per cent.

The value + 46.44 K. for the heat evolved per gram of hydrogen occluded may safely be taken as correct within 1 per cent.

In Experiment V. a portion of the hydrogen which had been occluded was exhausted by means of the pump, giving the number 45.5 K. *absorbed* per gram of hydrogen *removed*. The hydrogen comes off so slowly at 0° C. that the experiment could not be continued. The result does not differ very much from those already obtained, and no doubt the conditions under which the experiment was made account largely for the difference. In experimenting with the ice calorimeter it is essential, in order to obtain an accurate measurement of the deflection, that the normal creepage of the instrument should be in the same direction as the deflection to be measured. In this case, however, the two were opposed to each other. In the particular instrument with which we have worked, there was generally a slow creepage inwards, due to the slow melting of the ice. In any serious attempt, therefore, to measure the amount of heat absorbed, it would be advisable to alter the melting point of the ice in the interior of the calorimeter by increasing or diminishing the pressure at the open end of the capillary tube until the creepage was either entirely suspended, or, at least, in the same sense as the deflection to be measured.

FAVRE, by means of his mercury calorimeter ('Comptes Rend.,' vol. 78, p. 1262), determined the heat evolved on the occlusion of hydrogen by palladium sponge, and obtained numbers varying from 202.7 K. to 38.9 K. The results which exceed 60 K., however, were only obtained for the first fractions of hydrogen admitted, and were obviously due, as FAVRE himself recognised, to the simultaneous occurrence of a second reaction, viz., the formation of water from oxide of palladium.

If we exclude four\* of FAVRE's experiments, in which the result is greater than 60 K., then the general mean of the twenty-five remaining experiments, varying between 60 K. and 38.9 K., is 48.6 K. per gram of hydrogen occluded, a result which is approximately the same as that found by us, viz., 46.4 K. In another independent series of experiments ('Comptes Rend.,' vol. 68, p. 1306) FAVRE estimated the heat of occlusion of hydrogen at 41.6 K. from the difference between the quantities of heat evolved when an ordinary Smee cell was placed in the calorimeter (*i.e.*, when the hydrogen was allowed to escape) and when in the same cell

\* Three of the four are the first members in a series of seventeen experiments, whilst the fourth is the first of another series of eleven experiments.



the platinum plate was replaced by palladium, so that the heat evolved on the occlusion of the hydrogen was also included.

Independent of direct measurement in the calorimeter, the heat of a reaction can be calculated, as was first shown by HORSTMANN ('Berichte,' 1869, vol. 2, p. 137), by an application of the second law of thermodynamics to the dissociation pressures of the substance at different temperatures. From measurements of the dissociation pressures of palladium hydrogen by TROOST and HAUTEFEUILLE ('Ann. Chim. Phys.,' (5) vol. 2, p. 279), MOUTIER ('Comptes Rend.,' vol. 79, p. 1242) calculated that the heat of occlusion of hydrogen by palladium at 20° C. was 41.5 K. per gram, and more recently DEWAR ('Proc. Chem. Soc.,' 1897, No. 183, p. 197) obtained, from similar measurements by ROOZEBOOM (HOITSEMA, 'Zeitschr. Physikal. Chem.,' 1895, vol. 17, p. 1), the value  $(45.61 + 0.2378 T.)$  K. At 0° C. (273° abs.), therefore, the heat evolved per gram of hydrogen occluded becomes 46.26 K., which agrees very closely with the value, 46.44, found by us. The magnitude which is directly measured in the calorimeter, however, represents the true heat of the reaction plus the heat corresponding to the work done by the atmosphere, viz., 2.7 K.,\* and hence the true heat of occlusion of hydrogen by palladium black is only  $46.4 - 2.7 = 43.7$  K. per gram of hydrogen.

#### VIII. *Investigation of the Ratio Palladium : Hydrogen.*

From the analysis of palladium black (*v. p.* 108) it follows that the black as prepared contained 97.63 per cent. palladium, the remainder consisting of 1.65 per cent. oxygen and 0.72 per cent. water. Hence, in calculating the ratio of the number of atoms of palladium to the number of atoms of hydrogen in palladium fully charged with hydrogen, we ought to reduce the weighed amount of palladium black to palladium metal *per se*. Strictly speaking, this ought also to be done in expressing the number of volumes of hydrogen occluded by unit volume of palladium existing in the substance we call palladium black.

The following table contains the atomic ratio of palladium to hydrogen for fully charged palladium black, sponge, and foil (the atomic weight of palladium being taken as 107), and also the corrected number of volumes of hydrogen occluded by unit volume of palladium :—

\* *V. p.* 123.



	Palladium Hydrogen.	Volumes of Hydrogen occluded.
Palladium black . . . . .	1.42	873
" " I.* . . . .	1.41	881
" " II.* . . . .	1.40	889
" " III.* . . . .	1.40	889
" sponge . . . . .	1.46	852
" foil . . . . .	1.47	846
" wire I. (GRAHAM) . .	1.37	912 (936)
" " II. " . . . .	1.46	846 (867)
" " III. " . . . .	1.44	859 (888)
" metal (DEWAR) . . .	1.47	847

We have also included in the table the corresponding values which we have calculated from three experiments by GRAHAM ('Researches,' p. 291) and from an experiment on a larger scale by DEWAR ('Phil. Mag.,' (4) vol. 47, p. 334). The numbers given in brackets are those calculated by GRAHAM on the basis that the specific gravity of palladium wire is 12.38. In the last four experiments the palladium was charged electrolytically with hydrogen, and it follows directly from a consideration of the table that, no matter whether the palladium exists as black, sponge, foil, wire, or compact metal, or whether it is charged by exposure to hydrogen gas (the proper conditions being observed), or charged electrolytically, the amount of hydrogen occluded in each case is approximately the same.

If we wish to express, by means of a formula, the composition of palladium hydrogen, then, by choosing the nearest whole numbers, we arrive at the formula  $\text{Pd}_3\text{H}_2$ , which was first proposed by DEWAR (*loc. cit.*), and which corresponds with the ratio 1.5, instead of varying between 1.37 and 1.47, as above.

It must be observed, however, that in all cases in which the palladium has received the maximum charge of hydrogen the amount of hydrogen taken up is always in excess of that required for the formation of a definite chemical compound  $\text{Pd}_3\text{H}_2$ . If we admit—and this is not at all improbable—that such a compound is still capable of occluding or condensing hydrogen in the ordinary way, then the excess of hydrogen which is absorbed does not exclude the possibility of the formation of a definite compound of this composition. On the other hand, evidence in favour of its existence is of a very meagre kind, and is, we think, chiefly confined to the approximation of the ratio Pd/H to the theoretical value 1.5.

The views of chemists and physicists on the nature of palladium hydrogen have from time to time undergone considerable change. In 1869 GRAHAM ('Researches,' p. 290) wrote:—"The idea forces itself upon the mind that palladium, with its occluded hydrogen, is simply an alloy of this volatile metal (hydrogenium), in which

\* Details will be found on p. 125.

the volatility of the one element is restrained by its union with the other, and which owes its metallic aspect equally to both constituents."

In 1874, however, another view was suggested by TROOST and HAUTEFEUILLE ('Ann. Chim. Phys.,' (5) vol. 2, p. 279). From experiments on the vapour pressure of palladium hydrogen at different temperatures, they conclude that the first addition of hydrogen to palladium forms a definite chemical compound  $\text{Pd}_2\text{H}$ , and that the excess of hydrogen over and above that required to form this compound is then occluded or absorbed in the usual way.

By subjecting the results of TROOST and HAUTEFEUILLE to a critical analysis, and from more extended experiments by himself and ROOSEBOOM, HOITSEMA ('Zeitschr. Physikal. Chem.,' vol. 17, p. 1, 1895) has arrived at an entirely different conclusion, namely, that, as regards the vapour pressure experiments, there is no decisive evidence for the formation of a definite chemical compound; and he further suggests that the results are in better agreement with the view that two immiscible solid solutions are formed. This appears to resolve itself into the formation of two alloys instead of one, as suggested by GRAHAM.

In confirmation of the arguments of HOITSEMA against the supposed formation of a compound having the formula  $\text{Pd}_2\text{H}$ , we have also the fact, first observed by FAVRE ('Comptes Rend.,' vol. 77, p. 649, and vol. 78, p. 1257) and confirmed by ourselves, that the heat of occlusion of hydrogen by palladium remains constant throughout the whole range of absorption.

If a chemical compound  $\text{Pd}_2\text{H}$  were first formed, we should expect to get a certain definite evolution of heat per gram of hydrogen combined for the hydrogen first admitted, and then, after sufficient hydrogen had been added to form the compound  $\text{Pd}_2\text{H}$  (about 630 volumes), we should expect to find a different value for the heat evolved per gram of hydrogen occluded or dissolved or absorbed. We have, therefore, confidence in discrediting the supposed formation of  $\text{Pd}_2\text{H}$ .

It would seem that HOITSEMA's conclusions apply equally well to the supposed formation of  $\text{Pd}_3\text{H}_2$ . But before finally dismissing the possibility of its existence we consider that some additional evidence is desirable. It is proposed to attack the problem from another and independent point of view, and it is hoped that all doubts, which are liable to affect any process in which it is uncertain whether equilibrium has really set in or not, will be overcome.

#### IX. *On the Heat of Occlusion and the Heat of Condensation.*

It has sometimes been suggested (cf. FAVRE, 'Ann. Chim. Phys.,' (5) vol. 1, p. 209), that the heat of occlusion of a gas represents the heat of condensation or liquefaction of the gas in the capillary pores of the absorbing substance. Now if this is so, and we vary the absorbing substance, at the same time maintaining it always at the same temperature, say  $0^\circ \text{C}$ ., so that its specific heat shall play no

part in the reaction, then it would seem justifiable to suppose that in the liquefaction of one gram of one and the same gas, hydrogen for example, in different absorbing substances, the *same* amount of heat, viz., the heat of condensation of one gram of hydrogen, would always be evolved.

We have found, for the heat of occlusion of one gram of hydrogen in platinum and palladium, the following values :—

Platinum . . . .	+ 68·8 — 2·7 = 66·1 K. per gram.
Palladium . . . .	+ 46·4 — 2·7 = 43·7 K. „

In order to obtain the difference between the initial and final values of the internal energy of the system, we require to subtract the heat equivalent of the work done by the atmosphere. For one gram molecule of a gas this amounts to 0·02 T.K. (2 T. g. cal.), and hence for half a gram molecule of hydrogen at 0° C. we require to diminish the above values for the heat of occlusion by 2·7 K. Since the numbers which we thus obtain are by no means approximately equal, we are of opinion that the phenomenon of the occlusion of hydrogen by platinum and palladium black is not simply the liquefaction or condensation of the gas in the capillary pores of the metals.

The same arguments would, doubtless, be valid in a comparison of the heat of occlusion with the heat of solidification or fusion.

#### X. *The Influence of Increased Pressure on the Occlusion of Hydrogen by Palladium Black.*

We have already seen that the composition of palladium hydrogen, fully charged with hydrogen at ordinary atmospheric pressure, corresponds approximately to the formula  $\text{Pd}_3\text{H}_2$ . Although from the experiments of HOITSEMA and ROOSEBOOM it would seem that increase of pressure should produce little or no effect on the occlusion of hydrogen by palladium, an experimental demonstration was deemed desirable. The details of the experiment were the same as in the corresponding experiment with platinum black ('Phil. Trans.,' A, 1895, vol. 186, p. 675). A quantity of palladium black was charged with hydrogen and shut up with an excess of hydrogen in the shorter graduated limb of a glass U-tube. After equilibrium had been established, the volume, temperature, and pressure of the enclosed gas were noted. Successive quantities of mercury were then poured into the longer limb, and the measurements repeated.

The results are contained in the following table :—



Palladium black used, 0.878 gram.				
Volume of enclosed gas.	Temperature.	Pressure (millims. Hg. at 0° C.).	PV.	Volume reduced to 0° C. and 760 millims.
cub. centims.				cub. centims.
128.1	19.14	810	103,800	127.6
74.8	19.14	1380	103,200	126.9
48.0	19.14	2151	103,300	127.0
36.5	19.14	2821	103,000	126.6
29.5	19.14	3496	103,100	126.8

The fact that the volume energy (PV), or the reduced volume of the gas, except for the first increment of pressure, may be taken as constant, satisfactorily establishes the conclusion that increase of pressure, at least up to 4.6 atmospheres, practically has no effect on the occlusion of hydrogen by palladium black which has already absorbed the maximum quantity of hydrogen at the ordinary temperature. Platinum and palladium black thus behave in precisely the same way as regards the effect of pressure on the occlusion of hydrogen. The first increase of pressure caused the disappearance of several volumes (about ten volumes) of hydrogen, but as this does not continue, it may possibly be due to the presence of moisture, derived from the palladium black, in the gas.

The influence of pressure on the occlusion of hydrogen by palladium at high temperatures has recently been described by DEWAR ('Proc. Chem. Soc.,' 1897, No. 183, p. 192).

#### XI. *The Heat of Absorption of Oxygen by Palladium Black.*

It is fairly certain that, on the absorption of oxygen by palladium black, a definite oxide, or a mixture of definite oxides, is formed. A redetermination of the heats of formation of the different oxides of palladium is outside the scope of this inquiry, but it seemed desirable to determine the heat of absorption of the oxygen actually contained in palladium black, for the purpose of comparing the number so found with the corresponding number for platinum black. Since it was impossible to make the measurement directly, an indirect method was employed, viz., the estimation of the amount of heat *absorbed* on the *removal* of the oxygen from palladium black.

A quantity of palladium black was introduced into the calorimeter, and fully charged with hydrogen. A portion of the hydrogen absorbed was thus really occluded, whilst the remainder removed the oxygen contained in the palladium black as water. After the heat change had been measured, the occluded fraction was easily ascertained by exhausting at a red heat, and from this the volume of oxygen removed from the palladium black was found.



Assuming, as was necessary in the case of the corresponding experiment on platinum black (*q.v.*, 'Phil. Trans.' A, 1897, vol. 190, p. 149), that no appreciable heat change is involved in the mere wetting of the palladium by the water formed, and knowing the total heat of the reaction, the heat of formation of water, the heat of occlusion of the hydrogen, and the quantity of oxygen removed, we can calculate the heat absorbed on its removal.

The results obtained in three independent experiments are given below in tabular form. The amounts of hydrogen occluded and oxygen removed are expressed in cubic centimetres, volumes (Pd. bk. = 1), and grams; and it will be seen that there is a very satisfactory agreement among themselves of the values representing the number of volumes of hydrogen occluded and the number of volumes of oxygen removed.

#### CALORIMETRIC Experiments on the Absorption of Oxygen by Palladium Black.

		I.	II.	III.
Palladium black used . . . . .	Grams	0·4789	1·0237	1·0570
" " " " " " " " " " " "	Cub. centims.	0·0399	0·0853	0·0881
Hydrogen occluded . . . . .	Cub. centims.	34·32	74·08	76·51
" " " " " " " " " " " "	Volumes	860	868	868
" " " " " " " " " " " "	Grams	0·003089	0·006667	0·006886
Hydrogen burnt to water . . . . .	Cub. centims.	11·19	24·02	25·16
" " " " " " " " " " " "	Grams	0·001007	0·002162	0·002264
Oxygen removed . . . . .	Cub. centims.	5·595	12·01	12·58
" " " " " " " " " " " "	Volumes	140	141	143
" " " " " " " " " " " "	Grams	0·008000	0·01717	0·01799
Heat due to formation of water (calculated) . . . . .	K.	0·3444	0·7394	0·7743
Heat due to occlusion of hydrogen (calculated) . . . . .	K.	0·1433	0·3093	0·3195
Sum of last two . . . . .	K.	0·4877	1·0487	1·0938
Heat evolved, corresponding to { the deflection . . . . . }	K.	0·3927	0·8424	0·8931
Difference = heat absorbed on the removal of the oxygen . . . . .	Millims.	373·0	800·0	848·1
	K.	0·0950	0·2063	0·2007
Heat absorbed per gram of oxygen removed . . . . .	K.	11·9	12·0	11·2

The three values of the heat *absorbed* per gram of oxygen *removed*, or, what amounts to the same thing, the heat evolved per gram of oxygen absorbed, are thus + 11·9, + 12·0, and + 11·2 K., whilst the mean of the three is + 11·7 K. This number, referred to 16 grams of oxygen, becomes + 187 K. (+ 18700 gram calories).

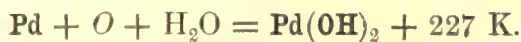
From the nature of these experiments it is difficult to say within what limits of error this number may be taken as correctly representing the heat of absorption of oxygen in palladium black. As far as such measurements go, however, the agreement between the individual numbers may be said to be satisfactory.

Whilst the heat of occlusion of hydrogen in palladium is less than the corresponding number for platinum, the heat of absorption of oxygen is slightly greater. It will be remembered that the heat of absorption of oxygen in platinum black, viz., + 176 K. per gram atom, is almost identical with the number given by JULIUS THOMSEN for the heat of formation of platinous hydroxide,  $\text{Pt}(\text{OH})_2$ , viz., + 179 K., thereby furnishing presumptive evidence that the absorption of oxygen by platinum black is a true oxidation phenomenon, the water necessary for the formation of the hydroxide being always present in platinum black.

It therefore remains to be seen whether there is any sort of agreement between the numbers representing the heat of absorption of oxygen by palladium black and the heat of formation of an oxide or hydroxide of palladium.

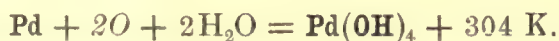
JOANNIS ('Comptes Rend.', vol. 95, p. 295) determined the heat of formation of palladous hydroxide,  $\text{Pd}(\text{OH})_2$  (oxide?), by precipitating the substance from a solution of (1) potassium palladobromide, (2) potassium palladochloride, by means of caustic potash, and obtained the values + 193.4 K. and + 208.8 K., the mean of the two being about + 200 K.

JULIUS THOMSEN ('Thermochem. Untersuch.', vol. 3, p. 436) also determined the same heat of formation by precipitating a solution of sodium palladochloride by caustic soda, and gives as its value + 227 K., thus:—



There is thus a discrepancy of about 27 K. between the results obtained by the two observers.

The heat of formation of palladic hydroxide was also determined by THOMSEN in a similar way, with the following result:—



This number refers to 32 grams of oxygen; for 16 grams it therefore becomes + 152 K.

The value found by us for the heat of absorption of oxygen in palladium black lies intermediate between the values given by THOMSEN for the heats of formation of palladous and palladic hydroxide, and, considering the difference between the numbers representing the heat of formation of the lower compound as determined by THOMSEN and JOANNIS, the heat of absorption of oxygen in presence of sufficient moisture may be consistent with the formation of either the higher or lower hydroxide, or possibly with a mixture of both. In any case, it is of the same order of magnitude, and taken in conjunction with the behaviour of palladium black when heated in an atmosphere of oxygen, is undoubtedly in harmony with the view that the absorption of oxygen by palladium black is a true phenomenon of oxidation.

IV. *Comparative Photographic Spectra of Stars to the  $3\frac{1}{2}$  Magnitude.*

By FRANK McCLEAN, F.R.S.

Received March 24,—Read April 8, 1897.

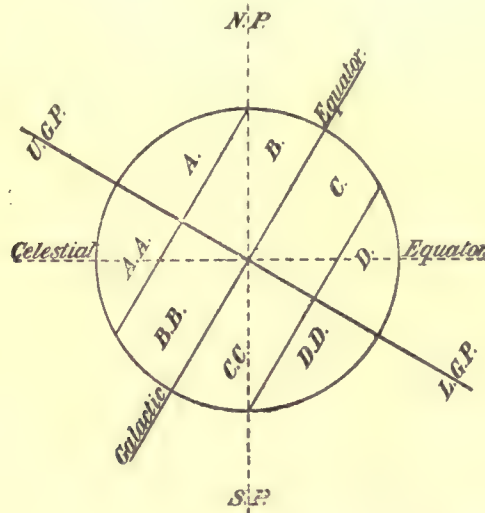
[PLATES 1-17.]

THE 160 photographs which accompany the paper include, with insignificant exceptions, all stars equal to and brighter than the  $3\frac{1}{2}$  magnitude, contained in five out of eight equal areas, into which the celestial sphere has been divided.

The diagram shows the position of the eight areas consisting of—

Two upper galactic polar regions, viz.,	A and AA.
„ „ „ zones	„ B „ BB.
Two lower galactic zones	„ C „ CC.
„ „ „ polar regions	„ D „ DD.

The galactic zones extend to  $30^\circ$  from the galactic plane.



The object of the division into equal areas is to bring out roughly any differences of distribution of the different types of spectra in relation to the galactic plane.



The areas A, B, C, D, and AA are included in the present series of photographs. Of these the first four constitute a complete hemisphere, symmetrically divided by the galactic plane, and the deductions as to distribution have been made from them.

The stellar spectra have been arranged in series, and classed separately for the respective areas to which they belong. It appears at once that SECCHI's Type I. requires further subdivision into distinct classes. To effect this, a series of divisions in parallel to SECCHI's types have been adopted, in which Divisions I., II., and III. correspond to Type I., and Divisions IV., V., and VI. to Types II., III., and IV. respectively.

The classified tables of the stellar spectra are given at the end of the paper, and previous classifications by SECCHI in 1866-1868, by VOGEL in 1883, by PICKERING in 1890, and by LOCKYER in 1892 are indicated in separate columns.

Division I. includes all stars whose spectra are characterised by the lines of hydrogen and of cleveite gas.

Subdivision I. ( $\alpha$ ) shows other special lines in addition. I have made a close comparison with the spectra of nearly all the elements, for the purpose of identifying these extra lines, but without definite result. The only suggestion that presents itself to me is the possibility of their being due to oxygen.

I have placed below the scale attached to these spectra RUNGE and PASCHEN's spectrum of cleveite gas, which forms the characteristic spectrum of the division. Also THALÉN's spark spectrum of oxygen. Although there is not a perfect agreement between the spectrum of oxygen and the special lines referred to, there is a very remarkable correspondence.

I have further placed below the scale CAMPBELL's bright-line spectrum of the nebula in Orion. The general coincidence with the stellar line leaves little doubt as to the close connexion between this class of stars and the gaseous nebulæ.

Evidence to the same effect is afforded by the wonderful photographs taken in recent years, showing the physical connexion of many of these stars with the nebulæ. Such are Dr. ROBERTS's photographs of the Pleiades and of the great nebula in Orion. Alongside copies of these I have shown the corresponding groups of the spectra of the involved stars; the respective stars and their spectra are easily identified on inspection. All six stars of the Pleiades show the cleveite gas spectrum, but the presence of the special extra lines is doubtful. The Orion stars, which include  $\iota$  and  $\theta$  Orionis, all show the cleveite spectrum, but in these the special lines are also recognised.

We have thus evidence of both physical and chemical connexion between these stars and the nebulæ.

[Note added 8th April, 1897.—There are two other instances of the connexion of stars of Subdivision I. ( $\alpha$ ) with nebulæ.

BARNARD, in 1895, photographed an extended nebula in the vicinity of  $\alpha$  Scorpii (Antares). He mentions  $\sigma$  Scorpii as connected with the nebulosity. The photo-

graphic spectra of the four stars,  $\beta$ ,  $\delta$ ,  $\pi$ , and  $\sigma$  Scorpii, all near Antares, belong to Division I. (a). They are in the area BB, and do not appear in the present series.

BARNARD also photographed, with a six-hour exposure, an extended nebula contiguous to  $\xi$  Persei, a star of the 4th magnitude. The spectra of  $\epsilon$  and  $\zeta$  Persei, on either side of it, appear in the photographs, and belong to Subdivision I. (a). A photograph of the spectra of  $\xi$  Persei was recently obtained, and it also belongs to the same division.

Both these instances point to the same conclusion, that stars of this type are in the first stage of stellar development from nebulae.

The star  $\xi$  Persei and also  $\iota$  Orionis give in their spectra three lines of the second hydrogen series, recently identified by PICKERING in  $\zeta$  Puppis. One of the three lines at wave-length 4027 corresponds to the characteristic helium line. The other two at wave-length 4201 and wave-length 4544 do not appear in any other spectra of Subdivision I. (a) in the present photographs. These lines appear to belong to the earliest stage of stellar development.]

There is further evidence to the same effect in the similarity of the distribution of this type of star, and of the gaseous and planetary nebulae.

The gaseous nebulae, given in the Table in FROST'S edition of SCHEINER'S 'Spectroscopy,' have been distributed into the same equal areas as the stars.

The following shows the relative distribution for the nebulae and for the stars:—

	A.	B.	C.	D.
Gaseous nebulae . . . .	3	7	16	6
Stars of Division I. . . .	3	6	17	3.

Thus it appears that the helium stars of Division I. and the gaseous nebulae are subject to a similar law of distribution in relation to the galactic plane.

All these facts afford grounds for accepting the conclusion that the helium stars of Subdivision I. (a) are in the first stage of stellar development from the gaseous nebulae.

Division I. has a second subdivision, viz., I. (b), which is also characterised by a few special lines. These lines have been attributed to calcium, barium, and magnesium. Since these special lines persist through the subsequent divisions, it may be concluded with tolerable certainty, that in order of development Subdivision I. (b) follows after Subdivision I. (a). The K line of calcium, which first appears in this subdivision, gradually increases in strength to such a marked extent in the subsequent divisions, that it may be practically taken (as suggested by Dr. HUGGINS), as a criterion of the type to which a spectrum belongs. Two bright line spectra have been treated as belonging respectively to the divisions to which their absorption spectra belong.

The spectrum given of  $\gamma$  Cassiopeiae, clearly places it in Subdivision I. (a). It is a helium star. The bright hydrogen lines are weak, and are placed centrally in the diffuse absorption lines.



$\beta$  Lyrae is also a helium star. Its spectrum belongs to Subdivision I. (*b*). Both the hydrogen and the helium lines appear bright. The peculiarities of this star are well known.

A third bright line star, of another division, may be mentioned here. The spectrum of Mira Ceti belongs to Type III. or Division V. The bright hydrogen lines appear periodically, for a brief time, with great brilliancy. The banded absorption spectrum brightens up simultaneously, a peculiarity which it is difficult to explain. The exceptionally good photograph of this spectrum was taken on the 1st January last.

Division II. is especially the hydrogen type. In it the hydrogen spectrum attains its full development both in the strength of the lines and in the extent of the spectrum. This is shown in the ultra violet series of lines, discovered by Dr. HUGGINS. The narrower and more sharply defined calcium line K, in itself distinguishes the division from those which precede and follow it. The delicate absorption lines, which also distinguish the type, are difficult to photograph, but they are well shown in the spectra of Sirius and Vega. The spectrum of  $\alpha$  Cygni has been sometimes classed differently, but a comparison with the spectrum of Sirius, which is placed next it, clearly shows their identity, and that they only differ in strength. The fine lines appear to be due to calcium and titanium rather than to iron, although that spectrum is also present in an incipient form. The distribution of this division is irregular, viz. :

A.	B.	C.	D.	AA.
10	7	0	3	3

Division III., or the hydrogen-iron type, is the last of the separate divisions into which Type I. has been divided. In its more advanced examples the iron spectrum is fully developed. That spectrum has been plotted on the scale for comparison. This type is more closely allied to the subsequent solar type, Division IV., than to the preceding hydrogen and helium types. The brightness of the violet end, and the obscurity of the red end of the spectrum, remain the same as in the preceding divisions. The hydrogen lines remain very strong, and the calcium lines K and H are generally subordinate to them.

This division completes the requisite subdivision of SECCHI's Type I., which has not hitherto been fully established. The subsequent divisions remain as defined by SECCHI in 1868. It must also be remembered that SECCHI fully recognised the special character of the spectra of the Orion stars.

Division IV. is equivalent to Type II., or the solar type. The characteristics of this type are well defined. They are also elucidated by our more intimate knowledge of the solar spectrum, which forms the basis of our knowledge of astronomical physics.

Division V. is equivalent to Type III., and is the first of the banded types investigated by Dunér. The photographs of this type of spectra are difficult to take. These stars are not numerous within the range of magnitude under con-



sideration. The spectra are closely allied to those of Division IV., and have been placed consecutively with them. The coincidences with the spectra of calcium and manganese have been marked on the scale.

Division VI. is equivalent to Type IV. There are no stars in this division brighter than the  $5\frac{1}{2}$  magnitude, but to complete the series, photographs of the spectra of SECCHI's Superba, and of 19 Piscium have been included. The photographs are poor owing to the faintness of the stars, but their interest lies in the hydrocarbon absorption bands, and further in the line spectrum, also recognizable, and similar to that of  $\alpha$  Tauri (Aldebaran).

There remains the question of how far the distribution of the stars into the eight equal areas discloses any information as to the distribution of their respective types of spectra in space.

The dimensions of the sphere enclosing stars to the  $3\frac{1}{2}$  magnitude is obtained from the light ratio. The decrease of light for each magnitude beyond the 1st is inversely as the light ratio 2.51. The corresponding increase of distance is as the square root of this ratio, or as 1 to 1.58. Taking the distances of the  $3\frac{3}{4}$  magnitude stars arrived at in this way as the radius of the sphere containing the stars of the  $3\frac{1}{2}$  magnitude, its radius will be, in terms of the mean distance of 1st magnitude stars from the sun,

$$= 1 \times (1.58)^{\frac{1}{2}} = 1.25 \text{ approximately.}$$

The mean distance of 1st magnitude stars from the sun has been determined from parallax observations to be approximately  $36\frac{1}{2}$  light years. Thus the diameter of the enclosing sphere for stars of the  $3\frac{1}{2}$  magnitude is approximately 255 light years.

This gauge block of space, although large, might be so small, compared with the dimensions of the galaxy, as to disclose nothing with regard to its structure. The following table of distribution, however, shows that within this block there are indications of differences in the distribution of the different stellar types, relatively to the plane of the galaxy. The table shows the distribution of the gaseous nebulae, both planetary and extended, for the same areas.

TABLE of Distribution of Gaseous Nebulæ and of Stellar Types.  
Stars to the  $3\frac{1}{2}$  Magnitude.

	A.	B.	C.	D.	Total.	AA.	BB.	CC.	DD.	Total.
Planetary nebulæ . .	2	3	8	2	(15)	2	7	3	0	(12)
Extended „ . .	1	4	8	4	(17)	1	4	3	1	(9)
Total gaseous nebulæ	3	7	16	6	(32)	3	11	6	1	(21)
<i>Stellar Types.</i>										
Division I. . . . .	3	6	17	3	(29)	6				
„ II. . . . .	10	7	0	3	(20)	3				
„ III. . . . .	7	8	8	4	(27)	9				
„ IV. . . . .	14	8	9	13	(44)	9				
„ V. . . . .	1	2	4	3	(10)	3				
Total stellar spectra .	35	31	38	26	(130)	30				

NOTE.—The gaseous nebulæ are those given in the table in FROST's edition of SCHEINER's 'Astronomical Spectroscopy.'

We gather from this table that as the stellar types of spectra become more advanced they are found to be more evenly distributed in space. It suggests the idea that stars of the solar type—Division IV.—started on their career as helium stars of Division I., before the condensation of the galaxy.

The Procyon stars of Division III. possibly followed in the same course after the galaxy was formed. The paucity of stars of this type in the lower polar region, coupled with their even distribution in the other areas, suggests the idea that the sun itself is situated near the lower boundary of the galaxy.

The distribution of the Sirian stars—Division II.—is irregular, and further information is required as to their distribution in the southern areas.

The Orion stars—Division I.—are mostly confined to the galactic zones. It has been already conjectured that they are still in the first stage of stellar development from the gaseous nebulæ.

It has been throughout assumed that the successive types or divisions are merely the manifestations of the successive physical states, through which every star naturally passes in the course of its career.

*Index to the Tables.**Andromeda.*

$\alpha$  D. 3, I.  
 $\beta$  C. 35, V.  
 $\gamma$  C. 32, IV.  
 $\delta$  D. 16, IV.

*Aquarius.*

$\alpha$  D. 12, IV.  
 $\beta$  D. 11, IV.  
 $\delta$  D. 6, II.

*Aquila.*

$\alpha$  C. 21, III.  
 $\beta$  C. 27, IV.  
 $\gamma$  C. 33, IV.  
 $\delta$  C. 18, III.  
 $\zeta$  B. 7, II.  
 $\theta$  C. 15, I.

*Aries.*

$\alpha$  D. 22, IV.  
 $\beta$  D. 9, III.

*Auriga.*

$\alpha$  B. 23, IV.  
 $\beta$  B. 9, II.  
 $\epsilon$  B. 20, III.  
 $\theta$  B. 8, II.  
 $\iota$  C. 34, IV.

*Bootes.*

$\alpha$  AA. 25, IV.  
 $\beta$  A. 28, IV.  
 $\gamma$  A. 19, III.  
 $\delta$  A. 23, IV.  
 $\epsilon$  A. 27, IV.  
 $\eta$  AA. 19, IV.

*Canes.*

$\alpha$  A. 4, II.

*Canis Major.*

$\alpha$  B. Compn.

*Canis Minor.*

$\alpha$  B. 17, III.  
 $\beta$  B. 4, I.

*Capricornus.*

$\delta$  D. 10, III.

*Cassiopeia.*

$\alpha$  C. 31, IV.  
 $\beta$  C. 23, III.  
 $\gamma$  C. 1, I.  
 $\delta$  C. 22, III.  
 $\epsilon$  B. 3, I.

*Cepheus.*

$\alpha$  B. 15, III.  
 $\beta$  B. 2, I.  
 $\gamma$  B. 25, IV.  
 $\delta$  B. 18, III.

*Cetus.*

$\alpha$  D. 25, V.  
 $\beta$  D. 13, IV.  
 $\gamma$  D. 7, III.  
 $\zeta$  D. 18, IV.  
 $\eta$  D. 21, IV.  
 $\theta$  D. 15, IV.  
 $\iota$  D. 17, IV.  
 $\circ$  D. 26, V.  
 $\tau$  D. 14, IV.

*Corona.*

$\alpha$  A. 6, II.

*Corvus.*

$\gamma$  AA. 5, I.  
 $\delta$  AA. 3, I.

*Cygnus.*

$\alpha$  B. 13, II.  
 $\beta$  B. 28, IV.  
 $\gamma$  B. 21, III.  
 $\delta$  B. 6, I.  
 $\epsilon$  C. 29, IV.  
 $\zeta$  C. 30, IV.

*Delphinus.*

$\beta$  C. 24, III.

*Draco.*

$\alpha$  A. 9, II.  
 $\beta$  A. 21, IV.  
 $\gamma$  B. 29, IV.  
 $\delta$  B. 26, IV.  
 $\zeta$  A. 3, I.  
 $\eta$  A. 30, IV.  
 $\iota$  A. 32, IV.  
 $\kappa$  A. 2, I.

*Eridanus.*

$\beta$  C. 19, III.  
 $\gamma$  D. 24, V.  
 $\delta$  D. 19, IV.  
 $\epsilon$  D. 20, IV.

*Gemini.*

$\alpha$  B. 12, II.  
 $\beta$  B. 24, IV.  
 $\gamma$  B. 10, II.  
 $\delta$  B. 16, III.  
 $\epsilon$  B. 27, IV.  
 $\theta$  AA. 16, III.  
 $\mu$  B. 30, V.

*Hercules.*

$\alpha$  B. 31, V.  
 $\beta$  A. 24, IV.  
 $\gamma$  A. 16, III.  
 $\delta$  A. 14, III.  
 $\epsilon$  A. 8, II.  
 $\zeta$  A. 22, IV.  
 $\eta$  A. 26, IV.  
 $\mu$  B. 22, IV.  
 $\pi$  A. 33, IV.

*Hydra.*

$\alpha$  AA. 27, IV.  
 $\zeta$  AA. 21, IV.

*Leo.*

$\alpha$  AA. 2, I.  
 $\beta$  AA. 14, III.  
 $\gamma$  AA. 24, IV.  
 $\delta$  AA. 15, III.  
 $\epsilon$  A. 25, IV.  
 $\zeta$  A. 20, III.  
 $\eta$  AA. 6, I.  
 $\theta$  AA. 9, II.

*Libra.*

$\alpha$  AA. 12, III.  
 $\beta$  AA. 4, I.

*Lyra.*

$\alpha$  B. 11, II.  
 $\beta$  B. 1, I.  
 $\gamma$  B. 5, I.

*Ophiuchus.*

$\alpha$  B. 14, III.  
 $\delta$  AA. 28, V.  
 $\epsilon$  AA. 23, IV.

*Orion.*

$\alpha$  C. 38, V.  
 $\beta$  C. 17, I.  
 $\gamma$  C. 9, I.  
 $\delta$  C. 2, I.  
 $\epsilon$  C. 3, I.  
 $\zeta$  C. 5, I.  
 $\iota$  C. 6, I.  
 $\kappa$  C. 4, I.

*Pegasus.*

$\alpha$  D. 4, II.  
 $\beta$  C. 37, V.  
 $\gamma$  D. 1, I.  
 $\epsilon$  D. 23, IV.  
 $\zeta$  D. 2, I.  
 $\eta$  C. 28, IV.  
 $\theta$  D. 5, II.

*Perseus.*

$\alpha$  C. 25, III.  
 $\beta$  C. 13, I.  
 $\gamma$  C. 26, IV.  
 $\delta$  C. 14, I.  
 $\epsilon$  C. 7, I.  
 $\zeta$  C. 8, I.

*Pisces.*

$\alpha$  D. 8, III.

*Serpens.*

$\alpha$  AA. 26, IV.  
 $\beta$  AA. 11, III.  
 $\delta$  AA. 17, III.  
 $\epsilon$  AA. 13, III.  
 $\mu$  AA. 7, II.

*Taurus.*

$\alpha$  C. 36, V.  
 $\beta$  C. 12, I.  
 $\zeta$  C. 10, I.  
 $\eta$  C. 16, I.  
 $\lambda$  C. 11, I.

*Ursa Major.*

$\alpha$  A. 29, IV.  
 $\beta$  A. 12, II.  
 $\gamma$  A. 7, II.  
 $\delta$  A. 10, II.  
 $\epsilon$  A. 5, II.  
 $\zeta$  A. 13, II.  
 $\eta$  A. 1, I.  
 $\theta$  A. 18, III.  
 $\iota$  A. 17, III.  
 $\lambda$  A. 11, II.  
 $\mu$  A. 35, V.  
 $\psi$  A. 31, IV.

*Ursa Minor.*

$\alpha$  B. 19, III.  
 $\beta$  A. 34, IV.  
 $\gamma$  A. 15, III.

*Virgo.*

$\alpha$  AA. 1, I.  
 $\beta$  AA. 20, IV.  
 $\gamma$  AA. 18, III.  
 $\delta$  AA. 29, V.  
 $\epsilon$  AA. 22, IV.  
 $\zeta$  AA. 10, III.  
 $\eta$  AA. 8, II.



## PHOTOGRAPHIC Stellar Spectra. Stars to Magnitude 3.5.

## Upper Galactic Polar Region, N.

## A (Plates 1 to 3).

SECCHI's Types.	Present classification.	No. of photograph.	List of Stars.	Magnitude. Uranometria Oxoniensis.	LOCKYER'S "Tables," Nov. 1892, 'Phil. Trans.,' A, 1893.	PICKERING, 1890, 'DRAPER Catalogue.'	VOGEL, 1883, 'Catalogue of Stellar Spectra Zone - 1° to + 20°.'	SECCHI, 'Stellar Types,' 1866-1868.
Type I.	Division I.	1	$\eta$ Ursæ Majoris .	1.8	A ( $\alpha$ )	A	..	1°
		2	$\kappa$ Draconis . . .	3.7	..	A	..	2
		3	$\zeta$ " . . .	3.3	..	A	..	1
	Division II.	4	$\alpha$ Can. Venat . .	3.3	A ( $\delta$ )	..	..	..
		5	$\epsilon$ Ursæ Majoris .	1.8	*	A	..	1
		6	$\alpha$ Coronæ . . .	2.2	A ( $\delta$ )	A	..	1
		7	$\gamma$ Ursæ Majoris .	2.3	A ( $\delta$ )	A	..	1
		8	$\epsilon$ Herculis . . .	3.8	*	A	..	1
		9	$\alpha$ Draconis . . .	3.6	..	A	..	1
		10	$\delta$ Ursæ Majoris .	3.4	..	A	..	1
		11	$\lambda$ " . . .	3.5	..	A	..	..
		12	$\beta$ " . . .	2.2	A ( $\gamma$ )	A	..	1
		13	$\zeta$ " . . .	2.1	*	A	..	1
	Division III.	14	$\delta$ Herculis . . .	3.2	*	A	..	1
		15	$\gamma$ Ursæ Minoris .	3.0	..	A	..	..
		16	$\gamma$ Herculis . . .	3.6	A ( $\beta$ )	A	Ia!	1
		17	$\epsilon$ Ursæ Majoris .	3.2	..	A	..	..
		18	$\theta$ " . . .	3.1	..	F	..	..
		19	$\gamma$ Bootis . . .	3.2	*	A	..	1
		20	$\zeta$ Leonis . . .	3.4	..	F	..	1
Type II.	Division IV.	21	$\beta$ Draconis . . .	3.0	..	K	..	2
		22	$\zeta$ Herculis . . .	2.6	B ( $\beta$ )	G	..	2
		23	$\delta$ Bootis . . .	3.4	..	K	..	2
		24	$\beta$ Herculis . . .	2.7	B ( $\beta$ )	K	..	2
		25	$\epsilon$ Leonis . . .	3.4	B ( $\beta$ )	K	..	2
		26	$\eta$ Herculis . . .	3.6	B ( $\beta$ )	K?	..	2
		27	$\epsilon$ Bootis . . .	2.5	B ( $\beta$ )	G?	..	2
		28	$\beta$ " . . .	3.6	B ( $\beta$ )	K?	..	2
		29	$\alpha$ Ursæ Majoris .	1.9	B ( $\beta$ )	K	..	2
		30	$\eta$ Draconis . . .	2.8	..	K	..	2
		31	$\psi$ Ursæ Majoris .	3.2	C ( $\beta$ )	K	..	..
		32	$\epsilon$ Draconis . . .	3.3	..	K	..	2
		33	$\pi$ Herculis . . .	3.6	B ( $\beta$ )	K?	..	2
		34	$\beta$ Ursæ Minoris .	2.3	C ( $\beta$ )	L?	..	2
Type III.	Division V.	35	$\mu$ Ursæ Majoris .	3.1	..	M	..	..

NOTE.—For LOCKYER's Notation, see 'Phil. Trans.,' A, 1893.

For PICKERING's Notation, see 'Introduction to DRAPER Catalogue,' 1890.

For VOGEL's Notation, see 'Potsdam Observatory Publications,' No. 11, 1883.

For SECCHI's Types, see 'Mem. Soc. Italiana,' 1867. For present order of Types 2 and 3, see Preliminary Notice in 'Catalogo delle Stelle,' Paris, 1867.

## PHOTOGRAPHIC Stellar Spectra. Stars to Magnitude 3.5.

## Upper Galactic Zone, N.

## B (Plates 4 to 7).

Secchi's Types.	Present classification.	No. of photograph.	List of Stars.	Magnitude. Uranometria Oxoniensis.	LOCKYER'S "Tables," Nov., 1892, 'Phil. Trans.,' A, 1893.	PICKERING, 1890, 'DRAPER Catalogue.'	VOGEL, 1888, 'Catalogue of Stellar Spectra Zone 1' to +20'.	SECCHI, 'Stellar Types,' 1866-1868.
Type I.	Division I.	1	$\beta$ Lyrae . . . . .	{ 3.4 4.4 } Var.	D	G	..	..
		2	$\beta$ Cephei . . . . .	3.4	..	A	..	2
		3	$\epsilon$ Cassiopeiæ . . . . .	3.5	..	A	..	1
		4	$\beta$ Canis Minoris . . . . .	3.1	A ( $\gamma$ )	A	Ia !	1
		5	$\gamma$ Lyrae . . . . .	3.2	A ( $\gamma$ )	A	..	1
		6	$\delta$ Cygni . . . . .	2.8	A ( $\alpha$ )	A	..	1
	Division II.	7	$\zeta$ Aquilæ . . . . .	3.1	*	A	Ia	1
		8	$\theta$ Aurigæ . . . . .	3.0	A ( $\epsilon$ )	A	..	1
		9	$\beta$ " . . . . .	1.9	A ( $\gamma$ )	A	..	1
		10	$\gamma$ Geminorum . . . . .	2.1	A ( $\epsilon$ )	A	Ia !!	1
		11	$\alpha$ Lyrae (Vega) . . . . .	(+0.9)	A ( $\epsilon$ )	A	..	1
		12	$\alpha$ Geminorum (Castor) . . . . .	1.5	A ( $\epsilon$ )	A	..	1
		Compn.	$\alpha$ Canis Majoris (Sirius) . . . . .	(+1.9)	A ( $\epsilon$ )	A ?	..	1
		13	$\alpha$ Cygni . . . . .	1.3	A ( $\alpha$ )	A	..	2
	Division III.	Compn.	$\alpha$ Geminorum (Castor) . . . . .	1.5	A ( $\epsilon$ )	A	..	1
		14	$\alpha$ Ophiuchi . . . . .	2.2	A ( $\beta$ )	A	Ia !!	1
		15	$\alpha$ Cephei . . . . .	2.6	A ( $\beta$ )	A	..	1
		16	$\delta$ Geminorum . . . . .	3.5	..	A	..	1
		17	$\alpha$ Canis Min. (Procyon) . . . . .	(+0.5)	A ( $\beta$ )	F	Ia !	1
		18	$\delta$ Cephei . . . . .	{ 3.7 4.9 } Var.	..	F ?	..	2
		19	$\alpha$ Ursæ Min. (Polaris) . . . . .	2.0	B ( $\beta$ )	..	..	2
		20	$\epsilon$ Aurigæ . . . . .	{ 3.0 4.5 } Var.	..	F	..	2
		21	$\gamma$ Cygni . . . . .	2.3	B ( $\alpha$ )	Q	..	2
Type II.	Division IV.	22	$\mu$ Herculis . . . . .	3.5	..	I ?	..	1
		23	$\alpha$ Aurigæ (Capella) . . . . .	(+0.9)	B ( $\beta$ )	F	..	2
		24	$\beta$ Geminorum (Pollux) . . . . .	1.4	B ( $\beta$ )	A	..	2
		25	$\gamma$ Cephei . . . . .	3.5	..	K ?	..	2
		26	$\delta$ Draconis . . . . .	3.0	..	L ?	..	..
		27	$\epsilon$ Geminorum . . . . .	3.3	..	K	..	..
		28	$\beta$ Cygni . . . . .	3.0	..	Q ?	..	2
		29	$\gamma$ Draconis . . . . .	2.4	C ( $\beta$ )	K	..	2
Type III.	Division V.	30	$\mu$ Geminorum . . . . .	3.4	C ( $\alpha$ )	M	..	..
		31	$\alpha$ Herculis . . . . .	{ 3.0 3.9 } Var.	C ( $\alpha$ )	M	IIIa !!!	3
		Compn.	$\circ$ Ceti (Miræ) $\alpha$ Orionis (Betelgeuse)					

## PHOTOGRAPHIC Stellar Spectra. Stars to Magnitude 3.5.

## Lower Galactic Zone, N.

## C (Plates 8 to 11).

Secchi's Types.	Present classification.	No. of photograph.	List of Stars.	Magnitude. Uranometria Oxoniensis.	LOCKYER'S "Tables," Nov. 1892, 'Phil. Trans., A, 1893.	PICKERING, 1890. 'DRAPER Catalogue.'	VOGEL, 1883, 'Catalogue of Stellar Spectra Zone - 1° to + 20°.'	SECCHI, 'Stellar Types,' 1860-1868.
Type I.	Division I.	1	$\gamma$ Cassiopeiae . . .	2.2	D	Q	..	..
		2	$\epsilon$ Orionis . . .	$\left\{ \begin{array}{l} 2.2 \\ 2.7 \end{array} \right\}$ Var.	A ( $\alpha$ )	B	Ib!	1*
		3	" . . .	1.8	A ( $\alpha$ )	A	Ib?	1*
		4	" . . .	2.4	A ( $\alpha$ )	A	..	1*
		5	$\zeta$ " . . .	1.8	A ( $\alpha$ )	A	..	1*
		6	" . . .	3.2	A ( $\alpha$ )	B	..	..
		7	$\epsilon$ Persei . . .	3.1	A ( $\alpha$ )	A	..	2
		8	$\zeta$ " . . .	3.1	A ( $\alpha$ )	A?	..	2
		9	$\gamma$ Orionis (Bellatrix)	1.8	A ( $\alpha$ )	B	Ia	1*
		10	$\zeta$ Tauri . . .	3.0	A ( $\alpha$ )	A	..	..
		11	$\lambda$ " . . .	$\left\{ \begin{array}{l} 3.4 \\ 4.2 \end{array} \right\}$ Var.	..	B	Ia!	..
		12	$\beta$ " . . .	1.8	A ( $\alpha$ )	A	..	1
		13	$\beta$ Persei (Algol) . .	$\left\{ \begin{array}{l} 2.3 \\ 3.5 \end{array} \right\}$ Var.	A ( $\gamma$ )	A	..	1
		14	$\delta$ " . . .	3.1	..	A	..	1
		15	$\theta$ Aquilæ . . .	3.3	..	A	Ia!	..
		16	$\eta$ Tauri (Alcyone)	3.1	A ( $\gamma$ )	A	..	1
		17	$\beta$ Orionis (Rigel) .	(+1.0)	A ( $\alpha$ )	F	..	1*
	Division II.	..	None					
	Division III.	18	$\epsilon$ Aquilæ . . .	3.4	..	A	Ia!	1
		19	$\beta$ Eridani . . .	2.8	..	A	..	..
		20	$\beta$ Trianguli . . .	3.1	..	A	..	1
		21	$\alpha$ Aquilæ (Altair) .	1.0	A ( $\beta$ )	A	Ia!	1
		22	$\epsilon$ Cassiopeiae . . .	2.9	A ( $\beta$ )	A	..	1
		23	$\beta$ " . . .	2.3	A ( $\beta$ )	F	..	1
		24	$\beta$ Delphini . . .	3.5	■	F	Ia!	2
		25	$\alpha$ Persei . . .	1.9	A ( $\beta$ )	F	..	1
Type II.	Division IV.	Compu.	$\gamma$ " . . .	3.1	..	G	..	1
		26	$\gamma$ Persei . . .	3.1	..	G	..	1
		27	$\beta$ Aquilæ . . .	3.7	..	I?	IIa	2
		28	$\eta$ Pegasi . . .	2.9	B ( $\beta$ )	G	..	2
		29	$\epsilon$ Cygni . . .	2.4	C ( $\beta$ )	K	..	2
		30	$\zeta$ " . . .	3.1	B ( $\beta$ )	K?	..	2
		31	$\alpha$ Cassiopeiae . . .	$\left\{ \begin{array}{l} 2.2 \\ 2.8 \end{array} \right\}$ Var.	B ( $\beta$ )	K	..	2
		32	$\gamma$ Andromedæ . . .	2.1	C ( $\beta$ )	K	..	2
		33	$\gamma$ Aquilæ . . .	2.8	B ( $\beta$ )	K	IIa!!	2
		34	$\epsilon$ Aurigæ . . .	2.9	..	K	..	..
Type III.	Division V.	35	$\beta$ Andromedæ . . .	2.2	C ( $\alpha$ )	K?	..	2
		36	$\alpha$ Tauri (Aldebaran)	1.1	C ( $\beta$ )	K?	IIa!!!	2
		37	$\beta$ Pegasi . . .	2.5	C ( $\alpha$ )	M?	..	3
		38	$\alpha$ Orionis . . .	$\left\{ \begin{array}{l} 1.0 \\ 1.4 \end{array} \right\}$ Var.	C ( $\alpha$ )	M?	IIIa!!!	3



## PHOTOGRAPHIC Stellar Spectra. Stars to Magnitude 3.5.

## Lower Galactic Polar Region, N.

## D (Plates 12 to 14).

SECCHI'S Types.	Present classification.	No. of photograph.	List of Stars.	Magnitude, Uranometria Oxoniensis.	LOCKYER'S "Tables," Nov. 1892, 'Phil. Trans.,' A, 1893.	PICKERING, 1890, 'DRAPER Catalogue.'	VOGEL, 1883, 'Catalogue of Stellar Spectra Zone 1° to + 20°.'	SECCHI, 'Stellar Types,' 1866-1868.	
Type I.	Division I.	1	$\gamma$ Pegasi (Algenib) .	2.5	A ( $\alpha$ )	A	Ia	0	
		2	$\zeta$ " . . . . .	3.3	*	A	Ia !	1	
		3	$\alpha$ Andromedæ . . .	2.0	A ( $\gamma$ )	A	..	1	
	Division II.	4	$\alpha$ Pegasi (Markab) .	2.3	A ( $\gamma$ )	A	Ia !!	1	
		5	$\theta$ " " . . . . .	3.5	..	A	Ia !!	..	
		6	$\delta$ Aquarii . . . . .	3.0	..	A	..	..	
	Division III.	7	$\gamma$ Ceti . . . . .	3.4	*	A	Ia !	1	
		8	$\alpha$ Piscium . . . . .	3.7	..	A	..	1	
		9	$\beta$ Arietis . . . . .	2.7	A ( $\beta$ )	A	..	1	
		10	$\delta$ Capricorni . . . .	3.0	A ( $\gamma$ )	A	..	1	
Type II.	Division IV.	11	$\beta$ Aquarii . . . . .	3.1	..	K ?	..	2	
		12	$\alpha$ " . . . . .	3.0	..	K ?	IIa	2	
		13	$\beta$ Ceti . . . . .	2.4	B ( $\beta$ )	K	..	2	
		14	$\tau$ " . . . . .	3.1	..	G ?	..	2	
		15	$\theta$ " . . . . .	3.4	..	I	..	2	
		16	$\delta$ Andromedæ . . . .	3.2	..	K	..	2	
		17	$\epsilon$ Ceti . . . . .	3.7	..	K	..	2	
		18	$\zeta$ " . . . . .	3.5	..	K ?	..	2	
		19	$\delta$ Eridani . . . . .	3.0	..	I	..	2	
		20	$\epsilon$ " . . . . .	3.4	..	K	..	2	
		21	$\eta$ Ceti . . . . .	3.5	C ( $\beta$ )	K ?	..	2	
		22	$\alpha$ Arietis . . . . .	2.1	B ( $\beta$ )	K	..	2	
		23	$\epsilon$ Pegasi . . . . .	2.4	C ( $\beta$ )	K	IIa !!!	2	
Type III.	Division V.	24	$\gamma$ Eridani . . . . .	3.0	..	M ?	..	2	
		25	$\alpha$ Ceti . . . . .	2.2	C ( $\alpha$ )	M	IIIa !!!	3	
		Compn.	$\alpha$ Orionis . . . . .	$\left\{ \begin{array}{l} 1.0 \\ 1.4 \end{array} \right\}$	Var.	C ( $\alpha$ )	M ?	IIIa !!!	3
		"	$\alpha$ Herculis . . . . .	$\left\{ \begin{array}{l} 3.0 \\ 3.9 \end{array} \right\}$	Var.	C ( $\alpha$ )	M	IIIa !!!	3
		26	$\circ$ Ceti (Mira) . . . .	$\left\{ \begin{array}{l} 3.3 \\ 8.7 \end{array} \right\}$	Var.	..	M	..	3
		Compn.	$\alpha$ Aurigæ (Capella)						

## PHOTOGRAPHIC Stellar Spectrum. Stars to Magnitude 3.5.

## Upper Galactic Polar Region, S.

## AA (Plates 15 to 17).

SECCHI'S Types.	Present classification.	No. of photograph.	List of Stars.	Magnitude. Uranometria Oxoniensis.	LOCKYER'S "Tables," Nov., 1892, 'Phil. Trans., A, 1893.	PICKERING, 1890, 'DRAPER Catalogue.'	VOGEL, 1883, 'Catalogue of Stellar Spectra Zone - 1° to + 20°.'	SECCHI, 'Stellar Types,' 1866-68.
Type I.	Division I.	1	$\alpha$ Virginis (Spica) . . . . .	1.0	A ( $\alpha$ )	A	..	1
		2	$\alpha$ Leonis (Regulus)) . . . . .	1.2	A ( $\gamma$ )	A	Ia !!!	1
		3	$\delta$ Corvi . . . . .	3.1	*	A	..	1
		4	$\beta$ Libræ . . . . .	2.7	*	A	..	1
		5	$\gamma$ Corvi . . . . .	2.8	..	A	..	1
		6	$\eta$ Leonis . . . . .	3.5	A ( $\gamma$ )	A	Ia	1
	Division II.	7	$\mu$ Serpentis . . . . .	3.3	..	A	..	1
		8	$\eta$ Virginis . . . . .	3.8	..	A	Ia !	..
		9	$\theta$ Leonis . . . . .	3.4	..	A	Ia !!!	1
	Division III.	10	$\zeta$ Virginis . . . . .	3.4	*	A	Ia !!	..
		11	$\beta$ Serpentis . . . . .	3.5	..	A	Ia !	..
		12	$\alpha$ Libræ . . . . .	3.0	*	A	..	1
		13	$\epsilon$ Serpentis . . . . .	3.6	..	A	Ia	1
		14	$\beta$ Leonis . . . . .	2.1	*	A	Ia !!!	1
		15	$\delta$ " . . . . .	2.5	A ( $\beta$ )	A	..	1
		16	$\phi$ Geminorum . . . . .	3.6	..	A	..	1
		17	$\delta$ Serpentis . . . . .	3.8	..	A	I ! ?	2
		18	$\gamma$ Virginis . . . . .	2.7	A ( $\beta$ )	F	Ia !	1
	Compn.	$\eta$ Bootis . . . . .	2.7	B ( $\beta$ )	G	Ia !	2	
	Type II.	Division IV.	19	$\eta$ Bootis . . . . .	2.7	B ( $\beta$ )	G	Ia !
20			$\beta$ Virginis . . . . .	3.6	..	G	Ia	2
21			$\zeta$ Hydræ . . . . .	3.4	B ( $\beta$ )	K	IIa	2
22			$\epsilon$ Virginis . . . . .	3.0	B ( $\beta$ )	K	IIa	2
23			$\epsilon$ Ophiuchi . . . . .	3.3	B ( $\beta$ )	K ?	..	2
24			$\gamma$ Leonis . . . . .	2.1	B ( $\beta$ )	K	..	2
25			$\alpha$ Bootis (Arcturus) . . . . .	(+0.7)	B ( $\beta$ )	K	IIa !!	2
26			$\alpha$ Serpentis . . . . .	2.7	C ( $\beta$ )	K ?	IIa !!!	2
27			$\alpha$ Hydræ . . . . .	2.2	C ( $\beta$ )	K	..	2
Type III.	Division V.	28	$\delta$ Ophiuchi . . . . .	2.6	C ( $\alpha$ )	M	..	..
		29	$\delta$ Virginis . . . . .	3.5	C ( $\alpha$ )	M	IIIa !!!	3
		Compn.	$\alpha$ Orionis . . . . .					





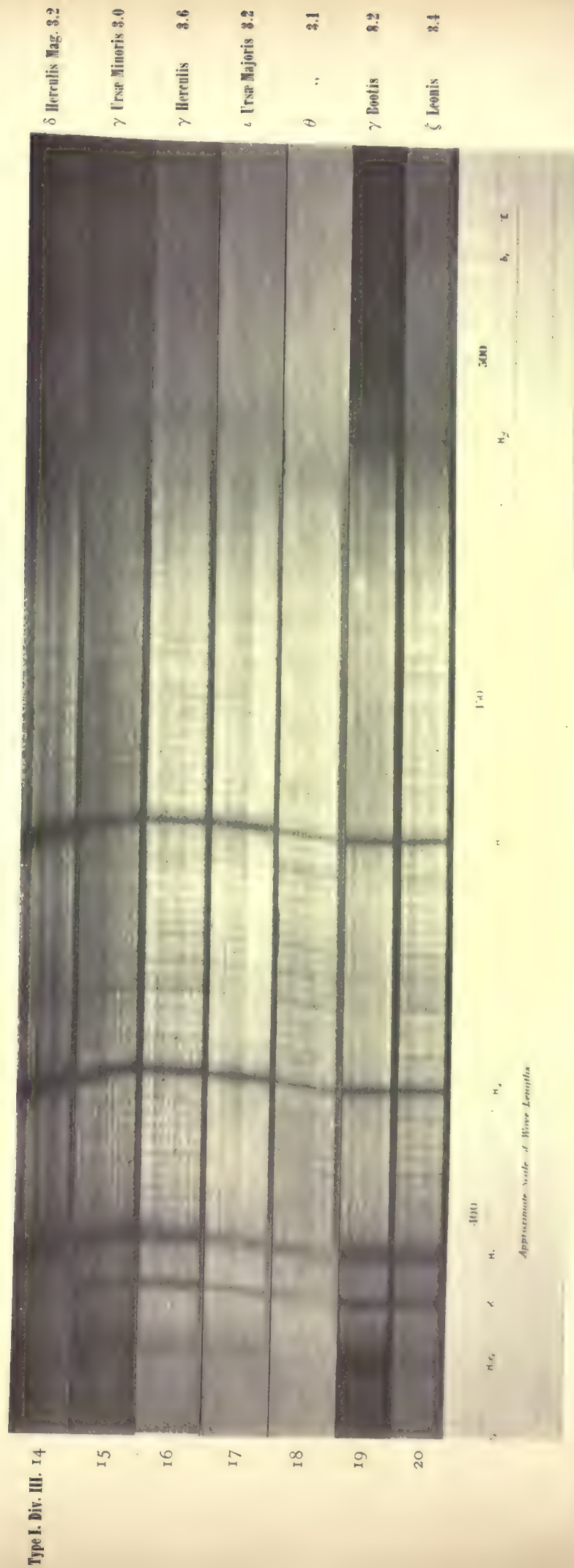


# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.

UPPER GALACTIC POLAR REGION, N.

A



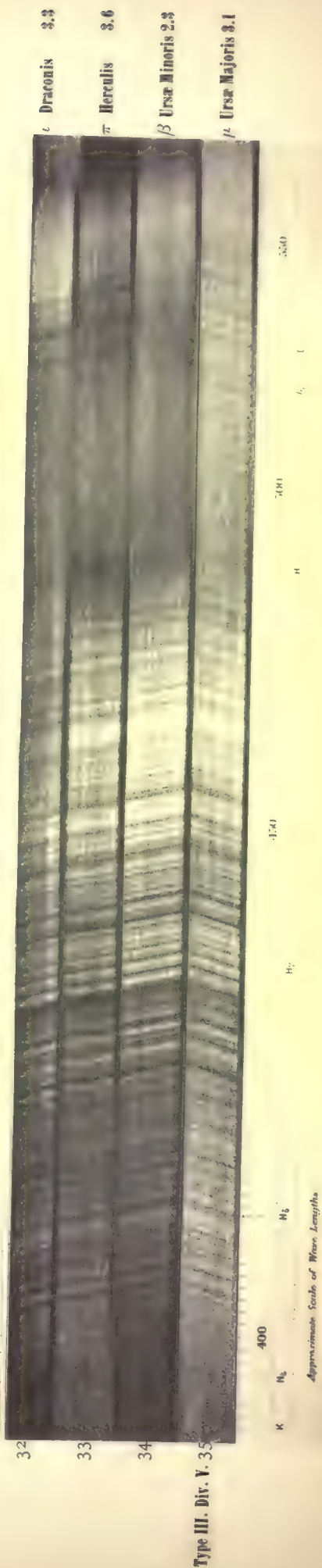
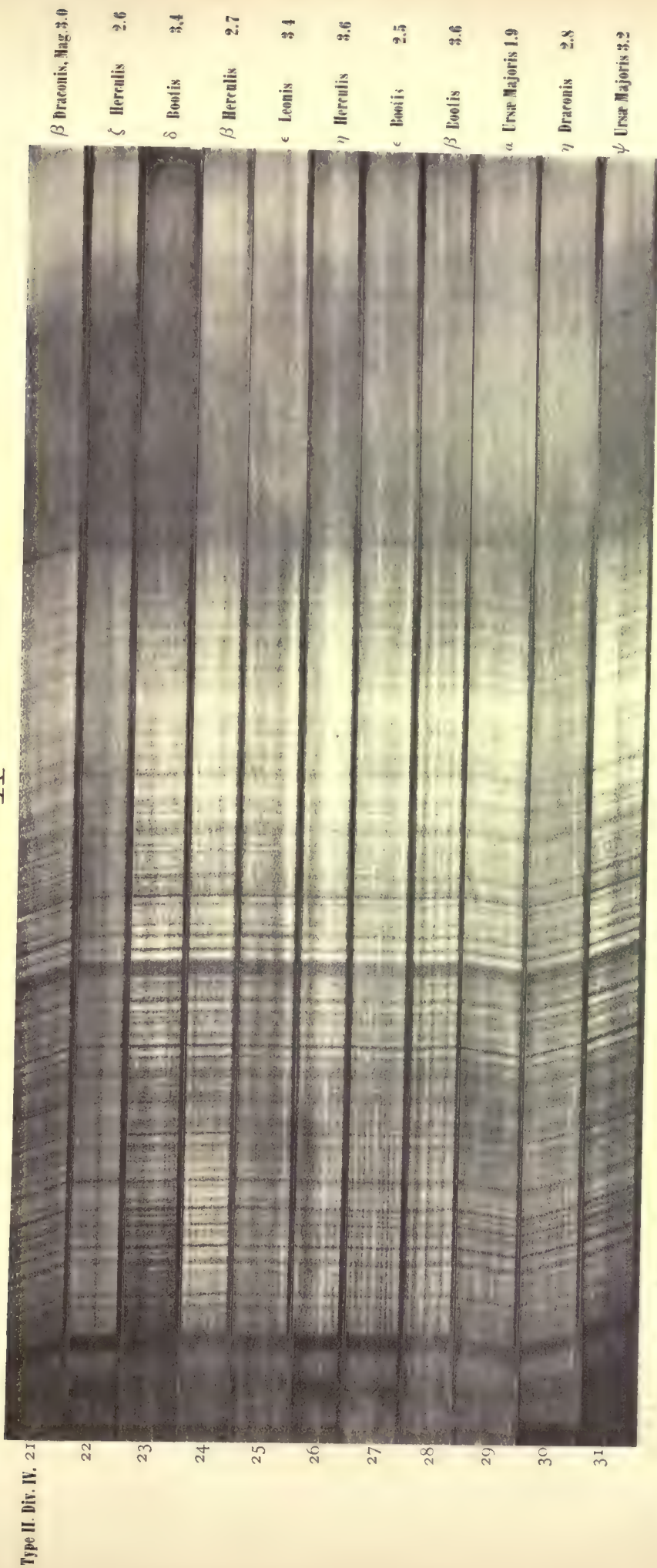




# PHOTOGRAPHIC STELLAR SPECTRA, Stars to Magnitude 3.5. UPPER GALACTIC POLAR REGION, N.

McClean.

Phil. Trans. A. 191, Plate 3.







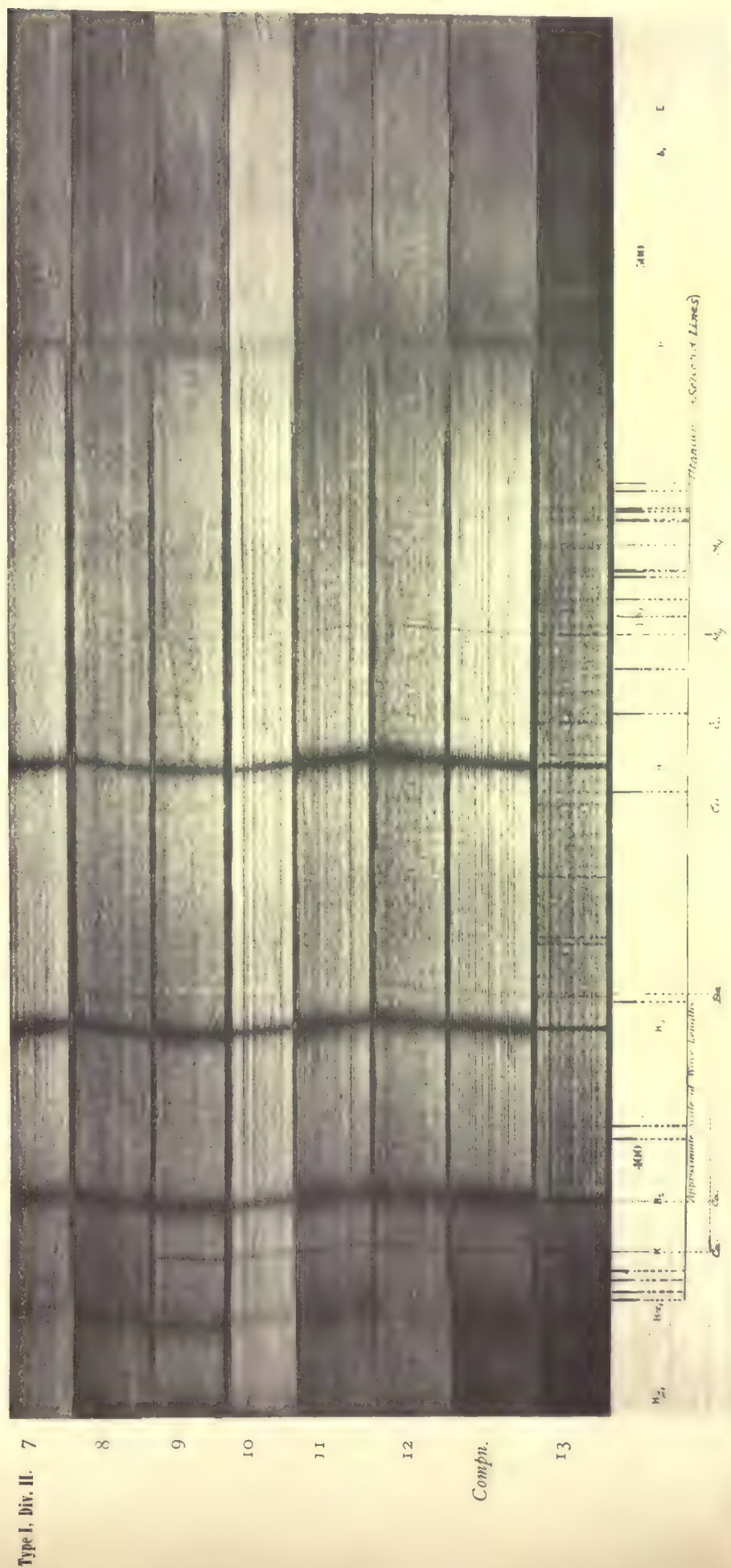




Stars to Magnitude 35.

UPPER GALACTIC ZONE, N.

М





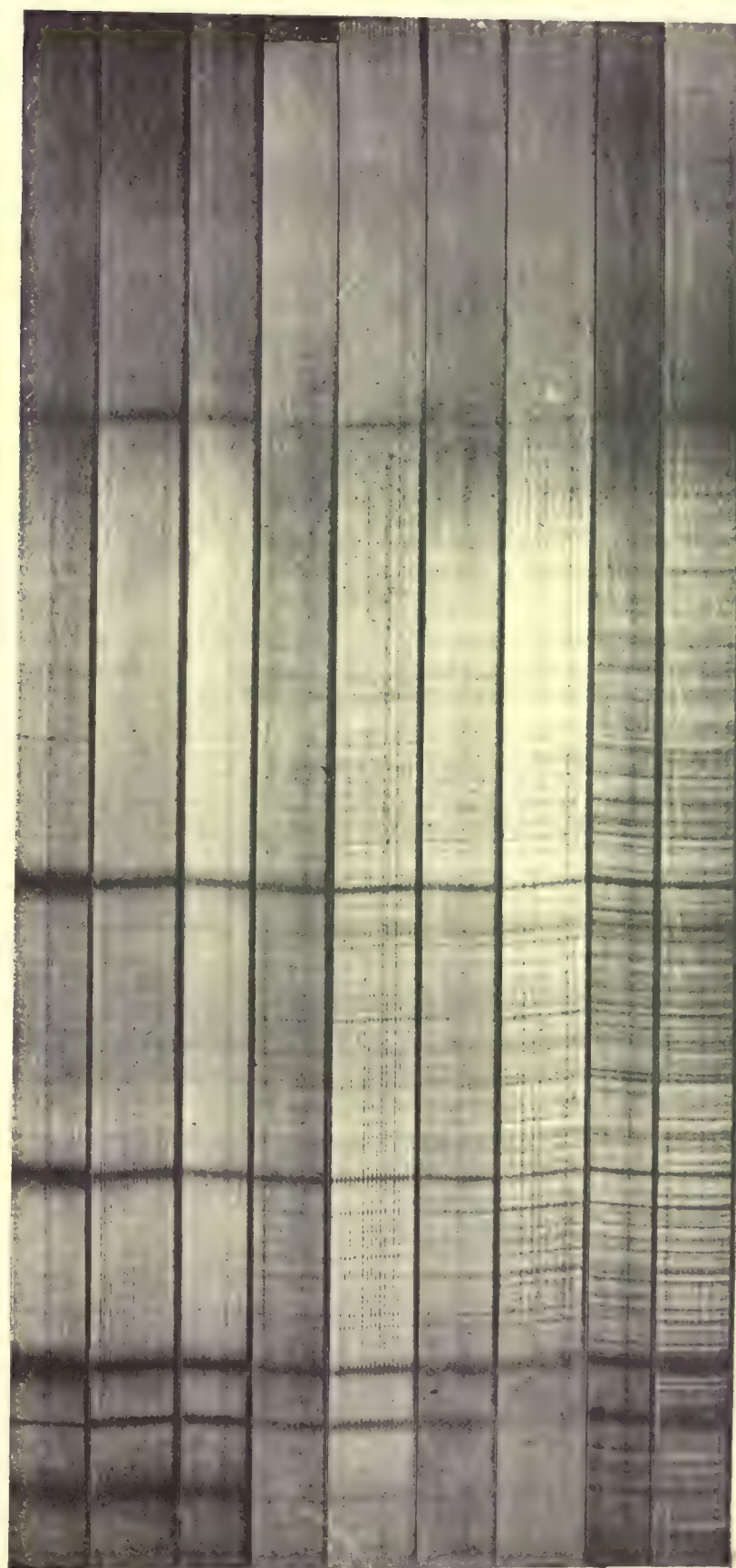


# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.

UPPER GALACTIC ZONE, N.

B



Mag.  
α Geminorum 1.5  
*Castor*

α Ophiuchi 2.2

α Cephei 2.6

δ Geminorum 3.5

α Canis Minoris  
*Procyon (+0.5)*

δ Cephei 3.7 Var.  
4.9

α Urse Minoris 2.0  
*Polaris*

ε Aurigae 3.0 Var.  
4.5

γ Cygni 2.3

680

500

1.40

100

H<sub>γ</sub> K H<sub>β</sub> H<sub>γ</sub>  
Approximate Scale of Wave Lengths

H<sub>2</sub>

H<sub>2</sub>

1.40

H<sub>2</sub>

H<sub>2</sub>

H<sub>2</sub>

H<sub>2</sub>

H<sub>2</sub>

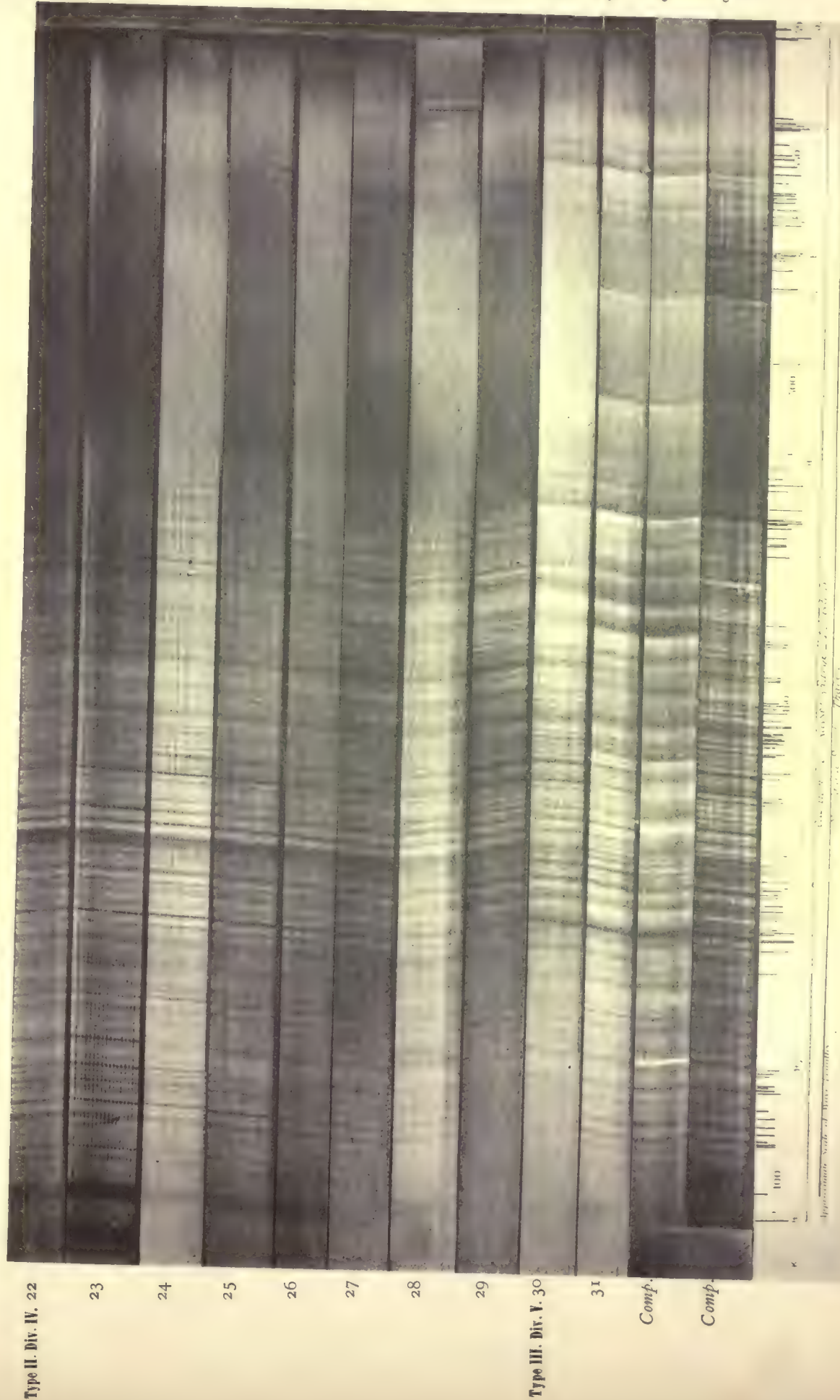


# PHOTOGRAPHIC STELLAR SPECTRA,

Stars to Magnitude 3.5.

UPPER GALACTIC ZONE, N.

B



Type II. Div. IV. 22

23

24

25

26

27

28

29

Type III. Div. V. 30

31

Comp.

Comp.

μ Herculis Mag. 3.5

α Aurigae (+0.9)  
Capella.

β Geminorum 1.4  
Pollux

γ Cephei 3.5

δ Draconis 3.0

ε Geminorum 3.3

β Cygni 3.0

γ Draconis 2.4

μ Geminorum 3.4

α Herculis 3.0 var.  
3.9

o Cell 8.7 var.  
Mira

α Orionis 1.0 var.  
1.4 Betelgeuse



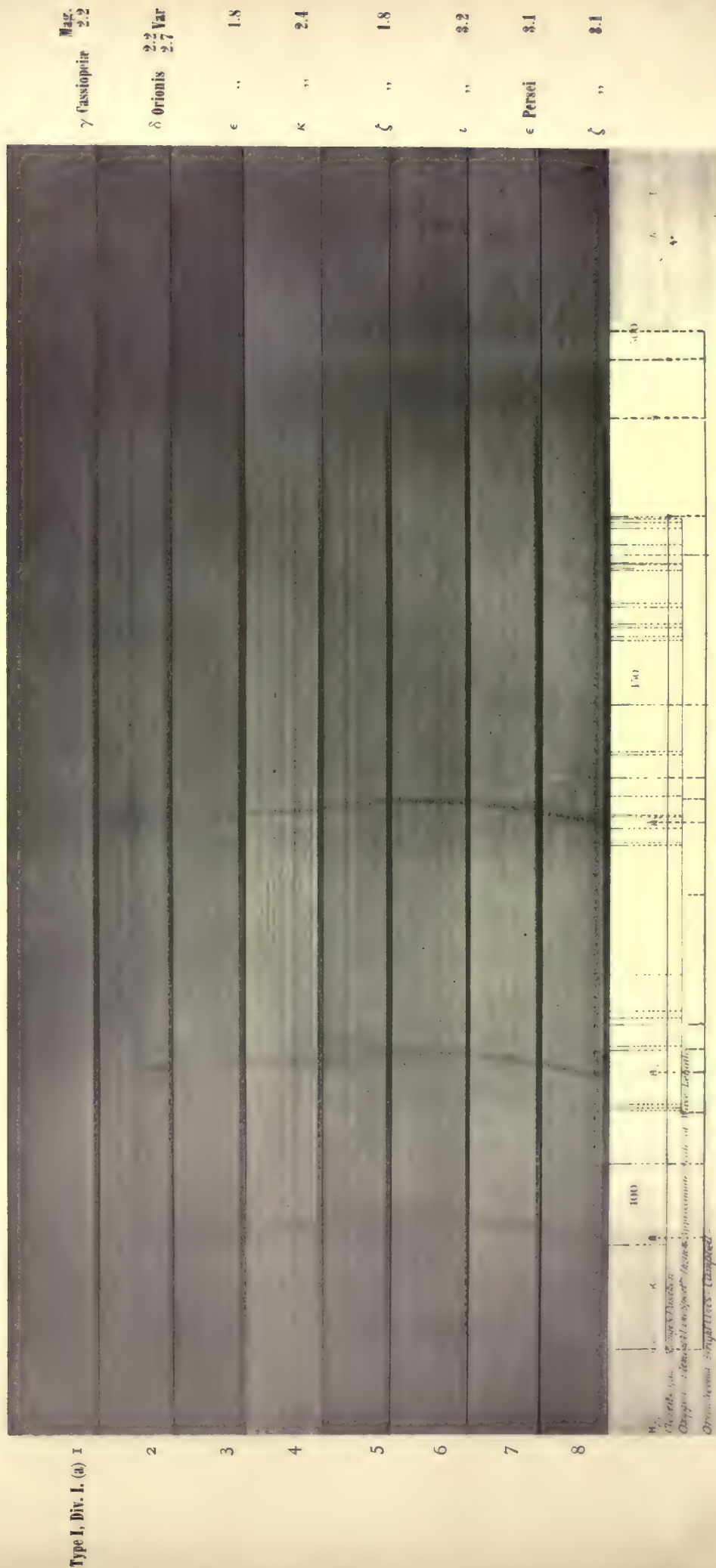


## PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.

## LOWER GALACTIC ZONE, N.

U

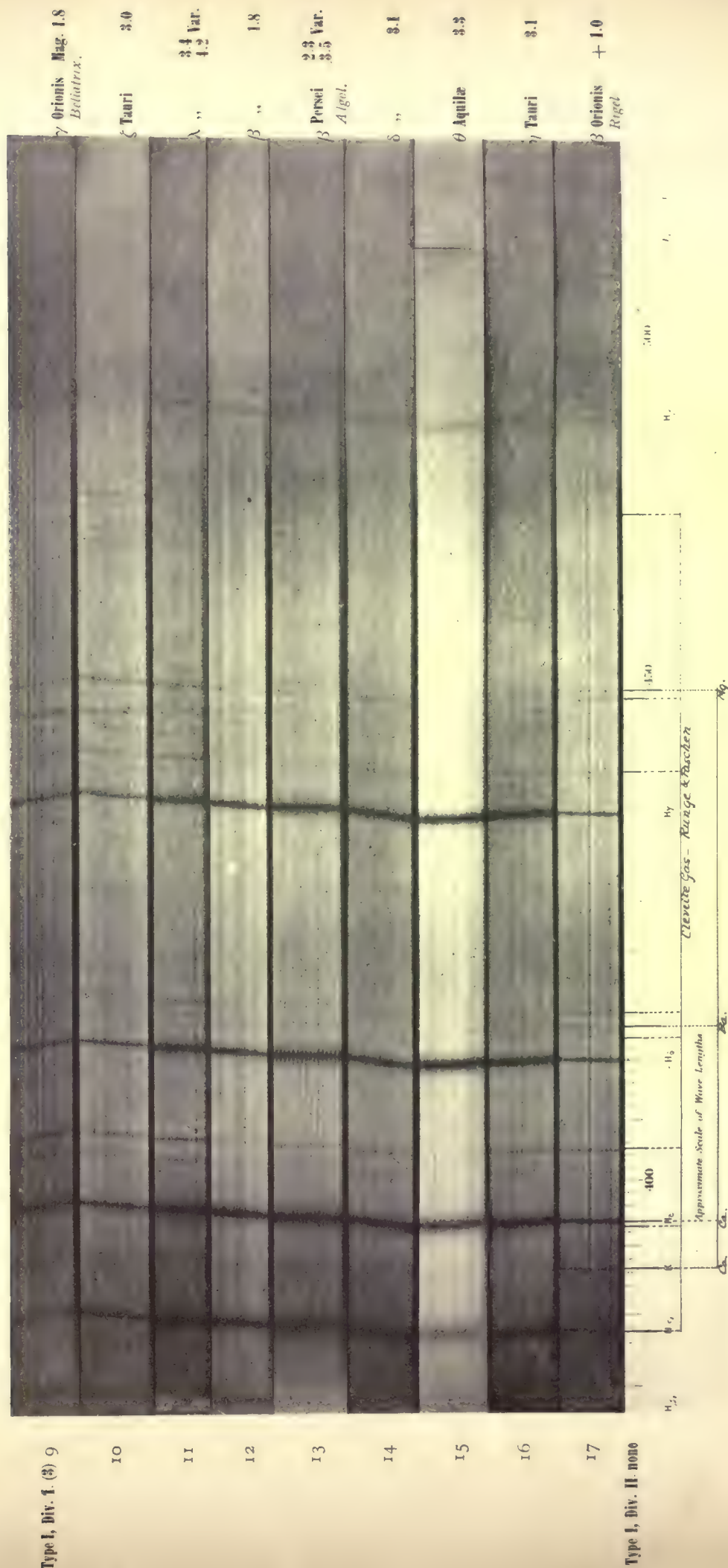






## Stars to Magnitude 35

LOWER GALACTIC ZONE N.

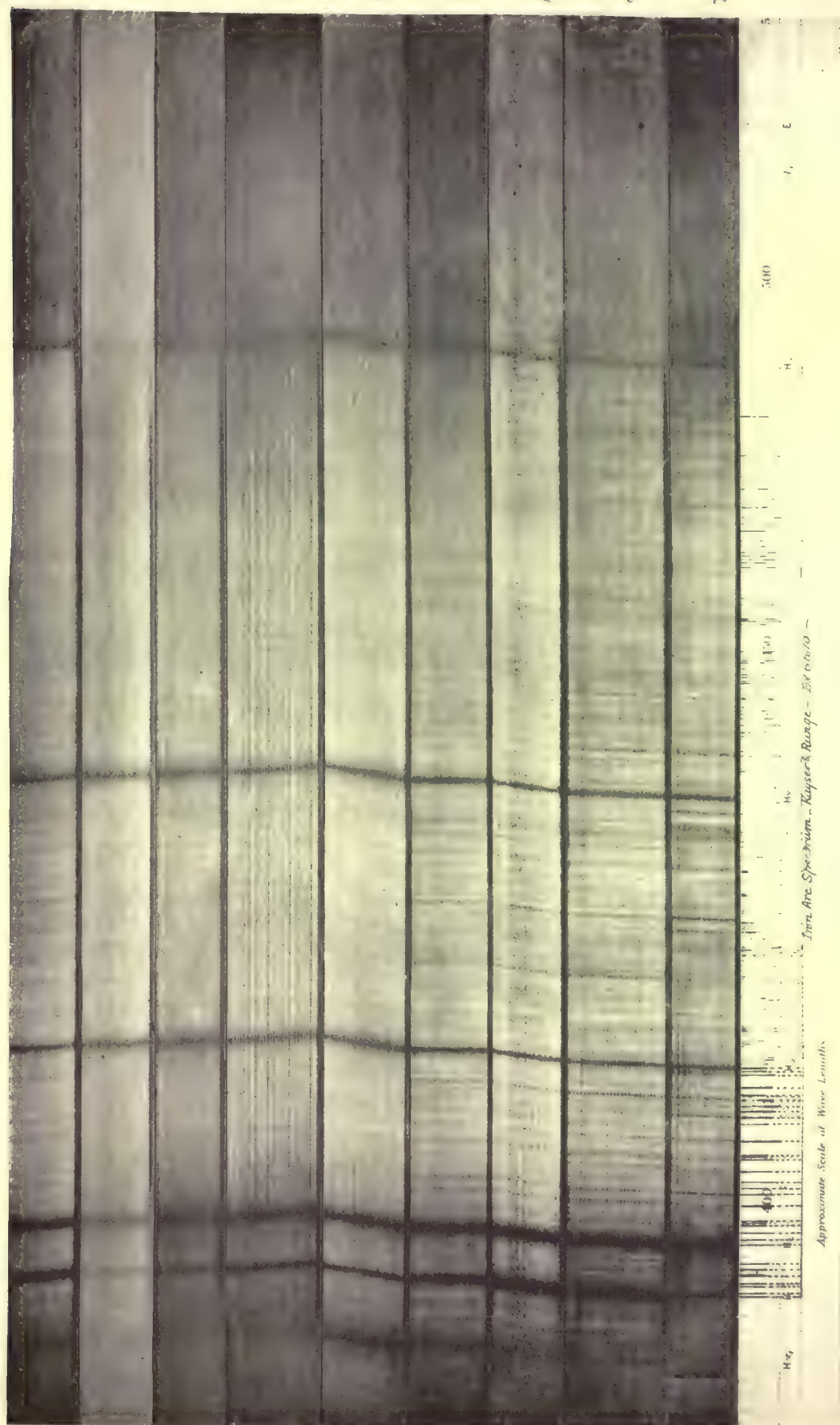




# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.  
LOWER GALACTIC ZONE, N.

C



Type I. Div. III. 18

19

20

21

22

23

24

25

Comp.

$\delta$  Aquila Mag. 3.4

$\beta$  Eridani 2.8

$\beta$  Trianguli 3.1

$\alpha$  Aquile  
*Altair* 1.0

$\delta$  Cassiopeie 2.8

$\beta$  " 2.3

$\beta$  Delphini 3.5

$\alpha$  Persi 1.9

$\gamma$  " 3.1





# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.

UPPER GALACTIC ZONE, N.

C



Type II, Div. IV. 26

27

28

29

30

31

32

33

34

Type III, Div. V. 35

36

37

38

$\gamma$  Persei Mag 3.1

$\beta$  Aquila 3.7

$\eta$  Pegasi 2.9

$\epsilon$  Cygni 2.4

$\zeta$  .. 3.1

$\alpha$  Cassiopeia<sup>2.2</sup><sub>5</sub> Var

$\gamma$  Andromeda 2.1

$\gamma$  Aquila 2.8

$\epsilon$  Auriga 2.9

$\beta$  Andromeda 2.2

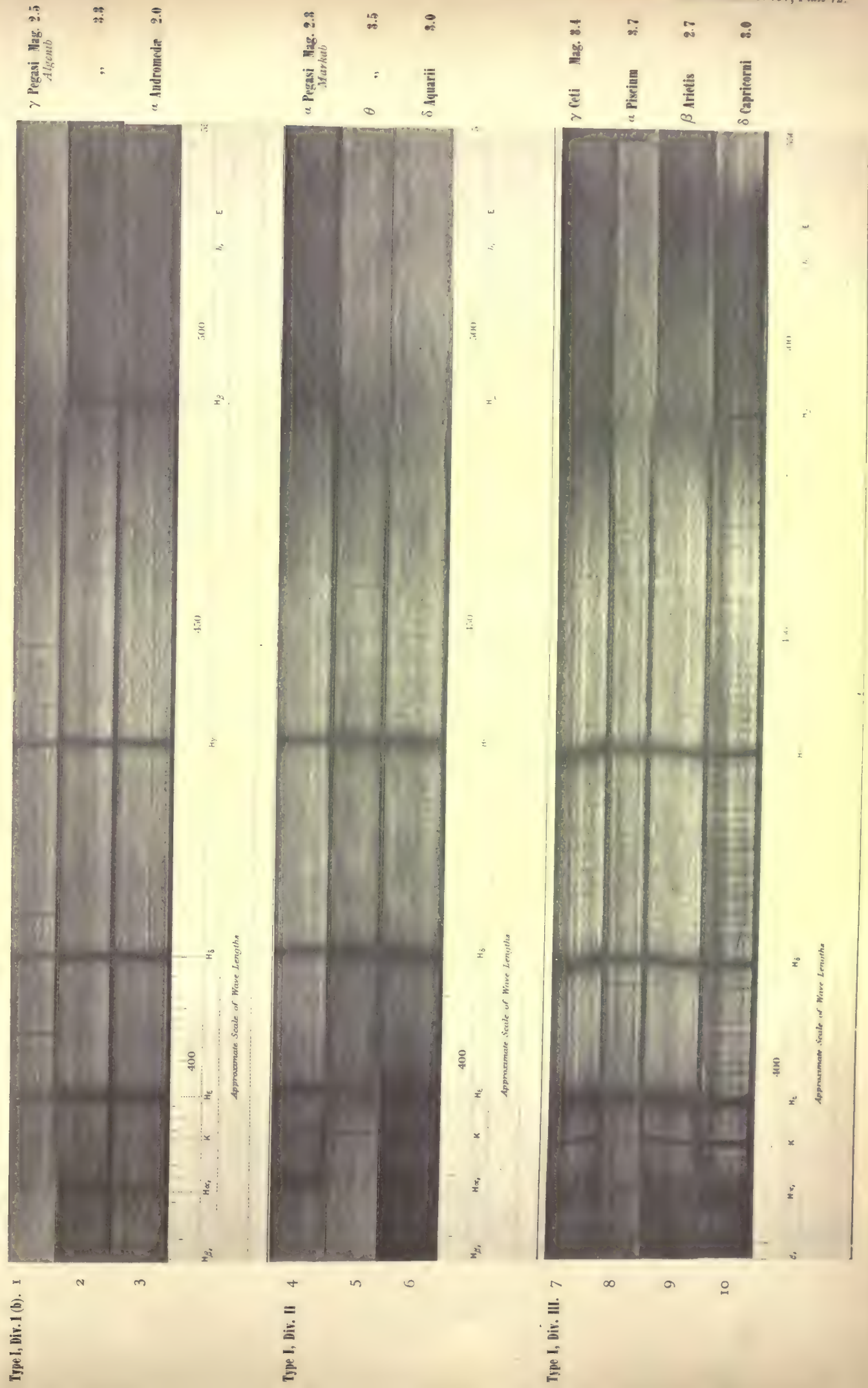
$\alpha$  Tauri 1.1  
*Aldebaran.*

$\beta$  Pegasi 2.5

$\alpha$  Orionis 1.0 Var.  
1.4



PHOTOGRAPHIC STELLAR SPECTRA.  
Stars to Magnitude 3.5  
LOWER GALACTIC POLAR REGION, N.  
D

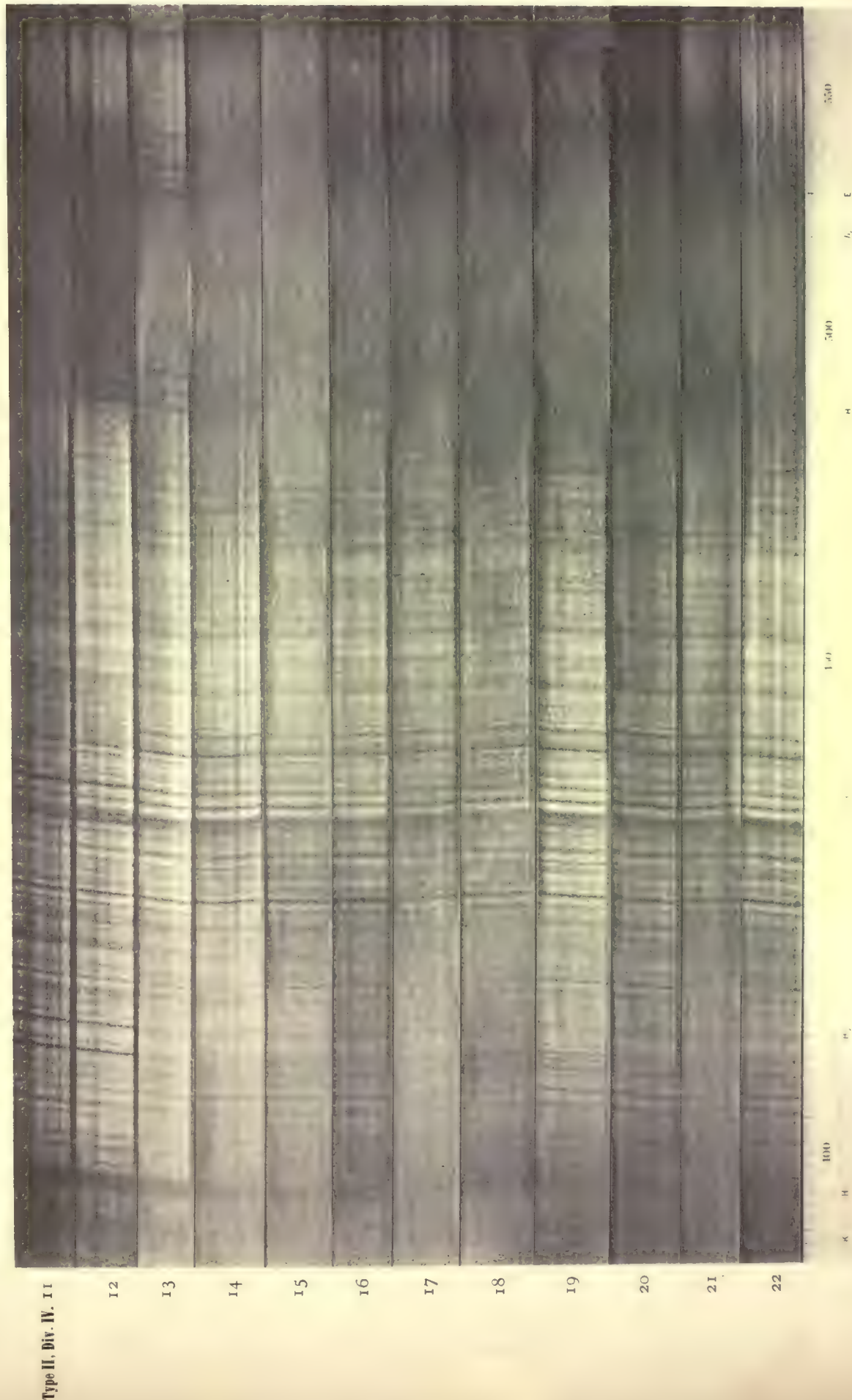






# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.  
LOWER GALACTIC POLAR REGION, N.  
D





# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.

LOWER GALACTIC POLAR REGION, N.

D

Type II. Div. IV. 23  
*contd.*

Type III Div. V. 24

25

Comp.

Comp.

26

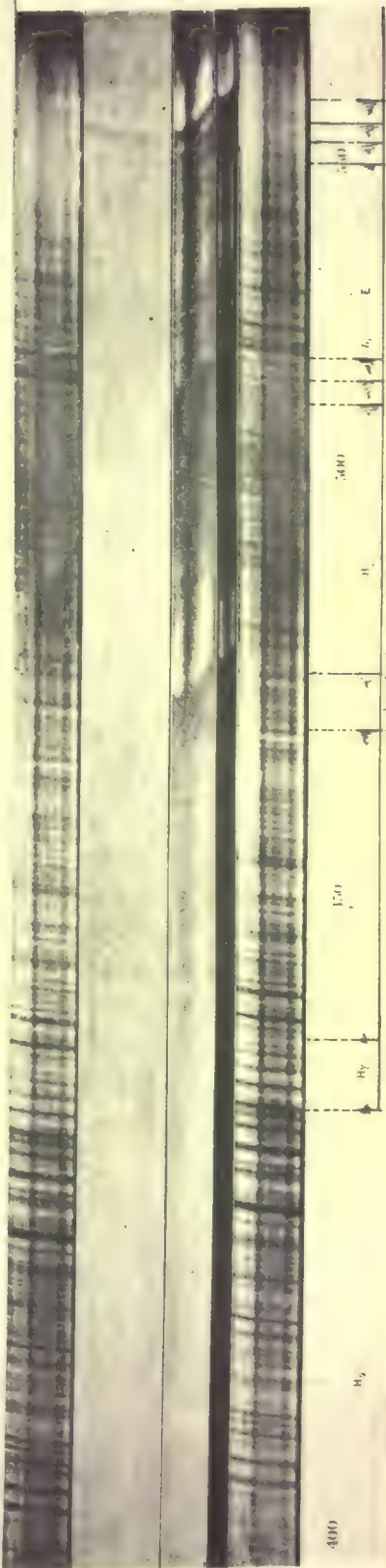
Comp.



Comp.

Type IV. iv. VI.

Comp.



Approximate Scale of Wave Lengths

Hydro-Gen. Ion. Gas Spectrum



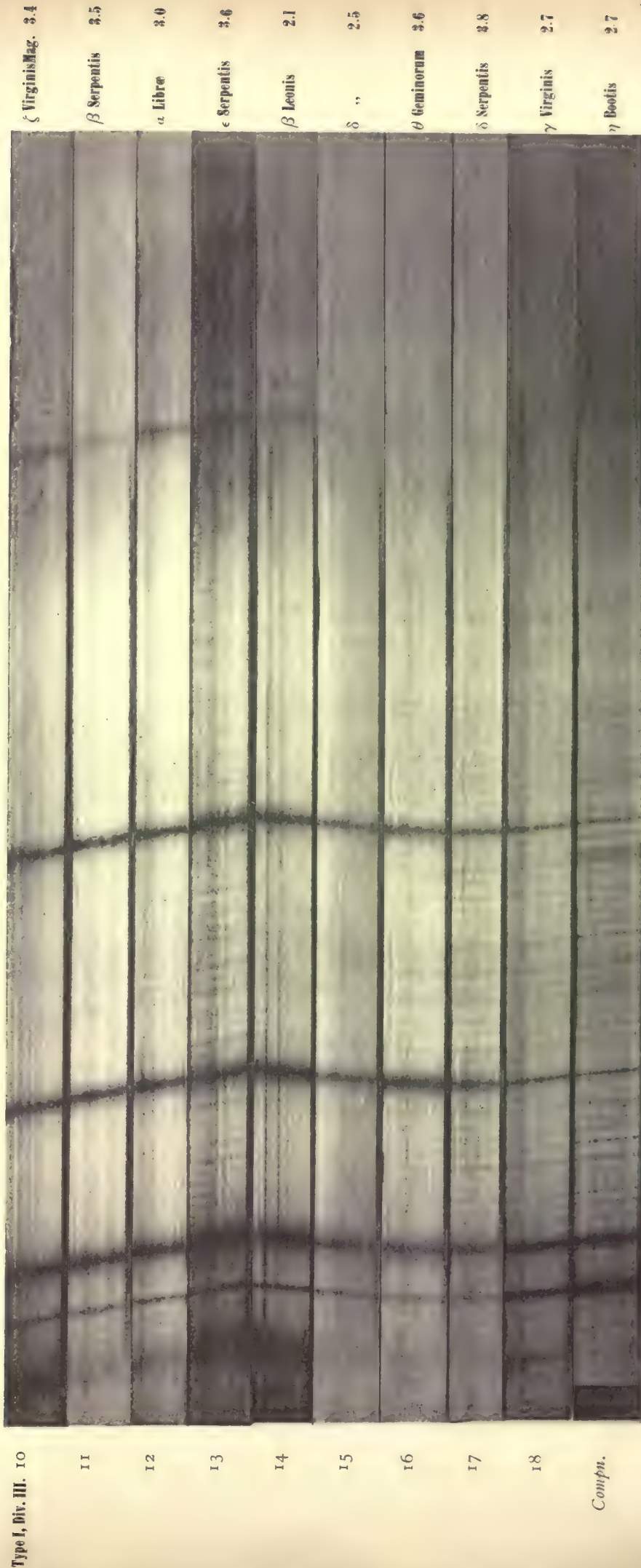






# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.  
UPPER GALACTIC POLAR REGION, S.  
A.A



100

100

100

Approximate scale of Wave Lengths.

Type I, Div. III. 10

11

12

13

14

15

16

17

18

Compt.

ζ Virginis Mag. 3.4

β Serpentis 3.5

α Libræ 3.0

ε Serpentis 3.6

β Leonis 2.1

δ " 2.5

θ Geminorum 3.6

δ Serpentis 3.8

γ Virginis 2.7

η Bootis 2.7



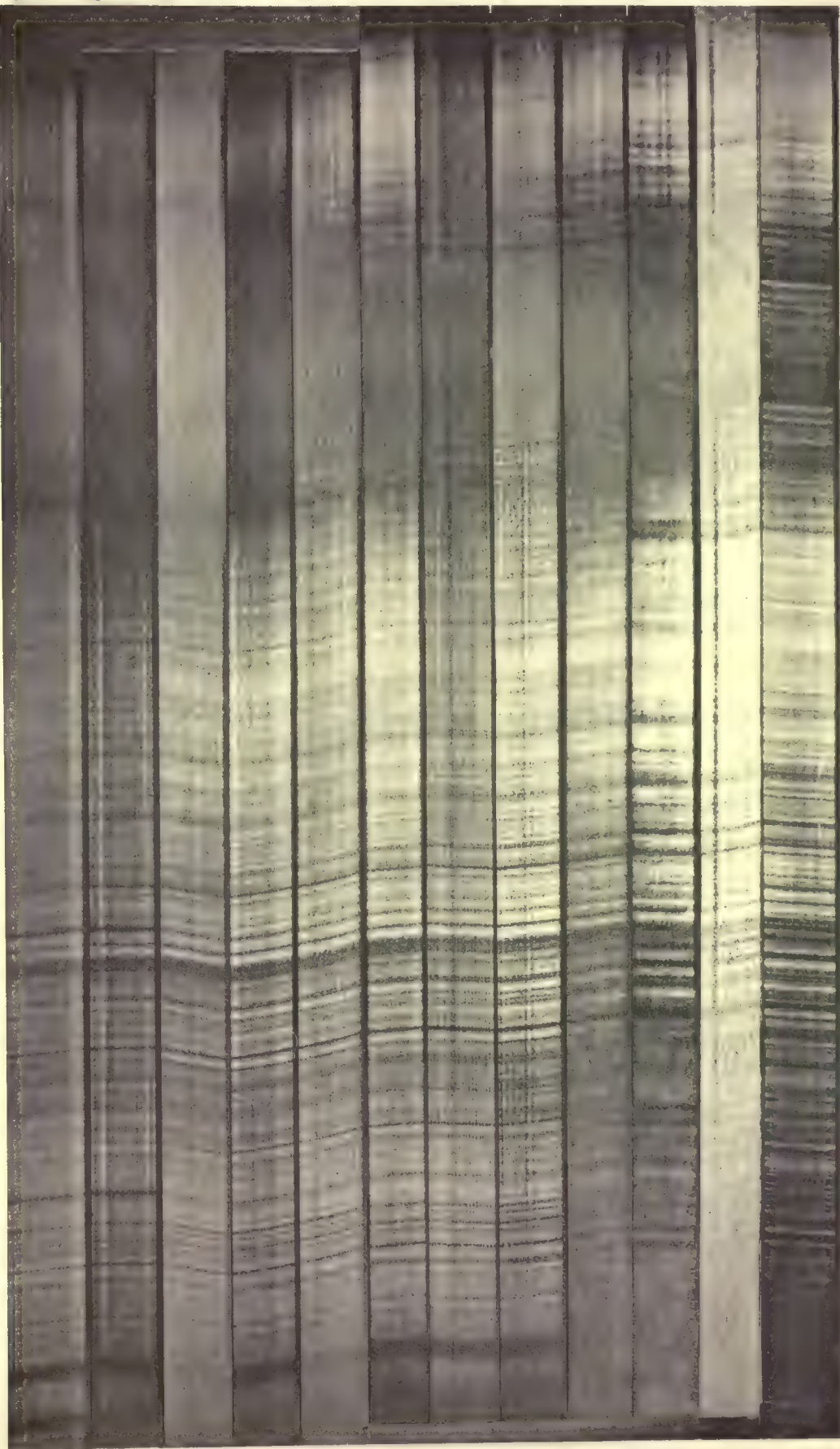


# PHOTOGRAPHIC STELLAR SPECTRA.

Stars to Magnitude 3.5.  
UPPER GALACTIC POLAR REGION, S.

AA

$\eta$ Bootis	Magn. 2.7
$\beta$ Virginis	3.6
$\zeta$ Hydrae	3.4
$\epsilon$ Virginis	3.0
$\epsilon$ Ophiuchi	3.3
$\gamma$ Leonis	2.1
$\alpha$ Bootis ( $+0.7$ ) <i>Arcturus</i>	
$\alpha$ Serpentis	2.7
$\alpha$ Hydrae	2.2
$\delta$ Ophiuchi	2.6
$\delta$ Virginis	3.5
$\alpha$ Orionis	1.0 Var 1.4



300 350 400 450 500

H<sub>2</sub>

H<sub>γ</sub>

H<sub>β</sub>

Approximate Scale of Wave Lengths

Type II, Div. IV. 19

20

21

22

23

24

25

26

27

Type III, Div. V. 28

29

Compn.



V. *On the Application of Harmonic Analysis to the Dynamical Theory of the Tides.*—  
Part II. *On the General Integration of LAPLACE'S Dynamical Equations.*

By S. S. HOUGH, M.A., *Fellow of St. John's College and Isaac Newton Student in the University of Cambridge.*

*Communicated by Professor G. H. DARWIN, F.R.S.*

Received October 27,—Read December 9, 1897.

IN the former paper on this subject I have dealt with the formation of LAPLACE'S dynamical equation for the tides, and the integration of it, subject to the limitation that the solutions obtained should be symmetrical with respect to the axis of rotation. In the present paper I propose to extend the method of solution so as to free it from this restriction.

The difficulties experienced by LAPLACE in his attempts to integrate the equation in question were so great that he abandoned all efforts to obtain a general solution, and confined his discussion to a few of the special cases which present the greatest interest from a practical point of view; even in these simple cases however his original attempts to express the solutions by means of the coefficients associated with his name were discarded in favour of series proceeding according to powers of a certain variable used to define the position of a point on the Earth's surface. These power-series have been further employed by Lord KELVIN\* to obtain a more general solution of the problem, but the results obtained, though of considerable analytical interest, do not lend themselves well to a numerical discussion. Both AIRY† and KELVIN condemn the employment of the surface-harmonic functions as inappropriate, but a profound conviction that the efforts of LAPLACE, though unsuccessful, were well directed, has led me to take up the problem again from his point of view; with what success will be seen hereafter.

I was originally led to attack the problem by a totally different method from that of LAPLACE based on the work of POINCARÉ‡ and BRYAN§, and the principal analytical results, both in this paper and in the preceding, were at first obtained by

\* "On the General Integration of LAPLACE'S Differential Equation of the Tides." 'Phil. Mag.,' 1875.

† 'Encyc. Metropolitana.' Art. "Tides and Waves," Section III., § 116.

‡ "Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation." 'Acta Math.,' vol. 7, p. 355, *et seq.*

§ "The Waves on a Rotating Liquid Spheroid of Finite Ellipticity." 'Phil. Trans.,' A, 1889.



a very lengthy analysis similar to that used by the latter writer. The comparative simplicity of these results seemed, however, to point to the fact that they might be more easily obtained by less pretentious means. The deduction of the formulæ in the former paper from the differential equation of LAPLACE presented no serious difficulties, but in attempting to apply a like method to obtain the more general formulæ of the present paper, I found that formidable obstacles had to be overcome. The method of integration now adopted seems to leave little to be desired for simplicity, considering its generality, but the fact that it has been built up partly by working forwards from the differential equation, and partly by working backwards from the results, must account for the apparent artificiality of the procedure.

In all previous attempts at the solution of the dynamical equations for the tides, the integration has been effected by assuming that the expression for the tide-height could be expressed by an infinite series of terms of known form associated with undetermined numerical coefficients. The differential equations then lead to a difference-relation between a certain number of these coefficients from which their numerical values are to be evaluated. The numerical determination of the coefficients will be facilitated when this difference-relation contains as few terms as possible. Now it is found in the present paper that, without imposing any restriction on the period of the disturbing force, if the form we assign to the terms of the series for the tide-height is that of the tesseral harmonics or LAPLACE'S functions, a linear relation involving three successive coefficients only may be deduced, provided that the law of depth is such that both the internal and external surfaces of the ocean are spheroids of revolution about the polar axis. This however appears to be the most general law of depth which can be employed without obtaining more than three successive coefficients in the linear relation in question, and consequently our discussion deals only with cases where the law of depth is subject to this limitation.

In § 1 I have collected the principal properties of the functions used in the analysis. These properties are for the most part well known, but in consideration of the want of agreement in the notation employed by different writers, I have thought it best to briefly prove such of them as are required in preference to giving references to places where they may be found. Moreover I have thus been enabled to write the results in the exact form required for subsequent application.

§§ 2-4 deal with the integration of the differential equations and the deduction of the linear equations (31), (40) connecting the coefficients in the expansion of the tide-height. These equations, the analogy of which with the equations (23), (23A) of Part I. will be at once apparent, constitute the chief analytical results of the paper, and the remainder is occupied with the application of these formulæ to the discussion of the free and forced vibrations on lines similar to those adopted in Part I.

§§ 5-11 treat of the free oscillations, the discussion being confined to the case where the depth is uniform. A period-equation is obtained, and an approximate



method of determining the higher roots is given. The approximations will not however be sufficiently close for the earlier roots, and consequently it is necessary to evaluate these earlier roots by trial and error. The method of procedure is indicated by numerical examples, and several of the more important roots are tabulated for four different depths of the ocean. The most interesting result is the existence of a second class of free oscillations besides those whose existence may be at once inferred by analogy from the simpler problem of the oscillations of an ocean covering a non-rotating globe. The characteristics of the oscillations of this class are discussed in § 11.

In § 12 a general analytical solution of the problem of the forced vibrations due to any disturbing force is given, but as the analytical expressions obtained are too intricate to afford much indication of the nature of the forced tides, the various types of oscillation which occur on the earth are afterwards treated numerically.

In certain cases, intimately associated with those actually occurring, the analytical expressions however admit of considerable reductions. These cases are discussed in § 14, where theorems due to LAPLACE and Professor DARWIN are obtained and generalized.

§§ 15-18 contain numerical examples of the evaluation of the semi-diurnal and diurnal tidal constituents. The arithmetic is considerably simplified when the period of the disturbing force is rigorously equal to half a sidereal day or a sidereal day, and consequently these cases are first dealt with and the results compared with those of LAPLACE. Additional examples are however also given to illustrate the effects of the departure of the periods from exact coincidence with half a sidereal day and a sidereal day respectively, the cases selected for investigation corresponding with the leading lunar constituents.

### § 1. *Properties of Tesseral Harmonics.*

Let  $P_n(\mu)$  denote the zonal harmonic of order  $n$ . Then  $P_n$  is the solution which remains finite when  $\mu = \pm 1$  of the differential equation

$$\frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{dP_n}{d\mu} \right\} + n(n+1) P_n = 0 \quad . \quad . \quad . \quad (1).$$

Let

$$P_n^s(\mu) = (1 - \mu^2)^{\frac{1}{2}s} \frac{d^s P_n}{d\mu^s} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then, on differentiating the equation (1)  $s$  times, we obtain

$$(1 - \mu^2) \frac{d^{s+2} P_n}{d\mu^{s+2}} - 2(s+1) \mu \frac{d^{s+1} P_n}{d\mu^{s+1}} + (n-s)(n+s+1) \frac{d^s P_n}{d\mu^s} = 0,$$

or,

$$(1 - \mu^2)^{\frac{1}{2}(s+2)} \frac{d^2}{d\mu^2} \{ (1 - \mu^2)^{-\frac{1}{2}s} P_n^s \} - 2(s+1) \mu (1 - \mu^2)^{\frac{1}{2}s} \frac{d}{d\mu} \{ (1 - \mu^2)^{-\frac{1}{2}s} P_n^s \} + (n-s)(n+s+1) P_n^s = 0,$$

which on reduction gives

$$\frac{d}{d\mu} \left\{ (1 - \mu^2) \frac{dP_n^s}{d\mu} \right\} + \left\{ n(n+1) - \frac{s^2}{1 - \mu^2} \right\} P_n^s = 0 \quad (3).$$

Thus  $P_n^s$  is a solution of the equation (3); the form (2) shows that it does not become infinite when  $\mu = \pm 1$ , while from (3) we see that the two functions  $P_n^s \cos s\phi$ ,  $P_n^s \sin s\phi$ , or what is equivalent, the two functions  $P_n^s e^{\pm is\phi}$ , are spherical surface-harmonics of order  $n$ . In our subsequent work the latter forms involving the imaginary exponential will be more convenient than the real trigonometrical forms. We shall therefore describe the functions  $P_n^s(\mu) e^{\pm is\phi}$  as the tesseral harmonics of order  $n$  and rank  $s$ . In some cases it may be convenient to apply the same nomenclature to the "associated function"  $P_n^s(\mu)$ , but whenever this is done, it must be understood that an exponential factor is implied, though not expressed.

The tesseral harmonics of course include as special cases the zonal harmonics obtained by putting  $s = 0$ , and the sectorial harmonics obtained by putting  $s = n$ , while, in accordance with the definition (2), for values of  $s$  greater than  $n$  we may suppose that  $P_n^s(\mu) = 0$ .

The principal properties of the tesseral harmonics which we shall require may be derived from those of the zonal harmonics. Thus, if we differentiate  $s$  times the well-known relation

$$(n+1)P_{n+1} - (2n+1)\mu P_n + nP_{n-1} = 0 \quad (4),$$

we obtain

$$(n+1) \frac{d^s P_{n+1}}{d\mu^s} - (2n+1)\mu \frac{d^s P_n}{d\mu^s} - (2n+1)s \frac{d^{s-1} P_n}{d\mu^{s-1}} + n \frac{d^s P_{n-1}}{d\mu^s} = 0,$$

which, on making use of the formula

$$\frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu} = (2n+1)P_n \quad (5),$$

gives

$$(n-s+1) \frac{d^s P_{n+1}}{d\mu^s} - (2n+1)\mu \frac{d^s P_n}{d\mu^s} + (n+s) \frac{d^s P_{n-1}}{d\mu^s} = 0.$$

On multiplying by the factor  $(1 - \mu^2)^{\frac{1}{2}s}$  this may be written

$$(n-s+1)P_{n+1}^s - (2n+1)\mu P_n^s + (n+s)P_{n-1}^s = 0 \quad (6).$$

Again by differentiating the equation (2) we find

$$\begin{aligned} (1 - \mu^2) \frac{dP_n^s}{d\mu} &= -s\mu(1 - \mu^2)^{\frac{1}{2}s} \frac{d^s P_n}{d\mu^s} + (1 - \mu^2)^{\frac{1}{2}s+1} \frac{d^{s+1} P_n}{d\mu^{s+1}} \\ &= -s\mu P_n^s + (1 - \mu^2)^{\frac{1}{2}s} \left[ \frac{d^s}{d\mu^s} \left\{ (1 - \mu^2) \frac{dP_n}{d\mu} \right\} + 2s\mu \frac{d^s P_n}{d\mu^s} + s(s-1) \frac{d^{s-1} P_n}{d\mu^{s-1}} \right] \\ &= s\mu P_n^s - (1 - \mu^2)^{\frac{1}{2}s} \left[ n(n+1) \frac{d^{s-1} P_n}{d\mu^{s-1}} - s(s-1) \frac{d^{s-1} P_n}{d\mu^{s-1}} \right] \end{aligned}$$

in virtue of the differential equation (1) for  $P_n$ ; and therefore by means of (5)

$$\begin{aligned}(1 - \mu^2) \frac{dP_n^s}{d\mu} &= s\mu P_n^s - \frac{(n-s+1)(n+s)}{2n+1} (1 - \mu^2)^{\frac{s}{2}} \frac{d^s}{d\mu^s} \{P_{n+1} - P_{n-1}\} \\ &= s\mu P_n^s - \frac{(n-s+1)(n+s)}{2n+1} (P_{n+1}^s - P_{n-1}^s),\end{aligned}$$

which with the aid of (6) may be expressed in the form

$$(1 - \mu^2) \frac{dP_n^s}{d\mu} = - \frac{n(n-s+1)}{2n+1} P_{n+1}^s + \frac{(n+1)(n+s)}{2n+1} P_{n-1}^s \quad \dots \quad (7).$$

Let us write for brevity

$$\left. \begin{aligned}D &\equiv (1 - \mu^2) \frac{d}{d\mu} \\ \Delta &\equiv \frac{d}{d\mu} (D) - \frac{s^2}{1 - \mu^2} = \frac{1}{1 - \mu^2} (D^2 - s^2)\end{aligned} \right\} \dots \dots \dots (8).$$

Then the equation (3) may be written

$$\Delta P_n^s = -n(n+1) P_n^s \quad \dots \quad (9),$$

while, if  $\sigma$  denote any constant quantity, we obtain from (6), (7)

$$(D + \sigma\mu) P_n^s = - \frac{(n-\sigma)(n-s+1)}{2n+1} P_{n+1}^s + \frac{(n+\sigma+1)(n+s)}{2n+1} P_{n-1}^s \quad \dots \quad (10).$$

The relation between the operators  $D$ ,  $\Delta$  may be written in the forms

$$\left. \begin{aligned}(D - \sigma\mu)(D + \sigma\mu) - (s^2 - \sigma^2\mu^2) &= (1 - \mu^2)(\Delta + \sigma) \\ (D + \sigma\mu)(D - \sigma\mu) - (s^2 - \sigma^2\mu^2) &= (1 - \mu^2)(\Delta - \sigma)\end{aligned} \right\} \quad \dots \quad (11),$$

which will be useful hereafter.

## § 2. Transformation of the Dynamical Equations for the Tides.

The formation of the differential equations for the tidal oscillations of the ocean has been fully dealt with in Part I. It is there shown (§ 4) that, if  $U$ ,  $V$  denote the northward and eastward velocity-components in latitude  $\sin^{-1}\mu$  and longitude  $\phi$ , when the system is executing a simple harmonic vibration in period  $2\pi/\lambda$ , these velocity-components will be expressible in terms of a single function  $\psi$  by means of the equations

$$\left. \begin{aligned}U &= - \frac{i\lambda \sqrt{1 - \mu^2}}{a(\lambda^2 - 4\omega^2\mu^2)} \frac{\partial \psi}{\partial \mu} + \frac{2\omega\mu}{a\sqrt{1 - \mu^2}(\lambda^2 - 4\omega^2\mu^2)} \frac{\partial \psi}{\partial \phi} \\ V &= - \frac{2\omega\mu \sqrt{1 - \mu^2}}{a(\lambda^2 - 4\omega^2\mu^2)} \frac{\partial \psi}{\partial \mu} - \frac{i\lambda}{a\sqrt{1 - \mu^2}(\lambda^2 - 4\omega^2\mu^2)} \frac{\partial \psi}{\partial \phi}\end{aligned} \right\} \quad \dots \quad (12).$$



Supposing that  $U, V, \psi$  are each proportional to  $e^{i(\lambda t + s\phi)}$ , we may take  $\partial\psi/\partial\phi = is\psi$ , and therefore, if we put  $2\omega s/\lambda = \sigma$  and introduce the abridged notation of the previous section, we may write the above equations in the form—

$$\left. \begin{aligned} \sqrt{(1-\mu^2)} U &= -\frac{is\sigma}{2\omega a} \frac{1}{s^2 - \sigma^2\mu^2} (D - \sigma\mu)\psi \\ \sqrt{(1-\mu^2)} V &= -\frac{\sigma^2}{2\omega a} \left[ \frac{1}{s^2 - \sigma^2\mu^2} \mu D - \frac{s^2}{\sigma(s^2 - \sigma^2\mu^2)} \right] \psi \\ &= -\frac{\sigma^2}{2\omega a} \left[ \frac{1}{s^2 - \sigma^2\mu^2} \mu (D - \sigma\mu) - \frac{1}{\sigma} \right] \psi \end{aligned} \right\} \dots (13).$$

The equation of continuity is

$$\frac{\partial\zeta}{\partial t} = -\frac{1}{a} \left[ \frac{\partial}{\partial\mu} \{ \sqrt{(1-\mu^2)} hU \} + \frac{\partial}{\partial\phi} \left\{ \frac{hV}{\sqrt{(1-\mu^2)}} \right\} \right]$$

where  $h$  denotes the depth, and  $\zeta$  the height of the surface-waves. On substituting for  $U, V$  from (13) and performing the differentiations with regard to  $t, \phi$  this becomes

$$(1-\mu^2)\zeta = \frac{\sigma^2}{4\omega^2 a^2} \left[ D \left\{ \frac{h}{s^2 - \sigma^2\mu^2} (D - \sigma\mu)\psi \right\} \right] + \frac{\sigma^3}{4\omega^2 a^2} \left[ \frac{h\mu}{s^2 - \sigma^2\mu^2} (D - \sigma\mu)\psi \right] - \frac{h\sigma^2}{4\omega^2 a^2} \psi,$$

or

$$\frac{4\omega^2 a^2}{\sigma^2} (1-\mu^2)\zeta = (D + \sigma\mu) \left[ \frac{h}{s^2 - \sigma^2\mu^2} (D - \sigma\mu)\psi \right] - h\psi \dots (14).$$

This equation is equivalent to the equation (17) of Part I. A second equation for the determination of the two functions  $\psi, \zeta$  is obtained from the pressure-condition at the free surface. On reference to § 2 of Part I., this condition is seen to lead to

$$\psi = v' - g\zeta + v \dots (15),$$

where  $v'$  denotes the surface-value of the potential due to the harmonic inequalities, and  $v$  the surface-value of the disturbing potential.

In order to effect the integration of these equations, we introduce two auxiliary functions  $\Psi_1, \Psi_2$ , connected with  $\psi$  by the relation

$$\psi = (D + \sigma\mu)\Psi_1 + (s^2 - \sigma^2\mu^2)\Psi_2 \dots (16).$$

On applying the operator  $(D - \sigma\mu)$  to the two members of this equation, we obtain in virtue of (11)

$$\begin{aligned} (D - \sigma\mu)\psi &= (1-\mu^2)(D + \sigma)\Psi_1 + (s^2 - \sigma^2\mu^2)\Psi_1 \\ &\quad + (s^2 - \sigma^2\mu^2)(D - \sigma\mu)\Psi_2 - 2\sigma^2\mu(1-\mu^2)\Psi_2 \dots (17). \end{aligned}$$

Now the functions  $\Psi_1, \Psi_2$  have as yet been subjected only to the single condition



(16). We may therefore impose on them any other arbitrary condition not inconsistent with the former. Suppose we choose them so as to satisfy the relation

$$(\Delta + \sigma) \Psi_1 = 2\sigma^2 \mu \Psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The two equations (16), (18) serve for the complete definition of the two functions  $\Psi_1, \Psi_2$ ; making use of the latter, (17) reduces to

$$(D - \sigma\mu) \psi = (s^2 - \sigma^2 \mu^2) \{ \Psi_1 + (D - \sigma\mu) \Psi_2 \} \quad . \quad . \quad . \quad . \quad (19).$$

Thus on replacing  $\psi$  by its value in terms of  $\Psi_1, \Psi_2$  in the right-hand member of (14), we deduce

$$\frac{4\omega^2 a^2}{\sigma^2} (1 - \mu^2) \zeta = (D + \sigma\mu) \{ h\Psi_1 + h(D - \sigma\mu) \Psi_2 \} - h[(D + \sigma\mu) \Psi_1 + (s^2 - \sigma^2 \mu^2) \Psi_2].$$

If we suppose that  $h$  is constant, the terms involving  $\Psi_1$  will disappear, while in virtue of (11) we shall obtain

$$\frac{4\omega^2 a^2}{\sigma^2} (1 - \mu^2) \zeta = h(1 - \mu^2) (\Delta - \sigma) \Psi_2$$

or

$$(\Delta - \sigma) \Psi_2 = \frac{4\omega^2 a^2}{\sigma^2 h} \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

We have now for the determination of the functions  $\psi, \zeta, \Psi_1, \Psi_2$  the four simultaneous differential equations (15), (16), (18), (20).

### § 3. *Integration in Series of Tesseral Harmonics.*

Let us suppose that  $\psi, \zeta, v, \Psi_1, \Psi_2$  are each expressible as series of tesseral harmonics of the same rank  $s$ . Omitting the exponential factor  $e^{i(\lambda t + s\phi)}$ , we assume that

$$\left. \begin{aligned} \psi &= \sum_{n=s}^{\infty} \Gamma_n P_n^s(\mu), \\ \zeta &= \sum_{n=s}^{\infty} C_n^s P_n^s(\mu), \\ v &= \sum_{n=s}^{\infty} \gamma_n^s P_n^s(\mu), \\ \Psi_1 &= \sum_{n=s}^{\infty} a_n^s P_n^s(\mu), \\ \Psi_2 &= \sum_{n=s}^{\infty} \beta_n^s P_n^s(\mu), \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Then, if  $\rho$  denote the density of the water, and  $\sigma_0$  the mean density of the whole



Again, on replacing  $\Psi_1, \Psi_2$  by their expansions in (18), the two members may be expressed as series of associated functions by means of (9), (6). Thus we find

$$-\Sigma \alpha_n^s \{n(n+1) - \sigma\} P_n^s = 2\sigma^2 \Sigma \beta_n^s \left[ \frac{n-s+1}{2n+1} P_{n+1}^s + \frac{n+s}{2n+1} P_{n-1}^s \right],$$

whence, on equating coefficients, we deduce that

$$\alpha_n^s \{n(n+1) - \sigma\} = -2\sigma^2 \left\{ \frac{n-s}{2n-1} \beta_{n-1}^s + \frac{n+s+1}{2n+3} \beta_{n+1}^s \right\} \quad \dots \quad (26).$$

Finally from (20), on expressing the two members by their expansions and equating coefficients, we obtain

$$\{n(n+1) + \sigma\} \beta_n^s = -\frac{4\omega^2 a^2}{\sigma^2 h} C_n^s \quad \dots \quad (27).$$

The relations (26), (27), enable us to eliminate the auxiliary constants  $\beta_n^s, \alpha_{n-1}^s, \alpha_{n+1}^s$  from (25), and thus to express  $\Gamma_n^s$  as a linear function of  $C_{n-2}^s, C_n^s, C_{n+2}^s$ . On substituting for  $\alpha_{n-1}^s, \alpha_{n+1}^s$  from the formula (26) in (25), we obtain

$$\begin{aligned} \Gamma_n^s &= s^2 \beta_n^s - \sigma^2 \frac{(n-s)\{(n-1)(n-2) + \sigma\}}{(2n-1)\{(n-1)n - \sigma\}} \left\{ \frac{n-s-1}{2n-3} \beta_{n-2}^s + \frac{n+s}{2n+1} \beta_{n-1}^s \right\} \\ &\quad - \sigma^2 \frac{(n+s+1)\{(n+2)(n+3) + \sigma\}}{(2n+3)\{(n+1)(n+2) - \sigma\}} \left\{ \frac{n-s+1}{2n+1} \beta_n^s + \frac{n+s+2}{2n+5} \beta_{n+2}^s \right\} \\ &= -\sigma^2 \frac{(n-s)(n-s-1)\{(n-1)(n-2) + \sigma\}}{(2n-1)(2n-3)\{(n-1)n - \sigma\}} \beta_{n-2}^s \\ &\quad + \left[ s^2 - \sigma^2 \frac{(n-s)(n+s)\{(n-1)(n-2) + \sigma\}}{(2n-1)(2n+1)\{(n-1)n - \sigma\}} - \sigma^2 \frac{(n-s+1)(n+s+1)\{(n+2)(n+3) + \sigma\}}{(2n+1)(2n+3)\{(n+1)(n+2) - \sigma\}} \right] \beta_n^s \\ &\quad - \sigma^2 \frac{(n+s+1)(n+s+2)\{(n+2)(n+3) + \sigma\}}{(2n+3)(2n+5)\{(n+1)(n+2) - \sigma\}} \beta_{n+2}^s, \end{aligned}$$

and this, by means of (27), gives

$$\begin{aligned} \frac{h\Gamma_n^s}{4\omega^2 a^2} &= \frac{(n-s)(n-s-1)}{(2n-1)(2n-3)\{(n-1)n - \sigma\}} C_{n-2}^s - \Lambda_n^s C_n^s \\ &\quad + \frac{(n+s+1)(n+s+2)}{(2n+3)(2n+5)\{(n+1)(n+2) - \sigma\}} C_{n+2}^s \quad \dots \quad (28), \end{aligned}$$

where

$$\begin{aligned} \Lambda_n^s &= \frac{s^2}{\sigma^2 \{n(n+1) + \sigma\}} - \frac{(n-s)(n+s)\{(n-1)(n-2) + \sigma\}}{(2n-1)(2n+1)\{(n-1)n - \sigma\} \{n(n+1) + \sigma\}} \\ &\quad - \frac{(n+s+1)(n-s+1)\{(n+2)(n+3) + \sigma\}}{(2n+1)(2n+3)\{(n+1)(n+2) - \sigma\} \{n(n+1) + \sigma\}}. \end{aligned}$$

This expression for  $\Lambda_n^s$  may be somewhat simplified if we separate it into its component partial fractions; we thus find

$$\begin{aligned}\Lambda_n^s &= \frac{s^2}{\sigma^2 \{n(n+1) + \sigma\}} - \frac{(n-s)(n+s)}{(2n-1)(2n+1)} \left[ \frac{(n-1)^2}{n^2 \{n(n-1) - \sigma\}} - \frac{2n-1}{n^2 \{n(n+1) + \sigma\}} \right] \\ &\quad - \frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)} \left[ \frac{(n+2)^2}{(n+1)^2 \{(n+1)(n+2) - \sigma\}} + \frac{2n+3}{(n+1)^2 \{n(n+1) + \sigma\}} \right] \\ &= \frac{1}{\{n(n+1) + \sigma\}} \left[ \frac{s^2}{\sigma^2} + \frac{n^2 - s^2}{n^2(2n+1)} - \frac{(n+1)^2 - s^2}{(n+1)^2(2n+1)} \right] \\ &\quad - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{n(n-1) - \sigma\}} - \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2) - \sigma\}};\end{aligned}$$

whence finally, remembering that  $\sigma = 2\omega s/\lambda$ ,

$$\begin{aligned}\Lambda_n^s &= \frac{\lambda^2}{4\omega^2} \frac{n(n+1) - 2\omega s/\lambda}{n^2(n+1)^2} - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{n(n-1) - 2\omega s/\lambda\}} \\ &\quad - \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2) - 2\omega s/\lambda\}}. \quad (29).\end{aligned}$$

The relation (28) will hold for all values of  $n$  equal to or greater than  $s$ , provided we suppose that  $C_{s-2}^s = 0$  and  $C_{s-1}^s = 0$ . If we put for brevity

$$\left. \begin{aligned}x_n^s &= \frac{(n-s+1)(n-s+2)}{(2n+1)(2n+3)\{(n+1)(n+2) - 2\omega s/\lambda\}}, \\ y_n^s &= \frac{(n+s+1)(n+s+2)}{(2n+3)(2n+5)\{(n+1)(n+2) - 2\omega s/\lambda\}}\end{aligned} \right\} \quad \dots \quad (30),$$

it may be written

$$\frac{h\Gamma_n^s}{4\omega^2 a^2} = x_{n-2}^s C_{n-2}^s - \Lambda_n^s C_n^s + y_n^s C_{n+2}^s.$$

Replacing  $\Gamma_n^s$  by its value in terms of  $C_n^s$ ,  $\gamma_n^s$  [equation (22)], we obtain

$$x_{n-2}^s C_{n-2}^s - L_n^s C_n^s + y_n^s C_{n+2}^s = \frac{h\gamma_n^s}{4\omega^2 a^2} \quad \dots \quad (31),$$

where  $x_{n-2}^s$ ,  $y_n^s$  are defined by (30) and

$$L_n^s = -\frac{hg_n}{4\omega^2 a^2} + \Lambda_n^s \quad \dots \quad (32),$$

$\Lambda_n^s$  being defined by the equation (29).

On putting  $s = 0$  it may readily be verified that the equation (31) reduces to the equation (23) of Part I. The manner in which such an equation may be utilized for



the determination of the free and forced vibrations has been fully discussed in that paper. A strictly analogous procedure might be adopted in the present case, but it is found convenient to modify the previous treatment in some respects. We shall therefore only indicate briefly the course of procedure when our analysis corresponds with that given in Part I., giving greater detail as regards those points where a different method has been found desirable.

#### § 4. *Extension to the Case of Variable Depth.*

The formulæ developed in the preceding sections apply only to the case where the depth is uniform. We may however obtain a relation of the same nature as the equation (31) when the depth is a function of the latitude given by the formula

$$h = k + l(1 - \mu^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

that is, when both the internal and external surfaces of the ocean are spheroids of revolution.

The above expression for  $h$  may be written in the form

$$h = \kappa + l\left(\frac{\lambda^2}{4\omega^2} - \mu^2\right) = \kappa + \frac{l(s^2 - \sigma^2\mu^2)}{\sigma^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

where

$$\kappa = k + l\left(1 - \frac{\lambda^2}{4\omega^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Substituting the expression (34) for  $h$  in the equation (14) we obtain

$$\begin{aligned} \frac{4\omega^2 a^2}{\sigma^2} (1 - \mu^2) \zeta &= (D + \sigma\mu) \left[ \frac{\kappa}{s^2 - \sigma^2\mu^2} (D - \sigma\mu) \psi \right] - \kappa\psi \\ &+ \frac{l}{\sigma^2} [(D + \sigma\mu)(D - \sigma\mu) - (s^2 - \sigma^2\mu^2)] \psi, \end{aligned}$$

which, with the aid of (11), becomes

$$\frac{4\omega^2 a^2}{\sigma^2} (1 - \mu^2) \left[ \zeta - \frac{l}{4\omega^2 a^2} (\Delta - \sigma) \psi \right] = (D + \sigma\mu) \left[ \frac{\kappa}{\sigma^2 - \sigma^2\mu^2} (D - \sigma\mu) \psi \right] - \kappa\psi.$$

The right-hand member is of the same form as that of the equation (14), except that  $h$  is replaced by  $\kappa$ . We may therefore introduce two auxiliary functions  $\Psi_1$ ,  $\Psi_2$ , defined by (16), (18), and proceed as in § 2, and we shall obtain in place of (20) the equation

$$(\Delta - \sigma) \Psi_2 = \frac{4\omega^2 a^2}{\sigma^2 \kappa} \left[ \zeta - \frac{l}{4\omega^2 a^2} (\Delta - \sigma) \psi \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

But, if we introduce the expansions (21) for  $\psi$ ,  $\zeta$ , we find

$$\zeta - \frac{l}{4\omega^2 a^2} (\Delta - \sigma) \psi = \Sigma \left[ C_n^u + \frac{l}{4\omega^2 a^2} \{n(n+1) + \sigma\} \Gamma_n^u \right] P_n^s \dots (37),$$

whence the final equation which replaces (31) will be obtained by replacing  $C_n^u$  by

$$C_n^u + \frac{l}{4\omega^2 a^2} \{n(n+1) + \sigma\} \Gamma_n^u \dots (38)$$

in the right-hand member of (28), and  $h$  by  $\kappa$  in the left-hand member.

Thus we obtain

$$\begin{aligned} \frac{\kappa \Gamma_n^u}{4\omega^2 a^2} = x'_{n-2} & \left[ C'_{n-2} + \frac{l}{4\omega^2 a^2} \{(n-2)(n-1) + \sigma\} \Gamma'_{n-2} \right] \\ & - \Lambda^s \left[ C_n^u + \frac{l}{4\omega^2 a^2} \{n(n+1) + \sigma\} \Gamma_n^u \right] \\ & + y' \left[ C'_{n+2} + \frac{l}{4\omega^2 a^2} \{(n+2)(n+3) + \sigma\} \Gamma'_{n+2} \right] \dots (39): \end{aligned}$$

and therefore, on separating out the parts of  $\Gamma^u$  due to  $C^u$ ,  $\gamma^u$  respectively, we find

$$\xi'_{n-2} C'_{n-2} - \mathfrak{L}' C_n^u + \eta' C'_{n+2} = G' \dots (40),$$

where

$$\left. \begin{aligned} \xi' &= x' \left[ 1 - \frac{ly_n}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\} \right] \\ \eta' &= y' \left[ 1 - \frac{ly_{n+2}}{4\omega^2 a^2} \left\{ (n+2)(n+3) + \frac{2\omega s}{\lambda} \right\} \right] \\ \mathfrak{L}' &= -\frac{\kappa y_n}{4\omega^2 a^2} + \Lambda^s \left[ 1 - \frac{ly_n}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\} \right] \end{aligned} \right\} \dots (41),$$

and

$$\begin{aligned} G' &= -\frac{l}{4\omega^2 a^2} \left[ (n-2)(n-1) + \frac{2\omega s}{\lambda} \right] x'_{n-2} \gamma'_{n-2} \\ &+ \left[ \frac{\kappa}{4\omega^2 a^2} + \frac{l}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\} \Lambda^s \right] \gamma'_n \\ &- \frac{l}{4\omega^2 a^2} \left\{ (n+2)(n+3) + \frac{2\omega s}{\lambda} \right\} y'_n \gamma'_{n+2} \dots (42). \end{aligned}$$

For the special case  $s=0$  the equation (40) reduces to the equation (23A) of Part I.

### § 5. *The Period-equation for the Free Oscillations in an Ocean of Uniform Depth.*

To determine the periods of free oscillation we may proceed exactly as in § 6 of

Part I., making use of the equation (31) instead of the simpler equation (23) of the previous paper. On putting  $\gamma_n^s$  zero, the equation (31) gives

$$x_{n-2}^s C_{n-2}^s - L_n^s C_n^s + y_n^s C_{n+2}^s = 0 \quad . \quad . \quad . \quad (43),$$

an equation which must hold for all values of  $n$  equal to or greater than  $s$ , it being understood that  $C_{s-2}^s = 0$ , and  $C_{s-1}^s = 0$ . The series of equations typified by (43) may be divided into two groups, in the former of which the suffixes involved are such that  $n - s$  is even and in the latter odd. The types of motion resulting from these two groups may be treated independently, the former being characterised by symmetry with respect to the equator and the latter by asymmetry. The treatment of the two groups of equations will be exactly similar, and we shall therefore in the main confine our discussion to the former group.

If we introduce the notation

$$\left. \begin{aligned} H_n^s &= \frac{x_n^s y_n^s}{L_n^s} - \frac{x_{n-2}^s y_{n-2}^s}{L_{n-2}^s} - \dots - \frac{x_s^s y_s^s}{L_s^s} \\ K_n^s &= \frac{x_{n-2}^s y_{n-2}^s}{L_n^s} - \frac{x_n^s y_n^s}{L_{n+2}^s} - \dots \text{ad inf.} \end{aligned} \right\} \quad . \quad . \quad . \quad (44),$$

it may be shown as in Part I. that provided  $L_n^s = 0$ ,  
 $n = \infty$

$$\frac{x_n^s C_n^s}{C_{n+2}^s} = H_n^s, \quad \frac{y_{n-2}^s C_n^s}{C_{n-2}^s} = K_n^s \quad . \quad . \quad . \quad (45),$$

and therefore the equation (43) may be written

$$C_n^s [H_{n-2}^s - L_n^s + K_{n-2}^s] = 0,$$

whence the period-equation for the free oscillations of symmetrical type is obtainable in the form

$$L_n^s - H_{n-2}^s - K_{n-2}^s = 0 \quad . \quad . \quad . \quad (46),$$

when  $n - s$  is an even integer.

The same equation will apply to the asymmetrical types if we suppose that  $n - s$  is an odd integer, and that the continued fraction  $H_n^s$  terminates with the partial quotient  $\frac{x_{s+1}^s y_{s+1}^s}{L_{s+1}^s}$ .

In particular, putting  $n - s = 0$ , we can express the period-equation for the symmetrical types in the form

$$L_s^s - \frac{x_s^s y_s^s}{L_{s+2}^s} - \frac{x_{s+2}^s y_{s+2}^s}{L_{s+4}^s} - \dots \text{ad inf.} = 0 \quad . \quad . \quad . \quad (46a),$$

while, putting  $n - s = 1$ , that for the asymmetrical types may be written

$$L_{s+1}^s - \frac{x_{s+1}^s y_{s+1}^s}{L_{s+3}^s} - \frac{x_{s+3}^s y_{s+3}^s}{L_{s+5}^s} - \dots \text{ad inf.} = 0 \quad . \quad . \quad . \quad (46b).$$

On the analogy of the problem dealt with in Part I., we may anticipate that, when





The roots of the equation  $L_n^s = 0$ , regarded as an equation for the determination of  $\lambda/\omega$ , will be the abscissæ of the points of intersection of this curve with the line

$$y = hg_n/4\omega^2\alpha^2.$$

Since  $h$  is essentially positive, the roots will all be real, and they will lie in the intervals between

$$-\infty, \quad 0, \quad \frac{2s}{(n+1)(n+2)}, \quad \frac{2s}{n(n-1)}, \quad +\infty.$$

For large values of  $hg_n/4\omega^2\alpha^2$  the two extreme roots will approximate to the roots of the equation

$$-\frac{hg_n}{4\omega^2\alpha^2} + \frac{\lambda^2}{4\omega^2} \frac{n(n+1) - 2\omega s/\lambda}{n^2(n+1)^2} = 0,$$

while the remaining two roots will approximate to

$$\frac{2s}{(n+1)(n+2)}, \quad \frac{2s}{n(n-1)}.$$

The two former roots are those which have their analogue in the special case treated of in Part I., for which  $s = 0$ , in which case they have equal magnitudes but opposite signs. These roots, we may expect, will approximate to roots of the period-equation, at least when  $n$  is large.

In order to see the significance of the remaining roots, it is convenient to transform the period-equation into a different form, which moreover is far better adapted for the more accurate numerical determination of the earlier roots.

### § 6. *Modified Form of the Period-Equation.*

Referring back to the equations (29), (30), (32), which define  $x_n^s$ ,  $y_n^s$ ,  $L_n^s$ , we see that (31) may be written in the form

$$\begin{aligned} & \frac{(n-s)}{(2n-1) \{n(n-1) - 2\omega s/\lambda\}} \left[ \frac{(n-s-1)}{(2n-3)} C_{n-2}^s + \frac{(n-1)^2(n+s)}{n^2(2n+1)} C_n^s \right] \\ & - \left[ \frac{\lambda^2}{4\omega^2} \frac{n(n+1) - 2\omega s/\lambda}{n^2(n+1)^2} - \frac{hg_n}{4\omega^2\alpha^2} \right] C_n^s \\ & + \frac{(n+s+1)}{(2n+3) \{(n+1)(n+2) - 2\omega s/\lambda\}} \left[ \frac{(n+2)^2(n-s+1)}{(n+1)^2(2n+1)} C_n^s + \frac{(n+s+2)}{(2n+5)} C_{n+2}^s \right] = \frac{h\gamma_n^s}{4\omega^2\alpha^2} \quad (48). \end{aligned}$$

Hence, if we introduce a new set of auxiliary constants,  $D_n^s$ ,  $D_{n+1}^s$ , &c., such that for values of  $n$  equal to or greater than  $s$

$$\frac{(n+1)^2(n-s)}{2n-1} C_{n-1}^s + \frac{n^2(n+s+1)}{2n+3} C_{n+1}^s = \left\{ n(n+1) - \frac{2\omega s}{\lambda} \right\} D_n^s \quad (49),$$

the equation (48) reduces to

$$\begin{aligned} & \frac{(n+1)^2(n-s)}{2n-1} D_{n-1}^s + \frac{n^2(n+s+1)}{2n+3} D_{n+1}^s \\ &= \left[ \frac{\lambda^2}{4\omega^2} \left\{ n(n+1) - \frac{2\omega s}{\lambda} \right\} - n^2(n+1)^2 \frac{hg_n}{4\omega^2 a^2} \right] C_n^s + n^2(n+1)^2 \frac{h\gamma_n'}{4\omega^2 a^2} \quad (50). \end{aligned}$$

Thus, if we write for brevity

$$\left. \begin{aligned} M_n^s &= \frac{\lambda^2}{4\omega^2} \left\{ n(n+1) - \frac{2\omega s}{\lambda} \right\} - n^2(n+1)^2 \frac{hg_n}{4\omega^2 a^2} \\ N_n^s &= n(n+1) - \frac{2\omega s}{\lambda} \end{aligned} \right\} \dots \dots \dots (51),$$

and put  $\gamma_n' = 0$ , the equation (43) is replaced by the two following :

$$\left. \begin{aligned} \frac{(n+1)^2(n-s)}{2n-1} D_{n-1}^s - M_n^s C_n^s + \frac{n^2(n+s+1)}{2n+3} D_{n+1}^s &= 0 \\ \frac{(n+1)^2(n-s)}{2n-1} C_{n-1}^s - N_n^s D_n^s + \frac{n^2(n+s+1)}{2n+3} C_{n+1}^s &= 0 \end{aligned} \right\} \dots \dots (52).$$

For the determination of the symmetrical types we therefore have the series of equations

$$\begin{aligned} -M_s^s C_s^s + \frac{s^2(2s+1)}{2s+3} D_{s+1}^s &= 0, \\ \frac{(s+2)^2 \cdot 1}{2s+1} C_s^s - N_{s+1}^s D_{s+1}^s + \frac{(s+1)^2(2s+2)}{2s+5} C_{s+2}^s &= 0, \\ \frac{(s+3)^2 \cdot 2}{2s+3} D_{s+1}^s - M_{s+2}^s C_{s+2}^s + \frac{(s+2)^2(2s+3)}{2s+7} D_{s+3}^s &= 0, \\ &\dots \dots \dots \end{aligned}$$

On eliminating the quantities  $C_s^s$ ,  $D_{s+1}^s$ ,  $C_{s+2}^s$ , &c., by means of a continued fraction we find the period-equation in the form

$$M_s^s - \frac{a_s^s}{N_{s+1}^s} - \frac{a_{s+1}^s}{M_{s+2}^s} - \frac{a_{s+2}^s}{N_{s+3}^s} - \dots \text{ad inf.} = 0 \quad \dots \dots \dots (53),$$

where we have written for brevity  $a_n^s$  in place of

$$\frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)} \dots \dots \dots (54).$$

In like manner the period-equation for the asymmetrical types may be written

$$N_s^s - \frac{a_s^s}{M_{s+1}^s} - \frac{a_{s+1}^s}{N_{s+2}^s} - \frac{a_{s+2}^s}{M_{s+3}^s} - \dots \text{ad inf.} = 0 \quad \dots \dots \dots (55).$$

We may also write these equations in a variety of alternative forms in which prominence is given to any one we please of the quantities  $M_n^s$ ,  $N_n^s$ . These forms are obtained by giving  $n$  different integral values in the equations

$$M_n^s - \left[ \frac{\alpha_{n-1}^s}{N_{n-1}^s} - \frac{\alpha_{n-2}^s}{M_{n-2}^s} - \frac{\alpha_{n-3}^s}{N_{n-3}^s} - \dots \right] - \left[ \frac{\alpha_n^s}{N_{n+1}^s} - \frac{\alpha_{n+1}^s}{M_{n+2}^s} - \frac{\alpha_{n+2}^s}{N_{n+3}^s} - \dots \right] = 0. \quad (56),$$

$$N_n^s - \left[ \frac{\alpha_{n-1}^s}{M_{n-1}^s} - \frac{\alpha_{n-2}^s}{N_{n-2}^s} - \frac{\alpha_{n-3}^s}{M_{n-3}^s} - \dots \right] - \left[ \frac{\alpha_n^s}{M_{n+1}^s} - \frac{\alpha_{n+1}^s}{N_{n+2}^s} - \frac{\alpha_{n+2}^s}{M_{n+3}^s} - \dots \right] = 0. \quad (57).$$

In each case the former continued fraction terminates with a partial quotient involving  $\alpha_i^s$  in the numerator and either  $M_i^s$  or  $N_i^s$  in the denominator, while the latter proceeds to infinity.

For the symmetrical types, if we use the form (56) we must suppose  $n - s$  an even integer, whereas if we employ (57)  $n - s$  must be supposed odd. The reverse will of course be the case for the asymmetrical types.

The continued fractions of the present section will not converge so rapidly as those of the preceding, but in spite of this drawback they present considerable advantages. In the first place the numerators of the partial quotients, which are obtained by giving  $n$  different integral values in the expression (54), are independent of  $\lambda$ . These numerators, which further are in a convenient form for logarithmic computation, may therefore be tabulated once for all, whereas the numerators of the partial quotients in the continued fractions of the last section require to be re-determined at each successive trial in attempting to solve the period-equation by trial and error. Moreover, the evaluation of the denominators  $M_n^s$ ,  $N_n^s$  by means of the formulæ (51) may be very quickly effected, even though a fairly large number of these denominators is required, whereas the evaluation of the quantities  $L_n^s$  by means of (29) and (32) is extremely laborious.

Another disadvantage resulting from the use of the preceding form is that when  $\lambda/\omega$  is near the value  $2s/n(n+1)$  the functions  $x_{n-1}^s$ ,  $y_{n-1}^s$  both become large, while  $L_{n-1}^s$ ,  $L_{n+1}^s$  have both a zero and an infinity in the immediate proximity of this value. Hence, in order to evaluate  $L_{n-1}^s$ ,  $L_{n+1}^s$  for a value of  $\lambda$  in this region it is necessary to observe a very high degree of accuracy in the numerical work. The singularities which occur in the left-hand member of (46) when  $\lambda$  passes through one of these critical values no longer appear if we write the period-equation in the form (57) with the value of  $n$  appropriately chosen.

### § 7. Expressions for the Velocity-Components.

The auxiliary constants  $D_i^s$ ,  $D_{i+1}^s$ , &c., introduced in the last section, may be made use of to express the velocity-components by means of series of surface-harmonics.

Thus from the first of equations (13) we have, on replacing  $\psi$  by its value in terms of  $\Psi_1, \Psi_2$ ,

$$\sqrt{(1 - \mu^2)} U = -\frac{is\sigma}{2\omega a} [\Psi_1 + (D - \sigma\mu) \Psi_2].$$

On introducing the expansions for  $\Psi_1, \Psi_2$  on the right we find

$$\begin{aligned} \sqrt{(1 - \mu^2)} U &= -\frac{is\sigma}{2\omega a} \Sigma \left[ \alpha_n^s P_n^s + \beta_n^s \left\{ -\frac{(n + \sigma)(n - s + 1)}{2n + 1} P_{n+1}^s + \frac{(n - \sigma + 1)(n + s)}{2n + 1} P_{n-1}^s \right\} \right], \\ &= -\frac{is\sigma}{2\omega a} \Sigma P_n^s \left[ \alpha_n^s - \frac{(n + \sigma - 1)(n - s)}{2n - 1} \beta_{n-1}^s + \frac{(n - \sigma + 2)(n + s + 1)}{2n + 3} \beta_{n+1}^s \right], \end{aligned}$$

which, with the help of (26), (27), reduces to

$$\sqrt{(1 - \mu^2)} U = \frac{2\omega is a}{\sigma h} \Sigma P_n^s \left[ -\frac{(n - s)(n + \sigma - 1)}{(2n - 1)\{n(n + 1) - \sigma\}} C_{n-1}^s + \frac{(n + s + 1)(n - \sigma)}{(2n + 3)\{n(n + 1) - \sigma\}} C_{n+1}^s \right],$$

or from (49),

$$\sqrt{(1 - \mu^2)} U = \frac{i\lambda a}{h} \Sigma \left\{ \frac{n - s}{2n - 1} C_{n-1}^s + \frac{n + s + 1}{2n + 3} C_{n+1}^s - D_n^s \right\} P_n^s. \quad (58).$$

This may also be written in the form

$$\sqrt{(1 - \mu^2)} U = \frac{i\lambda a}{h} \Sigma \{ \mu C_n^s - D_n^s \} P_n^s. \quad (59).$$

Again, from (13)

$$\begin{aligned} \sqrt{(1 - \mu^2)} V &= -\frac{i\sigma}{s} \mu \sqrt{(1 - \mu^2)} U + \frac{\sigma}{2\omega a} \psi \\ &= +\frac{2\omega a}{h} \mu \Sigma (\mu C_n^s - D_n^s) P_n^s + \frac{s}{a\lambda} \Sigma \Gamma_n^s P_n^s. \quad (60), \end{aligned}$$

whence, by means of (6), we may express  $\sqrt{(1 - \mu^2)} V$  by a series of surface-harmonics.

The corresponding formulæ when the depth is variable may be obtained by replacing  $h$  by  $\kappa$  and  $C_n^s$  by

$$C_n^s + \frac{l}{4\omega^2 a^2} \left\{ n(n + 1) + \frac{2\omega s}{\lambda} \right\} \Gamma_n^s,$$

so that we find

$$\begin{aligned} \sqrt{(1 - \mu^2)} U &= \frac{i\lambda a}{\kappa} \Sigma \left( \mu \left[ C_n^s + \frac{l}{4\omega^2 a^2} \left\{ n(n + 1) + \frac{2\omega s}{\lambda} \right\} \Gamma_n^s \right] - D_n^s \right) P_n^s \\ \sqrt{(1 - \mu^2)} V &= \frac{2\omega a \mu}{\kappa} \Sigma \left( \mu \left[ C_n^s + \frac{l}{4\omega^2 a^2} \left\{ n(n + 1) + \frac{2\omega s}{\lambda} \right\} \Gamma_n^s \right] - D_n^s \right) P_n^s + \frac{s}{a\lambda} \Sigma \Gamma_n^s P_n^s \end{aligned} \quad (61)$$

where the quantities  $D_n^s$  are now defined by



$$N_n^s D_n^s = \frac{(n+1)^2(n-s)}{2n-1} \left[ C_{n-1}^s + \frac{l}{4\omega^2 a^2} \left\{ n(n-1) + \frac{2\omega s}{\lambda} \right\} \Gamma_{n-1}^s \right] \\ + \frac{n^2(n+s+1)}{2n+3} \left[ C_{n+1}^s + \frac{l}{4\omega^2 a^2} \left\{ (n+1)(n+2) + \frac{2\omega s}{\lambda} \right\} \Gamma_{n+1}^s \right] \quad (62).$$

The formulæ (61) cease to be of use in a special case which will present itself hereafter for which  $\kappa = 0$ . It will be seen in a later section that the expressions on the right become indeterminate in this case, so that the determination of the velocity-components must be effected by means of the formulæ (12) or (13). These latter formulæ seem at first sight to indicate that the velocity-components become infinite in latitude  $\sin^{-1}(s/\sigma)$ , but the forms (61) indicate that such cannot be the case, at least when  $\kappa$  is different from zero.

### § 8. *Approximate Determination of the Higher Roots of the Period-Equation.*

If we take the period-equation in the form (56), and as a first approximation omit the continued fractions from the left-hand member, it reduces to the quadratic

$$M_n^s = 0,$$

or

$$\frac{\lambda^2}{4\omega^2} \left\{ n(n+1) - \frac{2\omega s}{\lambda} \right\} - n^2(n+1)^2 \frac{h g_n}{4\omega^2 a^2} = 0.$$

For large values of  $n$  the roots of this equation will give a sufficiently accurate approximation to the roots of the period-equation, since it may be seen that the continued fractions tend to limits comparable with  $\frac{1}{4}n^2$ , and therefore small in comparison with  $n^2(n+1)^2 \frac{h g_n}{4\omega^2 a^2}$ , when  $n$  is very large and  $\lambda/\omega$  has as its value either of the roots of this equation. We may even obtain a fair approximation by omitting the term containing  $s$ , in which case the formula for  $\lambda$  corresponds with that obtained when the rotation is omitted.

A better approximation will however be obtained by representing the continued fractions by their first convergents instead of entirely neglecting them. The approximate form of the period-equation is then

$$M_n^s - \frac{a_{n-1}^s}{N_{n-1}^s} - \frac{a_n^s}{N_{n+1}^s} = 0,$$

or

$$\frac{\lambda^2}{4\omega^2} \frac{\{n(n+1) - 2\omega s/\lambda\}}{n^2(n+1)^2} - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{(n-1)n - 2\omega s/\lambda\}} \\ - \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2) - 2\omega s/\lambda\}} - \frac{h g_n}{4\omega^2 a^2} = 0.$$

We thus get back to the equation  $L_n^s = 0$ . The roots of this equation may be approximated to numerically by HORNER'S process, the significant roots being those which lie in the intervals between  $-\infty$  and 0, and between  $2s/n(n-1)$  and  $+\infty$ .

For the particular case  $s = 0$ , the biquadratic to which the equation  $L_n^s = 0$  is equivalent reduces to a quadratic, the two roots which remain finite being of equal magnitude and opposite sign. This special case has been examined in Part I., and it will be seen on reference to the tables there given (§§ 7-8), that the roots of the equation  $L_n = 0$  give a very good approximation to the roots of the period-equation except in the case of the earlier roots when  $hg/4\omega^2\alpha^2$  is small. In the present paper I have examined in some detail the special cases corresponding to the values 1 and 2 for  $s$ , and the approximation is found to be equally rapid, as will be seen from the tables given hereafter. Consequently, in these cases at least, all except the two or three smallest roots will be obtained with adequate accuracy by finding the roots which lie in the stated intervals of the equations  $L_n^s = 0$  with different integral values of  $n$ .

The roots so found will not however form the complete series of roots of the period-equation. We may in fact anticipate that the remaining roots of the equation  $L_n^s = 0$  will also approximate to roots of the period-equation. To obtain a better approximation of the roots of this class, it will however be preferable to make use of the period-equation in the form (57). As a first approximation we omit the continued fractions and obtain

$$N_n^s = 0 \quad \text{or} \quad n(n+1) - \frac{2\omega s}{\lambda} = 0.$$

This method of approximation will be valid if when  $\lambda/\omega = 2s/n(n+1)$  the two continued fractions involved in (57) are small in comparison with  $n(n+1)$ . But it may readily be verified that with large values of  $n$  these continued fractions become comparable with  $\omega^2\alpha^2/hg$ , and therefore the desired condition will certainly be satisfied when  $n$  is sufficiently large.

A better approximation may be obtained by representing the continued fractions by their first convergents. We thus obtain as the approximate form of the period-equation for the determination of the root which lies near  $\frac{2s}{n(n+1)}$

$$N_n^s - \frac{a_{n-1}^s}{M_{n-1}^s} - \frac{a_n^s}{M_{n+1}^s} = 0,$$

or

$$\begin{aligned} n(n+1) - \frac{2\omega s}{\lambda} = & \frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)} \\ & - \frac{\lambda^2}{4\omega^2} \left\{ n(n-1) - \frac{2\omega s}{\lambda} \right\} - \frac{n^2(n-1)^2 hg_{n-1}}{4\omega^2\alpha^2} \\ & + \frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)} \\ & + \frac{\lambda^2}{4\omega^2} \left\{ (n+1)(n+2) - \frac{2\omega s}{\lambda} \right\} - \frac{(n+1)^2(n+2)^2 hg_{n+1}}{4\omega^2\alpha^2}. \end{aligned}$$

Using the first approximation in the terms on the right, we deduce

$$n(n+1) - \frac{2\omega s}{\lambda} = - \frac{\frac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)}}{n^2(n-1)^2 \frac{hg_{n-1}}{4\omega^2 a^2} + \frac{2s^2}{n(n+1)^2}} - \frac{\frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)}}{(n+1)^2(n+2)^2 \frac{hg_{n+1}}{4\omega^2 a^2} + \frac{2s^2}{n^2(n+1)}} \quad (63).$$

This formula is found to lead to the roots of the period-equation with a surprising degree of accuracy.

Our analysis is only applicable when  $h$  is small in comparison with  $a$ , but subject to this limitation the approximations of the present section will improve as  $hg/4\omega^2 a^2$  increases, that is, as the depth of the water increases or the angular velocity of rotation diminishes. They will give good results even with small values of  $n$  when  $\omega$  is sufficiently small, and they may be used to determine the limiting values assumed by the roots when the angular velocity of rotation is indefinitely reduced.

We see then that the roots of the period-equation are of two classes, which may be distinguished by their limiting forms when the rotation is annulled. The roots of the former class are such that the values of  $\lambda$  remain finite when  $\omega = 0$ , their limiting values being given by the formula

$$\lambda = \pm \sqrt{\frac{n(n+1)hg_n}{a^2}}.$$

There will be an equal number of positive and negative roots of this class, but though these approach the same limiting values their numerical values will not be equal as in the case where  $s = 0$ , and the positive and negative roots must therefore be determined independently.

The roots of the second class are all positive and are such that the values of  $\lambda/\omega$  tend to finite limits when  $\omega$  is reduced to zero, the limiting values being given by the formula

$$\frac{\lambda}{\omega} = \frac{2s}{n(n+1)},$$

whereas  $\lambda$  will tend to the limit zero.

The analogue of the types of motion which correspond with the former roots will still be oscillatory when the rotation is annulled, but the types of motion corresponding with the roots of the second class will cease to exist as oscillations when the angular velocity of rotation is reduced to zero. These types of motion will have their equivalent in steady motions, but an infinitesimal amount of rotation would immediately convert such steady motions into oscillatory motions of very long period.

For the particular case  $s = 0$  the roots of the second class are all zero even when the angular velocity of rotation is finite. Hence steady motions can exist on a rotating globe, but these are necessarily of zonal type. We have in fact



already seen in Part I.\* that the only forms of steady motion which can exist are of this character, and have explained the fact by stating that the steady motions not of zonal type which can exist on a globe without rotation must have their analogue in the more general case in oscillatory motions whose period bears a finite ratio to the rotation-period, no matter how great the latter may be. Our present work confirms this statement and throws further light on the nature of these oscillatory motions.

### § 9. *Evaluation of the Earlier Roots.*

The errors resulting from the use of the approximate formulæ of the last section may be considerable in the case of the earlier roots for which  $n$  has small values. To obtain these earlier roots we must therefore proceed by trial and error, the preceding method being made use of to obtain values with which to commence the trials.

As a concrete example we will discuss in detail the computation of the positive root of the first class corresponding to the case  $n = 4$ ,  $s = 1$ , when the depth is given by  $hg/4\omega^2\alpha^2 = \frac{1}{20}$ . Taking  $\rho/\sigma_0 = 0.18093$ , and introducing the numerical values of  $n$ ,  $s$ , and  $h$ , the equation  $L_4^1 = 0$  becomes

$$(\lambda/\omega)^4 - 0.3333 (\lambda/\omega)^3 - 5.5481 (\lambda/\omega)^2 + 1.0906 (\lambda/\omega) - 0.0418 = 0.$$

By HORNER'S process the greatest positive root of this equation is found to be

$$2.43265.$$

Now experience shows that the numerical value thus suggested is in general too small.† We therefore select for a first trial a value rather larger than that indicated, say, for example,

$$\lambda/\omega = 2.4400.$$

From the formula (54) we find

$$\begin{aligned} \log a_1^1 &= 0.2553, & \log a_2^1 &= 1.1652, & \log a_3^1 &= 1.7289, \\ \log a_4^1 &= 2.1450, & \log a_5^1 &= 2.4769, & \log a_6^1 &= 2.7537, \\ \log a_7^1 &= 2.9915, & \log a_8^1 &= 3.2001, & \log a_9^1 &= 3.3859, \end{aligned}$$

while the values of the expression

$$n^2(n+1)^2 \frac{hg_n}{4\omega^2\alpha^2}$$

for the values 2, 4, 6, 8, 10 of  $n$  are

$$1.6046, \quad 18.794, \quad 84.517, \quad 250.92, \quad 589.4.$$

\* §§ 14, 15.

† Compare the 2nd and 3rd columns of Tables I. and II., Part I.



Thus we find from the formulæ (51), with  $\lambda/\omega = 2.4400$ ,

$$\begin{aligned} M_2^1 &= 6.104, & M_4^1 &= 9.753, & M_6^1 &= -23.25, & M_8^1 &= -144.97, & M_{10}^1 &= -426.9, \\ N_1^1 &= 1.180, & N_3^1 &= 11.180, & N_5^1 &= 29.180, & N_7^1 &= 55.180, & N_9^1 &= 89.18. \end{aligned}$$

It will be convenient for us now to introduce the following abridged notation:—

$$\left. \begin{aligned} e_n^s &= \frac{\alpha_{n-1}^s}{M_n^s} - \frac{\alpha_n^s}{N_{n+1}^s} - \frac{\alpha_{n+1}^s}{M_{n+2}^s} - \dots \text{ad inf.} \\ f_n^s &= \frac{\alpha_{n-1}^s}{N_n^s} - \frac{\alpha_n^s}{M_{n+1}^s} - \frac{\alpha_{n+1}^s}{N_{n+2}^s} - \dots \text{ad inf.} \\ E_n^s &= \frac{\alpha_n^s}{M_n^s} - \frac{\alpha_{n-1}^s}{N_{n-1}^s} - \frac{\alpha_{n-2}^s}{M_{n-2}^s} - \dots \\ F_n^s &= \frac{\alpha_n^s}{N_n^s} - \frac{\alpha_{n-1}^s}{M_{n-1}^s} - \frac{\alpha_{n-2}^s}{N_{n-2}^s} - \dots \end{aligned} \right\} \dots \dots \dots (64),$$

the last two continued fractions terminating with the partial quotient which involves  $\alpha_n^s$  in the numerator, and either  $M_n^s$  or  $N_n^s$  in the denominator.

From these definitions of the quantities  $e, f, E, F$ , we have:

$$\left. \begin{aligned} e_n^s &= \frac{\alpha_{n-1}^s}{M_n^s - f_{n+1}^s}, & f_n^s &= \frac{\alpha_{n-1}^s}{N_n^s - e_{n+1}^s} \\ E_n^s &= \frac{\alpha_n^s}{M_n^s - F_{n-1}^s}, & F_n^s &= \frac{\alpha_n^s}{N_n^s - E_{n-1}^s} \end{aligned} \right\} \dots \dots \dots (65);$$

while the period-equation may be written in the forms:

$$\left. \begin{aligned} M_n^s - F_{n-1}^s - f_{n+1}^s &= 0 \\ N_n^s - E_{n-1}^s - e_{n+1}^s &= 0 \end{aligned} \right\} \dots \dots \dots (66).$$

Suppose that we neglect  $f_{11}^1$ ; making use of the numerical values obtained above for the quantities  $M, N, \alpha$ , by successive applications of the formulæ (65) we obtain

$$\begin{aligned} \log e_{10}^1 &= n0.7556, & \log f_9^1 &= 1.2229, & \log e_8^1 &= n0.7829, \\ \log f_7^1 &= 0.9666, & \log e_6^1 &= n0.9649, & \log f_5^1 &= 0.5606. \end{aligned}$$

In like manner, if we neglect  $e_{10}^1$ , we find

$$\begin{aligned} \log f_9^1 &= 1.2498, & \log e_8^1 &= n0.7801, & \log f_7^1 &= 0.9668, \\ \log e_6^1 &= n0.9649, & \log f_5^1 &= 0.5606. \end{aligned}$$

Now the two values of  $f_5^1$  obtained by these methods are respectively the 6th and

5th convergents of the continued fraction  $f_5^1$ . Since we find that to the degree of accuracy retained these are equal, it follows that all subsequent convergents are sensibly equal to either of them. Hence the infinite continued fraction  $f_5^1$  may be replaced by its fifth convergent without sensible error.

Similarly we find

$$\log F_1^1 = 0.1834, \quad \log E_2^1 = 0.5045, \quad \log F_3^1 = 0.8266,$$

and therefore

$$M_4^1 - F_3^1 - f_5^1 = 9.753 - 6.707 - 3.636 = -0.590.$$

As a second trial we take

$$\lambda/\omega = 2.4600.$$

Proceeding as before, we deduce

$$M_4^1 - F_3^1 - f_5^1 = 10.234 - 6.619 - 3.607 = 0.008.$$

We conclude that there is a root of the period-equation lying between 2.4400 and 2.4600; by interpolation its value is found to be

$$2.4597.$$

The same method may be used for the determination of the roots of the second class, the initial trial values being suggested by the formula (63). As a numerical example, if we put  $n = 5$ ,  $s = 1$ ,  $hg/4\omega^2 a^2 = \frac{1}{20}$  in (63), we find

$$2\omega/\lambda = 40.974, \quad \text{or} \quad \lambda/\omega = 0.04881.$$

For a first trial we take  $2\omega/\lambda = 41$ , and deduce

$$N_5^1 - E_4^1 - e_6^1 = -11 + 8.678 + 2.602 = 0.280.$$

As a second trial we take  $2\omega/\lambda = 41.280$ , and obtain

$$N_5^1 - E_4^1 - e_6^1 = -11.280 + 8.657 + 2.593 = -0.030,$$

and therefore, by interpolation,

$$N_5^1 - E_4^1 - e_6^1 = 0,$$

when

$$2\omega/\lambda = 41.253, \quad \text{or} \quad \lambda/\omega = 0.04848.$$

I have selected for special investigation the asymmetrical types when  $s = 1$ , and

the symmetrical types when  $s = 2$ , these types presenting special interest in relation to the diurnal and semi-diurnal forced tides. The annexed tables give the values of the computed roots, together with the corresponding periods of free oscillation for the four depths of 7260, 14,520, 29,040, 58,080 feet treated of in Part I., corresponding to the values  $\frac{1}{40}$ ,  $\frac{1}{20}$ ,  $\frac{1}{10}$ , and  $\frac{1}{5}$  for  $hg/4\omega^2 a^2$ .

I.—DEPTH 7260 feet ( $hg/4\omega^2 a^2 = \frac{1}{40}$ ,  $\rho/\sigma_0 = 0.18093$ ).

		$s = 1.$			$s = 2.$		
		Approximate root.	Corrected root.	Period.*	Approximate root.	Corrected root.	Period.
Class I.	$n = 2$	1.8331	1.6337	hrs. mins. 14 41	1.4745	1.3347	hrs. mins. 17 59
		— 1.0906	— 0.9834	24 24	— 0.6582	— 0.6221	38 34
	$n = 4$	2.0042	2.0685	11 36	2.0024	1.9866	12 5
		— 1.8472	— 1.8234	13 10	— 1.7006	— 1.6595	14 28
Class II.	$n = 1$	0.6932	0.6401	days hrs. 1 13	..	..	days hrs. ..
	$n = 3$	0.05702	0.06251	16 0	0.1532	0.1671	6 0
	$n = 5$	0.03853	0.03784	26 10	0.08091	0.08178	12 5

II.—DEPTH 14,520 feet ( $hg/4\omega^2 a^2 = \frac{1}{20}$ ,  $\rho/\sigma_0 = 0.18093$ ).

		$s = 1.$			$s = 2.$		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
Class I.	$n = 2$	1.9588	1.8677	hrs. mins. 12 51	1.6804	1.6133	hrs. mins. 14 52
		— 1.3021	— 1.2450	19 16	— 0.9128	— 0.8922	26 54
	$n = 4$	2.4327	2.4597	9 45	2.4363	2.4349	9 51
		— 2.2949	— 2.2907	10 29	— 2.1706	— 2.1547	11 8
Class II.	$n = 1$	0.7502	0.7283	days hrs. 1 9	..	..	days hrs. ..
	$n = 3$	0.08423	0.08673	11 13	0.2054	0.2129	4 17
	$n = 5$	0.04881	0.04848	20 15	0.10067	0.10081	9 22

\* Throughout these tables the periods are expressed in sidereal time.

III.—DEPTH 29,040 feet ( $hg/4\omega^2\alpha^2 = \frac{1}{10}$ ,  $\rho/\sigma_0 = 0.18093$ ).

		$s = 1.$			$s = 2.$		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
Class I.	$n = 2$	2.2027	2.1641	hrs. mins. 11 5	2.0241	1.9968	hrs. mins. 12 1
		— 1.6439	— 1.6170	14 50	— 1.2960	— 1.2855	18 40
	$n = 4$	3.1183	3.1274	7 40	3.1295	3.1293	7 40
		— 2.9958	— 2.9961	8 1	— 2.8911	— 2.8856	8 19
Class II.	$n = 1$	0.8213	0.8149	days hrs. 1 5	..	..	days hrs. 3 ..
	$n = 3$	0.1116	0.1124	8 21	0.2523	0.2554	3 22
	$n = 5$	0.05636	0.05625	17 19	0.11472	0.11472	8 17

IV.—DEPTH 58,080 feet ( $hg/4\omega^2\alpha^2 = \frac{1}{5}$ ,  $\rho/\sigma_0 = 0.18093$ ).

		$s = 1.$			$s = 2.$		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
Class I.	$n = 2$	2.6431	2.6288	hrs. mins. 9 8	2.5636	2.5535	hrs. mins. 9 24
		— 2.1726	— 2.1611	11 6	— 1.8623	— 1.8575	12 55
	$n = 4$	4.1626	4.1659	5 46	4.1820	4.1818	5 44
		— 4.0501	— 4.0506	5 56	— 3.9610	— 3.9592	6 4
Class II.	$n = 1$	0.8886	0.8873	days hrs. 1 3	..	..	days hrs. 3 ..
	$n = 3$	0.13354	0.13375	7 11	0.2865	0.2876	3 11
	$n = 5$	0.06108	0.06105	16 9	0.12332	0.12332	8 3

The approximate roots here given have been evaluated by the method of § 7, except in the case of the roots of the second class for  $n = 1$ ,  $s = 1$ , where, instead of replacing the continued fraction involved in the period-equation by its first convergent, I have made use of the second convergent, so that the approximate form of the period-equation from which this root is determined is

$$N_1^1 - \frac{a_1^1}{M_2^1} - \frac{a_2^1}{N_3^1} = 0,$$

or

$$M_2^1 - \frac{a_1^1}{N_1^1} - \frac{a_2^1}{N_3^1} = 0,$$

or, what is equivalent,  $L_2^1 = 0$ .



By a comparison of the approximate values given in these tables with the true values we see that for extreme cases here tabulated, the error involved in the approximation does not amount to more than about 3 per cent., even with a depth as small as 7260 feet. We have here a justification of the statements made in the last section as to the approximation to the higher roots.

### § 10. *Oscillations of the First Class. Determination of the Type.*

We shall describe as oscillations of the first class those whose periods remain finite when the rotation-period is indefinitely prolonged, that is, those for which the roots of the period-equation are of the first class. The types of motion whose periods become infinitely long with the rotation-period will be called oscillations of the second class.

The determination of the type involves the determination of the constants  $C_s^s$ ,  $D_{s+1}^s$ ,  $C_{s+2}^s$ , &c. (supposing for convenience that we are dealing with symmetrical types). For this purpose we may either make use of the formulæ of § 6, or we may make use of the formulæ of § 5 for the determination of the  $C$ 's, after which the  $D$ 's must be computed from equation (49). The latter method will be closely analogous to that used in § 9 of Part I., but the former is the more convenient when the numerical determination of the constants is required.

If we make use of the notation (64), the following relations may be deduced from the equations (52):—

$$\left. \begin{aligned} \frac{C_n^s}{D_{n-1}^s} &= \frac{2n+1}{(n+s)(n-1)^2} e_n^s, & \frac{C_n^s}{D_{n+1}^s} &= \frac{2n+1}{(n-s+1)(n+2)^2} E_n^s \\ \frac{D_n^s}{C_{n-1}^s} &= \frac{2n+1}{(n+s)(n-1)^2} f_n^s, & \frac{D_n^s}{C_{n+1}^s} &= \frac{2n+1}{(n-s+1)(n+2)^2} F_n^s \end{aligned} \right\} \dots (67).$$

But we have seen in the last section how the quantities  $e, f, E, F$  may be determined numerically. Hence the above formulæ allow us to compute the ratios of the constants  $C, D$ . One of these constants must be regarded as arbitrary, and the ratios of the others to it can then be computed. When the type under examination is that which corresponds to a root of the period-equation approximating to a root of  $M_n^s = 0$ , we select as the arbitrary constant of integration the quantity  $C_n^s$ , as the continued fractions  $e, f, E, F$  required to determine the ratios of the remaining constants to this one will then be free from singularities.

When these ratios have been determined we may substitute their values in the formulæ

$$\zeta = C_n^s e^{i(\lambda t + s\phi)} \left[ \dots + \frac{C_{n-4}^s}{C_n^s} P_{n-4}^s + \frac{C_{n-2}^s}{C_n^s} P_{n-2}^s + P_n^s + \frac{C_{n+2}^s}{C_n^s} P_{n+2}^s + \frac{C_{n+4}^s}{C_n^s} P_{n+4}^s + \dots \right]$$

$$\begin{aligned}
\sqrt{(1-\mu^2)} U &= \frac{i\lambda a}{h} C_n^s e^{i(\lambda t + s\phi)} \left[ \mu \left\{ \dots + \frac{C_{n-2}^s}{C_n^s} P_{n-2}^s + P_n^s + \frac{C_{n+2}^s}{C_n^s} P_{n+2}^s + \dots \right\} \right. \\
&\quad \left. - \left\{ \dots + \frac{D_{n-1}^s}{C_n^s} P_{n-1}^s + \frac{D_{n+1}^s}{C_n^s} P_{n+1}^s + \dots \right\} \right] \\
\sqrt{(1-\mu^2)} V &= \frac{2\omega a}{h} \mu C_n^s e^{i(\lambda t + s\phi)} \left[ \mu \left\{ \dots + \frac{C_{n-2}^s}{C_n^s} P_{n-2}^s + P_n^s + \frac{C_{n+2}^s}{C_n^s} P_{n+2}^s + \dots \right\} \right. \\
&\quad \left. - \left\{ \dots + \frac{D_{n-1}^s}{C_n^s} P_{n-1}^s + \frac{D_{n+1}^s}{C_n^s} P_{n+1}^s + \dots \right\} \right] \\
&\quad - \frac{s}{a\lambda} C_n^s e^{i(\lambda t + s\phi)} \left[ \dots + \frac{C_{n-2}^s}{C_n^s} g_{n-2} P_{n-2}^s + g_n P_n^s + \frac{C_{n+2}^s}{C_n^s} g_{n+2} P_{n+2}^s + \dots \right]
\end{aligned}$$

and we shall obtain expressions for the height of the surface-waves and the velocity-components.

I have not thought it worth while to compute any of these series in detail, as the general character of them may be inferred from the series computed in Part I. for the special case where  $s = 0$ . For large values of  $n$ , and even for comparatively small values of  $n$  when  $hg/4\omega^2 a^2$  is large, the quantities  $C$  will rapidly diminish as we pass away in either direction from  $C_n^s$ . Hence the most important term in the series for  $\zeta$  will be that involving  $P_n^s$ , and this term will in general sufficiently predominate to decide the number and approximate position of the nodal parallels of latitude.

If we neglect  $C_{r-1}^s$  in comparison with  $C_{r+1}^s$  and suppose that  $\omega$  is small in comparison with  $\lambda$ , the formula (49) gives

$$r(r+s+1)C_{r+1}^s = (r+1)(2r+3)D_r^s$$

and therefore  $D_r^s$  will be of the same order of magnitude as  $C_{r+1}^s$ . Hence, when  $r$  is less than  $n$ ,  $D_r^s$  will be of the same order of magnitude of  $C_{r+1}^s$ , and similarly it may be seen that when  $r$  is greater than  $n$ ,  $D_r^s$  will be of the same order of magnitude as  $C_{r-1}^s$ . Thus the predominant terms in the expressions for the velocity-components will be those involving  $C_n^s$ ,  $D_{n-1}^s$ ,  $D_{n+1}^s$ .

The exponential factors indicate that the type of motion involved will consist of waves propagated round the sphere with uniform angular velocity  $\lambda/s$  about the polar axis, and that there will be  $s$  crests or troughs on each parallel. Positive values of  $\lambda$  will correspond with waves propagated in the opposite direction to the rotation, that is westwards, while negative values will correspond with easterly waves. The paths of the fluid particles will be ellipses with their axes directed along the meridians and parallels.

### § 11. *Oscillations of the Second Class.*

In dealing with the oscillations of the second class we proceed in the same manner

as before, retaining  $D_n^s$  as the arbitrary constant of integration when the type under consideration is that whose period is approximately given by the formula

$$n(n+1) - 2\omega s/\lambda = 0.$$

For we may anticipate that this quantity will predominate over the others, at least when the depth is sufficiently large or the angular velocity of rotation sufficiently small. The ratios of the remaining constants to  $D_n^s$  may then be computed from the formulæ (67), and on substituting these ratios in the equations

$$\begin{aligned}\zeta &= D_n^s e^{i(\lambda t + s\phi)} \left[ \dots + \frac{C_{n-3}^s}{D_n^s} P_{n-3}^s + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \frac{C_{n+3}^s}{D_n^s} P_{n+3}^s + \dots \right] \\ \sqrt{(1-\mu^2)} U &= \frac{i\lambda a}{h} D_n^s e^{i(\lambda t + s\phi)} \left[ \mu \left\{ \dots + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \dots \right\} \right. \\ &\quad \left. - \left\{ \dots + \frac{D_{n-2}^s}{D_n^s} P_{n-2}^s + P_n^s + \frac{D_{n+2}^s}{D_n^s} P_{n+2}^s + \dots \right\} \right] \\ \sqrt{(1-\mu^2)} V &= \frac{2\omega a \mu}{h} D_n^s e^{i(\lambda t + s\phi)} \left[ \mu \left\{ \dots + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \dots \right\} \right. \\ &\quad \left. - \left\{ \dots + \frac{D_{n-2}^s}{D_n^s} P_{n-2}^s + P_n^s + \frac{D_{n+2}^s}{D_n^s} P_{n+2}^s + \dots \right\} \right] \\ &\quad - \frac{s}{a\lambda} D_n^s e^{i(\lambda t + s\phi)} \left[ \dots + \frac{C_{n-1}^s}{D_n^s} g_{n-1} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} g_{n+1} P_{n+1}^s + \dots \right],\end{aligned}$$

we obtain expressions for the height of the surface-waves and the velocity-components. The values of  $\lambda$  for the oscillations of this class being always positive, the direction of the wave-propagation will always be westwards.

By way of numerical illustration I have computed the series for  $\zeta$  corresponding to the case  $n=3, s=1$ , and to the values  $\frac{1}{40}, \frac{1}{20}, \frac{1}{10}, \frac{1}{5}$  for  $hg/4\omega^2 a^2$ . In these four cases the series within the square brackets are found to be

$$\begin{aligned}&-2.977 P_2^1 - 0.1880 P_4^1 + 0.3753 P_6^1 - 0.0916 P_8^1 + 0.01153 P_{10}^1 \\ &\quad - 0.00093 P_{12}^1 + 0.000052 P_{14}^1 - 0.000002 P_{16}^1 + \dots \\ &-1.4735 P_2^1 - 0.3260 P_4^1 + 0.11690 P_6^1 - 0.01309 P_8^1 + 0.00080 P_{10}^1 \\ &\quad - 0.000032 P_{12}^1 + 0.000001 P_{14}^1 - \dots \\ &-0.7296 P_2^1 - 0.2248 P_4^1 + 0.03159 P_6^1 - 0.00168 P_8^1 + 0.000051 P_{10}^1 \\ &\quad - 0.000001 P_{12}^1 + \dots \\ &-0.3617 P_2^1 - 0.1277 P_4^1 + 0.00814 P_6^1 - 0.00021 P_8^1 + 0.000003 P_{10}^1 - \dots\end{aligned}$$

It will be seen that as the depth increases or  $\omega$  diminishes the coefficients all become smaller, while the convergence of the series improves. It may be inferred as in the last section that  $C_r^s/D_{r+1}^s$  will tend towards the limit zero when  $r$  is less than



$n$ , while  $C_{r+1}^s/D_r^s$  will tend towards a finite limit. In like manner when  $r$  is greater than  $n$ ,  $C_r^s/D_{r-1}^s$  will tend towards zero while  $C_{r-1}^s/D_r^s$  will tend to a finite limit.

Let us examine the limiting forms assumed by  $\zeta$ ,  $U$ ,  $V$ , when the rotation is annulled. On putting  $n(n+1) - 2\omega s/\lambda = 0$  the relations (51) give

$$L' \omega^2 M_r^s = -r^2(r+1)^2 \frac{hg_r}{4a^2},$$

$$L' N_r^s = r(r+1) - n(n+1) = (r-n)(r+n+1).$$

Thus from (52) we obtain, if  $r < n$ ,

$$L' \frac{C_r^s}{\omega^2 D_{r+1}^s} = -\frac{4a^2}{g_r h} \frac{(r+s+1)}{(r+1)^2(2r+3)}, \quad L' \frac{C_{r+1}^s}{D_r^s} = \frac{(r-n)(r+n+1)(2r+3)}{r^2(r+s+1)},$$

and if  $r > n$ ,

$$L' \frac{C_r^s}{\omega^2 D_{r-1}^s} = -\frac{4a^2}{g_r h} \frac{(r-s)}{r^2(2r-1)}, \quad L' \frac{C_{r-1}^s}{D_r^s} = \frac{(r-n)(r+n+1)(2r-1)}{(r+1)^2(r-s)}.$$

Hence if we retain only the most significant terms and put  $\omega D_n^s = \Delta_n^s$ , we find

$$\begin{aligned} \zeta &= -\omega \Delta_n^s e^{i(\lambda t + s\phi)} \left[ \frac{4a^2}{hg_{n-1}} \frac{(n+s)}{n^2(2n+1)} P_{n-1}^s + \frac{4a^2}{hg_{n+1}} \frac{(n-s+1)}{(n+1)^2(2n+1)} P_{n+1}^s \right] \\ \sqrt{(1-\mu^2)} U &= -\frac{2isa}{n(n+1)h} \Delta_n^s e^{i(\lambda t + s\phi)} P_n^s \\ \sqrt{(1-\mu^2)} V &= -\frac{2a}{h} \mu \Delta_n^s e^{i(\lambda t + s\phi)} P_n^s \\ &\quad + \frac{n(n+1)}{2a} \Delta_n^s e^{i(\lambda t + s\phi)} \left[ \frac{4a^2}{h} \frac{n+s}{n^2(2n+1)} P_{n-1}^s + \frac{4a^2}{h} \frac{(n-s+1)}{(n+1)^2(2n+1)} P_{n+1}^s \right]. \end{aligned}$$

If therefore we suppose that  $\omega$  reduces to zero while  $\Delta_n^s$  remains finite,  $\zeta$  will reduce to zero, but the velocity-components will tend to finite limits given by

$$\begin{aligned} \sqrt{(1-\mu^2)} U &= -\frac{2isa}{n(n+1)h} \Delta_n^s P_n^s(\mu) e^{is\phi}, \\ \sqrt{(1-\mu^2)} V &= -\frac{2a}{h} \Delta_n^s \left[ \frac{n-s+1}{(n+1)(2n+1)} P_{n+1}^s - \frac{(n+s)}{n(2n+1)} P_{n-1}^s \right] e^{is\phi} \\ &= \frac{2a}{n(n+1)h} \Delta_n^s (1-\mu^2) \frac{dP_n^s}{d\mu} e^{is\phi} \end{aligned}$$

in virtue of (7).

Hence the steady motions to which the oscillations of the second class reduce when the rotation is annulled involve no deformation of the free surface. This of course may readily be verified by a direct method. For if  $u$ ,  $v$ ,  $w$  denote the velocity-components referred to fixed rectangular axes, it may be seen that all the conditions of the problem will be satisfied by



$$u = y \frac{\partial \chi}{\partial z} - z \frac{\partial \chi}{\partial y}, \quad v = z \frac{\partial \chi}{\partial x} - x \frac{\partial \chi}{\partial z}, \quad w = x \frac{\partial \chi}{\partial y} - y \frac{\partial \chi}{\partial x},$$

where  $\chi$  is an arbitrary function of  $x, y, z$  independent of  $t$ . These solutions make  $ux + vy + wz = 0$ , and therefore involve no deformation of the surface. If we refer them to polar co-ordinates, they are equivalent to

$$U = \frac{1}{\sin \theta} \frac{\partial \chi}{\partial \phi}, \quad V = \frac{\partial \chi}{\partial \theta}, \quad W = 0,$$

or

$$\sqrt{(1 - \mu^2)} U = \frac{\partial \chi}{\partial \phi}, \quad \sqrt{(1 - \mu^2)} V = - (1 - \mu^2) \frac{\partial \chi}{\partial \mu}.$$

This solution becomes identical with that found above if

$$\chi = -\frac{2a}{n(n+1)h} \Delta_n^* P_n^*(\mu) e^{i\pi b}.$$

For the special case  $s = 0$ , the values of  $\lambda$  corresponding with the roots of the second class will be zero, and the corresponding types of motion steady, even when  $\omega$  is finite. The types of steady motion will however involve a deformation of the free surface in all cases where the angular velocity of rotation is different from zero. These cases have been fully discussed in Part I., § 14.

### § 12. *Forced Oscillations. General Analytical Solution.*

The problem of the forced oscillations involves the determination of the quantities  $C_n^s$  in terms of  $\gamma_n^s$  from the equation (31) or (40). Dealing first with the case of uniform depth, and supposing that the disturbing potential involves only a single term  $\gamma_n^s P_n^s(\mu) e^{i(\lambda t + s\phi)}$ , we have to solve the simultaneous equations

[illegible]

with the condition that  $\lim_{r \rightarrow \infty} L^t C_r = 0$ .

From these we deduce, as in § 5, that

$$\frac{C_{r-2}^s}{C_r^s} = \frac{H_{r-2}^s}{x_{r-2}^s} (r < n+1),$$

$$\frac{C_{r+1}^s}{C_r^s} = \frac{K_{r+1}^s}{y_r^s} (r \geq n-1).$$

Thus the equation which involves  $\gamma_n^s$  becomes

$$C_n^s (H_{n-2}^s - L_n^s + K_{n+2}^s) = h\gamma_n^s / 4\omega^2 a^2;$$

whence

$$C_n^s = \frac{h\gamma_n^s / 4\omega^2 a^2}{H_{n-2}^s - L_n^s + K_{n+2}^s}.$$

We may now deduce  $C_{n-2}^s, C_{n-4}^s, \dots, C_{n+2}^s, C_{n+4}^s, \dots$  by means of the formulæ

$$\begin{aligned} \frac{C_{n-2}^s}{C_n^s} &= \frac{H_{n-2}^s}{\alpha_{n-2}^s}, \quad \frac{C_{n-4}^s}{C_n^s} = \frac{H_{n-2}^s H_{n-4}^s}{\alpha_{n-2}^s \alpha_{n-4}^s}, \quad \&c., \\ \frac{C_{n+2}^s}{C_n^s} &= \frac{K_{n+2}^s}{\beta_n^s}, \quad \frac{C_{n+4}^s}{C_n^s} = \frac{K_{n+2}^s K_{n+4}^s}{\beta_n^s \beta_{n+2}^s}, \quad \&c., \end{aligned}$$

and therefore

$$\begin{aligned} \zeta &= \frac{h\gamma_n^s}{4\omega^2 a^2} \frac{e^{i(\lambda t + s\phi)}}{H_{n-2}^s - L_n^s + K_{n+2}^s} \left[ \dots + \frac{H_{n-2}^s H_{n-4}^s}{\alpha_{n-2}^s \alpha_{n-4}^s} P_{n-4}^s \right. \\ &\quad \left. + \frac{H_{n-2}^s}{\alpha_{n-2}^s} P_{n-2}^s + P_n^s + \frac{K_{n+2}^s}{\beta_n^s} P_{n+2}^s + \frac{K_{n+2}^s K_{n+4}^s}{\beta_n^s \beta_{n+2}^s} P_{n+4}^s + \dots \right] \end{aligned}$$

It should be noticed that the term involving  $P_n^s$  need not here be the predominating term of the series within the square brackets; it will however be so when the value of  $\lambda$  for the disturbing force is in the neighbourhood of those roots of the period-equation which approximate to roots of  $L_n^s = 0$ . But if  $\lambda$  have as its value another root of the period-equation, say, for example, one which approximates to a root of  $L_m^s = 0$ , the series within square brackets will differ only by a constant multiplier from the series in the expression for the tide-height for the corresponding type of free oscillation, since the equations which determine the ratios of the  $C$ 's are evidently the same in both cases. Hence, for values of  $\lambda$  in the neighbourhood of this one, the predominating term will be that which involves  $P_m^s$ . Consequently, when  $n$  and  $m$  differ widely, the numerical computation of these series will become laborious; for as we proceed away from the term containing  $P_n^s$  towards that involving  $P_m^s$ , the terms will at first increase in magnitude, and the convergence of the series will not assert itself until the term depending on  $P_m^s$  has been passed. These circumstances will not however occur in any of the cases of more practical interest.

Whatever be the nature of the disturbing potential it will be possible to expand its surface-value in a series of surface-harmonics. Thus the most general value of  $v$  which can occur may be expressed in the form

$$\sum_{s=-\infty}^{\infty} \sum_{n=s}^{\infty} \left[ \gamma_n^s e^{i(\lambda t + s\phi)} + \delta_n^s e^{-i(\lambda t + s\phi)} \right] P_n^s(\mu).$$

The deformation of the surface due to each term may be calculated independently

and the results superposed, so that the deformation at time  $t$  resulting from this disturbing force will be given by

$$\zeta = \frac{h}{4\omega^2 a^2} \sum_{s=0}^{\infty} \sum_{n=s}^{\infty} \left[ \frac{\gamma_n^s e^{i(\lambda t + s\phi)} + \delta_n^s e^{-i(\lambda t + s\phi)}}{H_{n-2}^s - L_n^s + K_{n+2}^s} \left\{ \dots + \frac{H_{n-2}^s}{\mathcal{I}_{n-2}^s} P_{n-2}^s + P_n^s + \frac{K_{n+2}^s}{\mathcal{I}_{n+2}^s} P_{n+2}^s + \dots \right\} \right].$$

The corresponding formulæ when the depth is variable may be obtained by replacing  $h$ ,  $\gamma_n^s$ ,  $\delta_n^s$ ,  $H_n^s$ ,  $K_n^s$ ,  $L_n^s$ ,  $x_n^s$ ,  $y_n^s$ , by  $\kappa$ ,  $G_n^s$ ,  $\Delta_n^s$ ,  $\mathfrak{H}_n^s$ ,  $\mathfrak{K}_n^s$ ,  $\mathfrak{L}_n^s$ ,  $\xi_n^s$ ,  $\eta_n^s$  respectively, where  $\kappa$ ,  $G_n^s$ ,  $\mathfrak{L}_n^s$ ,  $\xi_n^s$ ,  $\eta_n^s$  are defined by the equations (35), (41), (42), and

$$\begin{aligned} \mathfrak{H}_n^s &= \frac{\xi_n^s \eta_n^s}{\mathfrak{L}_n^s} - \frac{\xi_{n-2}^s \eta_{n-2}^s}{\mathfrak{L}_{n-2}^s} - \dots \\ \mathfrak{K}_n^s &= \frac{\xi_{n-2}^s \eta_{n-2}^s}{\mathfrak{L}_n^s} - \frac{\xi_n^s \eta_n^s}{\mathfrak{L}_{n+2}^s} - \dots \end{aligned}$$

while  $\Delta_n^s$  is obtained by writing  $\delta_n^s$  in place of  $\gamma_n^s$  in the right-hand member of (42).

### § 13. *Classification of Tides.*

In the last section we have reduced the problem of the evaluation of the forced tides due to any disturbing force to that of the development of the disturbing potential as a series of surface-harmonics. This development for the case of the disturbance of the ocean due to the attraction of the sun and moon has been already dealt with, and reference may be made to Professor DARWIN's article in the 'Encyclopædia Britannica' for a full account of it. We give here a short summary of the principal results of which we propose to make use.

The principal part of the disturbing potential will consist of spherical harmonic functions of the second order, and when expressed by means of zonal and tesseral harmonics the terms which occur will be of three types, characterized by the rank of the harmonic involved.

For the first type  $s = 0$ , and the corresponding tides will be expressible as series of zonal harmonics. For these types the value of  $\lambda$  will be small in comparison with that of  $\omega$ , so that the period of the disturbance is long compared with a sidereal day. The terms will cease to be oscillatory when the orbital motion of the disturbing body is neglected. The tides generated by these parts of the disturbing potential have been already dealt with in Part I.

The terms of the second type are those for which  $s = 1$ ; in certain of these terms the value of  $\lambda$  will be equal to  $\omega$ , so that the period is rigorously equal to a sidereal day, while in the rest the period will reduce to a sidereal day when the orbital motion of the disturbing body is neglected. If  $n$  denote the mean orbital motion of the luminary, the "speeds" of the principal diurnal tides will be  $\omega$  and  $\omega - 2n$ . We propose to neglect the sun's orbital motion, so that for each of the principal solar



diurnal constituents we shall suppose that  $\lambda = \omega$  rigorously. The same analysis will then apply to one of the lunar diurnal constituents, while in order to illustrate the effect of the departure of the period from exact coincidence with a sidereal day we shall evaluate independently the lunar diurnal constituent for which

$$\lambda (= \omega - 2n) = 0.92700 \omega.$$

The principal part of the tidal oscillations will be due to the third part of the disturbing potential, which involves harmonics of rank 2. The period for the tides due to these terms will differ but slightly from half a sidereal day, and will reduce to half a day exactly when the orbital motion of the disturbing body is neglected. We shall therefore assume that  $\lambda = 2\omega$  rigorously for the solar semi-diurnal tides, while we shall take the value

$$\lambda/2\omega = 1 - n/\omega = 0.96350$$

as typical of the lunar semi-diurnal constituents. The analysis applied to the solar constituents will be rigorously applicable to the sidereal luni-solar semi-diurnal tide usually denoted by the symbol  $K_2$ .

#### § 14. *Special Cases.*

Instead of making use of the equation (31) as we have done in § 12, we may of course compute the forced tides by means of the equations (49), (50) of § 6, determining incidentally the constants  $D_s^s$ . Thus, if we suppose that all the  $\gamma$ 's are zero except  $\gamma_{s+1}^s$ , we have the following equations for the determination of  $D_s^s$ ,  $C_{s+1}^s$ ,  $D_{s+2}^s$ , &c.

$$\begin{aligned} -N_s^s D_s^s + \frac{s^2(2s+1)}{2s+3} C_{s+1}^s &= 0 \\ \frac{(s+2)^2 \cdot 1}{2s+1} D_s^s - M_{s+1}^s C_{s+1}^s + \frac{(s+1)^2(2s+2)}{2s+5} D_{s+2}^s &= (s+1)^2 (s+2)^2 \frac{h\gamma_{s+1}^s}{4\omega^2 a^2} \\ \frac{(s+3)^2 \cdot 2}{2s+3} C_{s+1}^s - N_{s+2}^s D_{s+2}^s + \frac{(s+2)^2(2s+3)}{2s+7} C_{s+3}^s &= 0 \\ \dots \dots \dots \end{aligned}$$

where the terms on the right are all zero, except in the second equation. Now it is evident that if  $N_s^s = 0$ , or

$$\lambda = \frac{2\omega}{s+1}$$

all these equations will be satisfied if

$$D_s^s = (s+1)^2 (2s+1) \frac{h\gamma_{s+1}^s}{4\omega^2 a^2}$$

and

$$C_{s+1}^s = D_{s+2}^s = C_{s+3}^s = \dots = 0.$$



It follows that if the disturbing potential be of order  $s + 1$  and rank  $s$ , and the period be  $\frac{1}{2}(s + 1)$  days, the tide will involve no rise and fall at the free surface, but will consist merely of horizontal currents. If we put  $s = 1$  the requisite period will be rigorously equal to a sidereal day, and the circumstances will correspond with those we have assumed to characterize the solar diurnal tides. We therefore conclude that in an ocean of uniform depth the solar diurnal tides will involve no rise and fall. We shall however see hereafter that for certain of the lunar diurnal tides the difference between the period and a sidereal day may be sufficiently great to render the rise and fall of considerable importance, unless the depth is very small.

We have seen in § 4 that the formulæ applicable to the case of variable depth may be deduced from those applicable to the case of uniform depth by replacing  $h$  by  $\kappa$  and  $C_n^s$  by  $C_n^s + \frac{l}{4\omega^2 a^2} \left[ n(n + 1) + \frac{2\omega s}{\lambda} \right] \Gamma_n^s$ .

But if  $\gamma_n^s = 0$ ,  $\Gamma_n^s = -g_n C_n^s$ , and therefore when  $\lambda = 2\omega/(s + 1)$  and all the  $\gamma$ 's are zero except  $\gamma_{s+1}^s$ ,  $C_{s+3}^s$ ,  $C_{s+5}^s$ , &c., will all be zero, while

$$C_{s+1}^s + \frac{l}{4\omega^2 a^2} \{ (s + 1)(s + 2) + s(s + 1) \} \Gamma_{s+1}^s = 0;$$

whence we obtain

$$C_{s+1}^s = - \frac{2(s + 1)^2 \frac{l\gamma_{s+1}^s}{4\omega^2 a^2}}{1 - 2(s + 1)^2 \frac{lg_{s+1}}{4\omega^2 a^2}}.$$

Thus the forced tide will be similar in type to the disturbing potential which produces it, though it will be inverted unless  $l < 0$  or  $> \frac{2\omega^2 a^2}{(s + 1)^2 g_{s+1}}$ .

This theorem will admit of application to the solar diurnal tides on putting  $s = 1$ , in which case it reduces to a theorem given by LAPLACE.\* The critical value of  $l$ , for a system comparable with the earth, is considerably greater than such depths as occur on the earth; hence, for depths comparable with that of the ocean, the diurnal tides will be inverted when the ocean is deeper at the equator than at the poles; they will however be direct when  $l$  is negative, so that the ocean is deeper at the poles than at the equator.

It is evident that the equations typified by (31) will all be satisfied with the  $C$ 's all zero if  $h = 0$ , since in this case the right-hand members will reduce to zero. In like manner the corresponding equations which apply to an ocean of variable depth will all be satisfied when  $\kappa = 0$ , if for all values of  $n$

$$C_n^s + \frac{l}{4\omega^2 a^2} \left\{ n(n + 1) + \frac{2\omega s}{\lambda} \right\} \Gamma_n^s = 0.$$

\* LAMB, 'Hydrodynamics,' § 212.

This equation leads to

$$C_n = - \frac{\frac{lg_n}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\}}{1 - \frac{lg_n}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\}}.$$

There exists then a certain law of depth, depending on the period, for which the tide will always be similar in type to the disturbing potential which produces it. This law of depth is expressed by the formula

$$h = l \left( \frac{\lambda^2}{4\omega^2} - \mu^2 \right).$$

If we suppose that  $\lambda = 2\omega$  rigorously, it reduces to

$$h = l (1 - \mu^2),$$

so that the depth will be a maximum at the equator, and will gradually decrease on passing away from the equator to zero at the poles.\*

For other values of  $\lambda$  the formula for  $h$  will make the depth negative at some parts of the surface unless  $l$  is positive and  $\lambda > 2\omega$ . The latter condition does not occur with any of the leading tidal constituents,† but it would hold good in the case of the semi-diurnal tides due to a satellite whose motion in its orbit was retrograde. If however we neglect the mutual attraction of the waters, the theorem under discussion may be supposed to apply to an ocean covering that part of the surface over which  $h$  is positive, the remaining parts of the surface being supposed to consist of continents. When  $l$  is positive, these continents must, for the lunar semi-diurnal tides, reduce to small circumpolar islands, while for the same tides when  $l$  is negative they will cover the whole globe with the exception of two small seas surrounding the poles.

For the diurnal-tides, the shores must coincide with parallels of latitude approximately  $30^\circ$  north and south of the equator, while for the tides of long period the appropriate forms of sea will be bounded by two parallels nearly coincident with the equator. A change in the sign of  $l$  in all cases involves an interchange between the seas and the land.

It should be noticed that the formulæ (12) make  $U, V$  infinite at the points where  $\mu = \pm \frac{\lambda}{2\omega}$ , that is, at the shores. This indicates that the neglect of the squares of the velocities is not allowable in the neighbourhood of the coasts no matter how small the amplitude of vibration may be, and seems to point to the existence of "breakers" as an essential accompaniment of the tides.

\* Cf. LAMB, 'Hydrodynamics,' § 213.

† There will be small tidal constituents depending on higher powers of the moon's parallax for which  $\lambda$  exceeds  $2\omega$ . Cf. DARWIN, "Harmonic Analysis of Tidal Observations." 'Brit. Assoc. Report,' 1883 (Southport), § 3.

§ 15. *Solar Semi-diurnal Tides.*

In the last section we have considered some special cases in which the tide-height is expressible by a single term. In general it will however only be expressible by a series of terms. It may be shown, as in § 5 of Part I., that this series will be finite when the law of depth is such that

$$1 - \frac{lg_n}{4\omega^2 a^2} \left\{ n(n+1) + \frac{2\omega s}{\lambda} \right\} = 0,^*$$

where  $n$  is an integer,  $n - s$  being even or odd according as we are dealing with the symmetrical or asymmetrical types of rank  $s$ . The values of  $l$  determined from this equation will in general involve  $\lambda$ , but for large values of  $n$ , they will approximate to the same values of  $l$  as those required for the expression in finite terms of the long-period tides, since for such values  $2\omega s/\lambda$  will be small compared with  $n(n+1)$ .

In other cases the expression for the tide-height will involve infinite series. We deal in the present section with the case where  $l$  is zero, so that the depth is uniform.

The numerical computation of the semi-diurnal tides admits of special simplicity when  $\lambda = 2\omega$  exactly. Putting  $\lambda = 2\omega$ ,  $s = 2$ , in the formulæ (30), (29) we obtain

$$x_n^2 = \frac{n-1}{(n+3)(2n+1)(2n+3)}, \quad y_n^2 = \frac{n+4}{n(2n+3)(2n+5)},$$

$$\Lambda_n^2 = \frac{(n-1)(n+2)}{n^2(n+1)^2} - \frac{(n-1)^2(n+2)}{n^2(n+1)(2n-1)(2n+1)} - \frac{(n+2)^2(n-1)}{n(n+1)^2(2n+1)(2n+3)},$$

the last of which gives on reduction

$$\Lambda_n^2 = \frac{2(n-1)(n+2)}{n(n+1)(2n-1)(2n+3)}.$$

This general formula for  $\Lambda_n^2$  fails to hold when  $n = 2$ ; for, in this case,  $n - s$  and  $n(n-1) - 2\omega s/\lambda$  are both zero, and therefore the second fraction involved in the expression for  $\Lambda_n^2$  is indeterminate. To determine its limiting form we must first suppose the period slightly different from half a day, so that  $\lambda$  is not rigorously equal to  $2\omega$ ; the formula (29) then gives

$$\Lambda_2^2 = \frac{\lambda^2}{4\omega^2} \frac{2.3 - 4\omega/\lambda}{2^2.3^2} - \frac{4^2.1.5}{3^2.5.7 \{3.4 - 4\omega/\lambda\}},$$

which, on putting  $\lambda = 2\omega$ , reduces to

$$\Lambda_2^2 = \frac{4}{2^2.3^2} - \frac{1.4^2}{3^2.7.10} = \frac{3}{5.7}.$$

The formulæ for  $x_n^2$ ,  $y_n^2$ ,  $\Lambda_n^2$  are now in a convenient form for logarithmic computation, and we may readily deduce the following numerical values

\* Cf. LAPLACE, 'Méc. Cél.,' Part I., Book IV., § 7.



$$\Lambda_2^2 = 0.085714$$

$$\Lambda_4^2 = 0.0233766$$

$$\Lambda_6^2 = 0.011544$$

$$\Lambda_8^2 = 0.006823$$

$$\Lambda_{10}^2 = 0.004494$$

$$\Lambda_{12}^2 = 0.003179$$

$$\Lambda_{14}^2 = 0.002367$$

$$\Lambda_{16}^2 = 0.001830$$

$$\log x_2^2 = 3.75696$$

$$\log x_4^2 = 3.63639$$

$$\log x_6^2 = 3.45469$$

$$\log x_8^2 = 3.29451$$

$$\log x_{10}^2 = \bar{3}.1564$$

$$\log x_{12}^2 = \bar{3}.0360$$

$$\log x_{14}^2 = \bar{4}.9297$$

$$\log y_2^2 = \bar{2}.67778$$

$$\log y_4^2 = \bar{2}.14569$$

$$\log y_6^2 = \bar{3}.81531$$

$$\log y_8^2 = \bar{3}.57512$$

$$\log y_{10}^2 = \bar{3}.3865$$

$$\log y_{12}^2 = \bar{3}.2312$$

$$\log y_{14}^2 = \bar{3}.0993$$

Taking  $hg/4\omega^2\alpha^2 = \frac{1}{40}$ , and  $\rho/\sigma_0 = 0.18093$ , the formula (32) leads to

$$L_2^2 = +0.063428$$

$$L_4^2 = -0.0001156$$

$$L_6^2 = -0.012412$$

$$L_8^2 = -0.017379$$

$$L_{10}^2 = -0.019860$$

$$L_{12}^2 = -0.02128$$

$$L_{14}^2 = -0.02216$$

$$L_{16}^2 = -0.0228$$

Thus if we neglect  $K_{18}^2$ , and make use of the formula

$$K_n^s = \frac{C_{n-2}^s y_{n-2}^s}{L_n^s - K_{n+2}^s},$$

we obtain in succession

$$\log K_{16}^2 = n \bar{5}.671$$

$$\log K_{14}^2 = n \bar{5}.9226$$

$$\log K_{12}^2 = n \bar{4}.2165$$

$$\log K_{10}^2 = n \bar{4}.5753$$

$$\log K_8^2 = n \bar{3}.03947$$

$$\log K_6^2 = n \bar{3}.72835$$

$$\log K_4^2 = \bar{2}.71588$$

and therefore, since  $C_{n+2}^s/C_n^s = K_{n+2}^s/y_n^s$ , we find

$$\log (C_4^2/C_2^2) = 0.03810$$

$$\log (C_6^2/C_4^2) = n \bar{1}.58266$$

$$\log (C_8^2/C_6^2) = n \bar{1}.22417$$

$$\log (C_{10}^2/C_8^2) = n \bar{1}.0002$$

$$\log (C_{12}^2/C_{10}^2) = n \bar{2}.8300$$

$$\log (C_{14}^2/C_{12}^2) = n \bar{2}.6914$$

$$\log (C_{16}^2/C_{14}^2) = n \bar{2}.572$$



But if  $\mathfrak{C}_n P_n^s(\mu) e^{i(\lambda t + s\phi)}$  denote the height of the 'equilibrium' tide resulting from a disturbing potential  $\gamma_n^s P_n^s(\mu) e^{i(\lambda t + s\phi)}$ , we have

$$\gamma_n^s = g_n \mathfrak{C}_n^s.$$

Therefore

$$C_2^2 = - \frac{h\gamma_2^2/4\omega^2 a^2}{L_2^2 - K_4^2} = - \frac{hg_2/4\omega^2 a^2}{L_2^2 - K_4^2} \mathfrak{C}_2^2$$

and on introducing the numerical values for  $h, g_2, L_2^2, K_4^2$  we obtain

$$C_2^2 = -1.9476 \mathfrak{C}_2^2.$$

Thus we have :

$$\begin{aligned} \log (C_4^2/\mathfrak{C}_2^2) &= n \, 0.32760 & \log (C_{12}^2/\mathfrak{C}_2^2) &= n \, \bar{4}.9646 \\ \log (C_6^2/\mathfrak{C}_2^2) &= \bar{1}.91025 & \log (C_{14}^2/\mathfrak{C}_2^2) &= \bar{5}.6560 \\ \log (C_8^2/\mathfrak{C}_2^2) &= n \, \bar{1}.13442 & \log (C_{16}^2/\mathfrak{C}_2^2) &= n \, \bar{6}.228; \\ \log (C_{10}^2/\mathfrak{C}_2^2) &= \bar{2}.1346 \end{aligned}$$

and therefore if we suppose the exponential or trigonometrical factor to be involved in  $\mathfrak{C}_2^2$ , so that the height of the equilibrium tide is expressed by  $\mathfrak{C}_2^2 P_2^2(\mu)$ , the height of the corresponding dynamical tide is given by

$$\begin{aligned} \zeta &= \mathfrak{C}_2^2 [-1.9476 P_2^2 - 2.12617 P_4^2 + 0.81331 P_6^2 - 0.13628 P_8^2 \\ &\quad + 0.01363 P_{10}^2 - 0.000922 P_{12}^2 + 0.000045 P_{14}^2 - 0.000002 P_{16}^2 + \dots]. \end{aligned}$$

For points lying on the equator we have  $\mu = 0$ , and it may be shown that in this case

$$P_{2n}^2 = (-)^{n+1} \frac{3.5 \dots (2n+1)}{2.4 \dots (2n-2)},$$

whence we deduce

$$\begin{aligned} \log P_2^2 &= 0.47712 & \log P_{10}^2 &= 1.4325 \\ \log P_4^2 &= n \, 0.87506 & \log P_{12}^2 &= n \, 1.5464 \\ \log P_6^2 &= 1.11810 & \log P_{14}^2 &= 1.643 \\ \log P_8^2 &= n \, 1.29419 & \log P_{16}^2 &= n \, 1.73. \end{aligned}$$

Thus the values of the successive terms of the series within the square brackets for points at the equator are

$$-5.8428 + 15.9463 + 10.6746 + 2.6829 + 0.3690 + 0.0324 + 0.0020 + 0.0001,$$

which, on addition, give 23.8645. But the height of the corresponding equilibrium-

tide at the equator is  $3\mathfrak{C}_2^2$ , and therefore the ratio of the height of the tide to that of the corresponding equilibrium tide at the equator is

$$+ 7.9548.$$

In like manner the tide-height in any other latitude may be compared with the equilibrium tide-height, but the process will be laborious in the absence of tables of the functions  $P_n^s$ .\*

The above example has been treated in some detail as illustrative of the method to be employed for the computation of the forced tides by infinite series; the chief part of the labour is involved in the determination of the quantities  $x_n^s$ ,  $y_n^s$ ,  $\Lambda_n^s$ , but when once these have been determined, since they do not involve the depth, it is easy without much additional labour to multiply cases for different depths. Besides the case already considered, which corresponds to a depth of about 7260 feet, I have computed the series for depths of 14,520, 29,040, and 58,080 feet, corresponding with the values  $\frac{1}{20}$ ,  $\frac{1}{10}$ , and  $\frac{1}{5}$  for  $hg/4\omega^2\alpha^2$ . For these depths the series within the square brackets is replaced by

$$- 0.83227P_2^2 + 0.21694P_4^2 - 0.02615P_6^2 + 0.00180P_8^2 - 0.000080P_{10}^2 \\ + 0.000003P_{12}^2 + \dots,$$

$$- 191.925P_2^2 + 15.696P_4^2 - 0.8082P_6^2 + 0.0256P_8^2 - 0.0005P_{10}^2 + \dots$$

and

$$1.9610P_2^2 - 0.06823P_4^2 + 0.00164P_6^2 - 0.000025P_8^2 + \dots$$

respectively, giving for the ratio of the tide-height to the equilibrium tide-height at the equator the values

$$- 1.5016, \quad - 234.87, \quad + 2.1389.$$

When the depth of the ocean is greater than 58,080 feet the tides are therefore direct at the equator. They gradually increase in magnitude as the depth decreases, and become infinite and change sign for some critical value of the depth rather in excess of 29,040 feet after which, for further decrease of the depth, they remain inverted until a second critical value is reached which is somewhat greater than 7260 feet, when a second change of sign occurs. The very large coefficients which appear when  $hg/4\omega^2\alpha^2 = \frac{1}{10}$  indicate that for this depth there is a period of free oscillation of semi-diurnal type whose period differs but slightly from half-a-day. On reference to the tables of § 9 it will be seen that we have, in fact, evaluated this period as 12 hours 1 minute, while for the case  $hg/4\omega^2\alpha^2 = \frac{1}{40}$  we have found a period of 12 hours 5 minutes. We see then that though, when the period of the forced oscillation differs from that of one of the types of free oscillation by as little as one minute, the forced tide may be nearly 250 times as great as the corresponding equilibrium tide, a

\* The zonal harmonics from  $P_1$  up to  $P_7$  have been tabulated by GLAISHER, 'Brit. Assoc. Reports,' 1879, but I do not know of the existence of any Tables of the Tesseral Harmonics, with the exception of a few given by THOMSON and TAIT, 'Nat. Phil.,' vol. 2, § 784.

difference of 5 minutes between these periods will be sufficient to reduce the tide to less than ten times the corresponding equilibrium tide. It seems then that the tides will not tend to become excessively large unless there is very close agreement with the period of one of the free oscillations.

The critical depths for which the forced tides here treated of become infinite are those for which a period of free oscillation coincides exactly with 12 hours. They may be ascertained by putting  $\lambda = 2\omega$  in the period-equation for the free oscillations and treating this equation as an equation for the determination of  $h$ . The roots may be found by trial and error as in § 9, the approximate values with which to commence the trials being suggested by the discussion already given. The two largest roots are found to be given by

$$hg/4\omega^2 a^3 = 0.10049, \quad hg/4\omega^2 a^2 = 0.02545,$$

and the corresponding critical depths are about 29,182 feet and 7375 feet.

We have hitherto supposed that  $\rho/\sigma_0 = 0.18093$ , but for purposes of comparison I have also examined the case where  $\rho/\sigma_0 = 0$ , that is where the mutual attraction of the waters is neglected. The series for  $\zeta$  in this case become

$$\zeta = \mathfrak{C}_2^2 [1.0927 P_2^2 + 1.91817 P_4^2 - 0.66909 P_6^2 + 0.10701 P_8^2 - 0.01036 P_{10}^2 \\ + 0.000683 P_{12}^2 - 0.000033 P_{14}^2 + 0.000001 P_{16}^2 - \dots]$$

$$\zeta = \mathfrak{C}_2^2 [-1.0733 P_2^2 + 0.24502 P_4^2 - 0.02790 P_6^2 + 0.00185 P_8^2 - 0.000080 P_{10}^2 \\ + 0.000002 P_{12}^2 - \dots]$$

$$\zeta = \mathfrak{C}_2^2 [9.34370 P_2^2 - 0.70311 P_4^2 + 0.03449 P_6^2 - 0.00106 P_8^2 + 0.000022 P_{10}^2 - \dots]$$

$$\zeta = \mathfrak{C}_2^2 [1.7739 P_2^2 - 0.05750 P_4^2 + 0.00132 P_6^2 - 0.00020 P_8^2 + \dots]$$

or the depths 7260, 14,520, 29,040, 58,080 feet respectively. From these series we deduce as the ratio of the tide to the equilibrium tide at the equator the four values

$$-7.4343, \quad -1.8208, \quad +11.2595, \quad +1.9236,$$

results which agree, except in the third case, with the numbers given by Professor LAMB\* deduced from the numerical formulæ of LAPLACE.

It will be seen that in three cases out of the four here considered the effect of the mutual gravitation of the waters is to increase the ratio of the tide to the equilibrium tide. In two of the cases the sign is also reversed. This of course results from the fact that, whereas when  $\rho/\sigma_0 = 0.18093$  one of the periods of free oscillation is

\* By a careful re-computation of the semi-diurnal tide for the case  $\beta=10$  (notation of Professor LAMB) I find the following series more accurate than that given for  $\zeta/H'''$  :—

$$\nu^2 + 6.1915\nu^4 + 3.2447\nu^6 + 0.7234\nu^8 + 0.0919\nu^{10} + 0.0076\nu^{12} + 0.0004\nu^{14} + \dots$$

This series reduces to 11.2595 when  $\nu = 1$ , thus agreeing with the result obtained above.



rather greater than 12 hours, when  $\rho/\sigma_0 = 0$  the corresponding period will be less than 12 hours

### § 16. *Lunar Semi-diurnal Tides.*

A similar method to that of the last section may be used to evaluate the lunar semi-diurnal tides for which we take  $\lambda/2\omega = 0.96350$ . The arithmetical work is, however, more severe, in consequence of the fact that the quantities  $x_n^s, y_n^s, \Lambda_n^s$  must be evaluated from the formulæ (29), (30) which do not assume the simple forms obtained in the last section. Substituting in these formulæ the value of  $\lambda/2\omega$  quoted above we deduce

$$\begin{array}{ll} \Lambda_2^s = 0.075603 & \Lambda_{10}^s = 0.003845 \\ \Lambda_4^s = 0.019864 & \Lambda_{12}^s = 0.002750 \\ \Lambda_6^s = 0.009856 & \Lambda_{14}^s = 0.002028 \\ \Lambda_8^s = 0.005834 & \Lambda_{16}^s = 0.001968 ; \\ \\ \log x_2^s = \bar{3}.76026 & \log x_{10}^s = \bar{3}.1556 \\ \log x_4^s = \bar{3}.63756 & \log x_{12}^s = \bar{3}.0362 \\ \log x_6^s = \bar{3}.45530 & \log x_{14}^s = \bar{4}.9299 ; \\ \log x_8^s = \bar{3}.29488 & \\ \\ \log y_2^s = \bar{2}.68108 & \log y_{10}^s = \bar{3}.3867 \\ \log y_4^s = \bar{2}.14687 & \log y_{12}^s = \bar{3}.2313 \\ \log y_6^s = \bar{3}.81592 & \log y_{14}^s = \bar{3}.0994. \\ \log y_8^s = \bar{3}.57549 & \end{array}$$

Our procedure is now exactly similar to that of the last section. Thus, if the height of the equilibrium tide be

$$\mathfrak{C}_2^s P_2^s(\mu),$$

we find, when  $hg/4\omega^2\alpha^2 = \frac{1}{40}$  and  $\rho/\sigma_0 = 0.18093$ ,

$$\begin{aligned} \zeta = \mathfrak{C}_2^s [ & 0.10395 P_2^s + 0.57998 P_4^s - 0.19273 P_6^s + 0.03054 P_8^s \\ & - 0.002960 P_{10}^s + 0.000196 P_{12}^s - 0.000010 P_{14}^s + \dots ]. \end{aligned}$$

Similarly, when  $hg/4\omega^2\alpha^2 = \frac{1}{20}$ ,

$$\begin{aligned} \zeta = \mathfrak{C}_2^s [ & -1.0647 P_2^s + 0.24038 P_4^s - 0.02774 P_6^s + 0.001867 P_8^s \\ & - 0.000082 P_{10}^s + 0.000003 P_{12}^s - \dots ]; \end{aligned}$$

when  $hg/4\omega^2\alpha^2 = \frac{1}{10}$ ,

$$\begin{aligned} \zeta = \mathfrak{C}_2^s [ & 9.1181 P_2^s - 0.71533 P_4^s + 0.03621 P_6^s \\ & - 0.001136 P_8^s + 0.000024 P_{10}^s - \dots ]; \end{aligned}$$



and when  $hg/4\omega^2 a^2 = \frac{1}{5}$ ,

$$\zeta = \mathcal{C}_2^2 [1.7646 P_2^2 - 0.06057 P_4^2 + 0.001447 P_6^2 - 0.000022 P_8^2 + \dots].$$

From these series we find for the ratio of the tide-heights to the equilibrium tide-heights at the equator the four values

$$-2.4187, \quad -1.8000, \quad +11.0725, \quad +1.9225.$$

On comparison of these numbers with those obtained for the solar tides in the preceding section, we see that for a depth of 7260 feet the solar tides will be direct while the lunar tides will be inverted, the opposite being the case when the depth is 29,040 feet. This is, of course, due to the fact that in each of these cases there is a period of free oscillation intermediate between twelve solar (or, more strictly, sidereal) hours and twelve lunar hours. The critical depths for which the lunar tides become infinite are found to be 26,044 feet and 6448 feet.

Consequently this phenomenon will occur if the depth of the ocean be between 29,182 feet and 26,044 feet, or between 7375 feet and 6448 feet. An important consequence would be that for depths lying between these limits the usual phenomena of spring and neap tides would be reversed, the higher tides occurring when the moon is in quadrature, and the lower at new and full moon.\*

There appears then to be a considerable range of depth comparable with the mean depth of the ocean over which the reversal of the spring and neap tide phenomena would take place, but in that the actual tides are highest in the neighbourhood of new and full moon we conclude that the effective depth of the ocean does not lie within this range, and that none of the periods of free oscillation of the actual ocean lie between twelve solar hours and twelve lunar hours. The true effective depth is almost certainly less than 26,044 feet, and therefore both solar and lunar tides will be in the main inverted, though the configuration of the land and of the ocean bed will probably give rise to considerable variations of phase in different places.

The shortest period of free oscillation of the second class for the case  $s=2$  approximates to, but is in excess of, three days. But if  $n$  denotes the moon's mean orbital motion, the speed of the lunar semi-diurnal tide is

$$2(\omega - n).$$

If we equate this to  $\frac{1}{3}\omega$ , we obtain

$$n = \frac{5}{6}\omega.$$

Hence, if the moon's orbital motion were accelerated, or the earth's rotation retarded, until the month and day were in a ratio less than 6:5, it would be possible for the period of the lunar semi-diurnal tide to confound itself with one of the periods of the oscillations of the second class, and the tides would then tend to become very large.

\* Cf. KELVIN, 'Popular Lectures,' vol. 2, p. 22 (footnote).

§ 17. *Solar Semi-diurnal Tides in an Ocean of Variable Depth.*

To evaluate the tides when the depth is a function of the latitude we must make use of the formulæ of § 4. The method will be sufficiently illustrated by the computation of the solar semi-diurnal tide for the case where

$$hg/4\omega^2 a^2 = \frac{1}{20} + \frac{1}{30} \sin^2 \theta,$$

or, where

$$h = (14,520 + 9680 \sin^2 \theta) \text{ feet,}$$

$\theta$  denoting the co-latitude.

Putting  $lg/4\omega^2 a^2 = \frac{1}{30}$ , and making use of the numerical values found in § 15 for  $x_n^2$ ,  $y_n^2$ ,  $\Lambda_n^2$ , we obtain from (41)

$$\begin{array}{ll} \log \xi_2^2 = \bar{3} \cdot 63907 & \log \xi_6^2 = n \bar{3} \cdot 06261 \\ \log \xi_4^2 = \bar{3} \cdot 12901 & \log \xi_8^2 = n \bar{3} \cdot 4369 \\ \log \eta_2^2 = \bar{2} \cdot 17040 & \log \eta_6^2 = n \bar{3} \cdot 95767 \\ \log \eta_4^2 = n \bar{3} \cdot 75361 & \log \eta_8^2 = n \bar{3} \cdot 9962 \end{array}$$

while, when  $\frac{\kappa f}{4\omega^2 a^2} = \frac{1}{20}$ , we obtain

$$\begin{array}{ll} \mathfrak{L}_2^2 = + 0 \cdot 020765 & \mathfrak{L}_8^2 = - 0 \cdot 05787 \\ \mathfrak{L}_4^2 = - 0 \cdot 039717 & \mathfrak{L}_{10}^2 = - 0 \cdot 0606 \\ \mathfrak{L}_6^2 = - 0 \cdot 052592. & \end{array}$$

From these we deduce in succession, on neglecting  $\mathfrak{R}_{12}^2$ ,

$$\begin{array}{ll} \log \mathfrak{R}_{10}^2 = n \bar{4} \cdot 651 & \log \mathfrak{R}_6^2 = \bar{4} \cdot 1632 \\ \log \mathfrak{R}_8^2 = n \bar{4} \cdot 2612 & \log \mathfrak{R}_4^2 = \bar{3} \cdot 20891 \end{array}$$

and

$$\log \mathfrak{H}_2^2 = \bar{3} \cdot 49214.$$

But the first two of equations (40) give

$$\begin{aligned} -\mathfrak{L}_2^2 C_2^2 + \eta_2^2 C_4^2 &= \left[ \frac{\kappa}{4\omega^2 a^2} + 8 \frac{l\Lambda_2^2}{4\omega^2 a^2} \right] \gamma_2^2 = \left[ \frac{\kappa g_2}{4\omega^2 a^2} + 8 \frac{lg_2}{4\omega^2 a^2} \Lambda_2^2 \right] C_2^2, \\ \xi_2^2 C_2^2 - (\mathfrak{L}_4^2 - \mathfrak{R}_6^2) C_4^2 &= -8 \frac{l\gamma_2^2}{4\omega^2 a^2} x_2^2 = -8x_2^2 \frac{lg_2}{4\omega^2 a^2} C_2^2, \end{aligned}$$

which, on solution, yield

$$\begin{aligned} C_2^2 &= - \left[ \frac{\kappa g_2}{4\omega^2 a^2} + 8 \frac{lg_2}{4\omega^2 a^2} \Lambda_2^2 \right] \frac{C_2^2}{\mathfrak{L}_2^2 - \mathfrak{R}_4^2} + 8x_2^2 \frac{lg_2}{4\omega^2 a^2} \frac{\mathfrak{H}_2^2}{\xi_2^2} \frac{C_2^2}{\mathfrak{L}_4^2 - \mathfrak{H}_2^2 - \mathfrak{R}_6^2}, \\ C_4^2 &= - \left[ \frac{\kappa g_2}{4\omega^2 a^2} + 8 \frac{lg_2}{4\omega^2 a^2} \Lambda_2^2 \right] \frac{\mathfrak{R}_4^2}{\eta_2^2} \frac{C_2^2}{\mathfrak{L}_2^2 - \mathfrak{R}_4^2} + 8x_2^2 \frac{lg_2}{4\omega^2 a^2} \frac{C_2^2}{\mathfrak{L}_4^2 - \mathfrak{H}_2^2 - \mathfrak{R}_6^2}. \end{aligned}$$

On substituting the numerical values for the quantities on the right, we obtain

$$C_2^2 = -2.9242 \mathfrak{C}_2^2, \quad C_4^2 = 0.28546 \mathfrak{C}_2^2.$$

The remaining constants may now be computed from the formulæ  $C_6^2/C_4^2 = \mathfrak{K}_6^2/\eta_4^2$ ,  $C_8^2/C_6^2 = \mathfrak{K}_8^2/\eta_6^2$  &c., and we finally obtain

$$\zeta = \mathfrak{C}_2^2 [-2.9242P_2^2 + 0.28546P_4^2 - 0.00733P_6^2 - 0.000147P_8^2 - 0.000007P_{10}^2 - \dots].$$

This makes the ratio of the height of the tide to that of the equilibrium tide at the equator

$$-3.6690.$$

The tide will evidently be in the main inverted, the longest period of free oscillation of the first class being in excess of twelve hours.

As a further example, I have computed the series for  $\zeta$  when the depth is given by the formula

$$\frac{hg}{4\omega^2 a^2} = \frac{1}{10} - \frac{1}{30} \sin^2 \theta,$$

that is, when the depth is 29,040 feet at the poles and shallows to 19,360 feet at the equator. This series is found to be

$$\begin{aligned} \zeta = \mathfrak{C}_2^2 [-2.3661P_2^2 + 0.35649P_4^2 - 0.03953P_6^2 + 0.00376P_8^2 \\ - 0.000333P_{10}^2 + 0.000029P_{12}^2 - 0.000002P_{14}^2 + \dots], \end{aligned}$$

making the ratio of the tide to the equilibrium tide at the equator

$$-3.4583.$$

If we put  $\frac{hg}{4\omega^2 a^2} = \frac{1}{30}$ , and replace  $\lambda$  by  $2\omega$  in the period-equation, regarding this as an equation for  $\kappa$ , the largest root is found to be

$$\kappa = 21,765.$$

Thus there will be a period of free oscillation coinciding exactly with twelve hours when

$$h = (21,765 + 9680 \sin^2 \theta) \text{ feet};$$

this formula makes the polar depth 21,765 feet, the equatorial depth 31,445 feet, and the mean depth 28,222 feet.

In like manner, when  $\frac{hg}{4\omega^2 a^2} = -\frac{1}{30}$ , there will also be a period of free oscillation



agreeing exactly with twelve hours, when the polar depth is 36,970 feet, the equatorial depth 26,290 feet, and the mean depth 29,517 feet.

### § 18. *Diurnal Tides.*

It has been shown in § 14 that the diurnal tidal constituents whose periods are equal to a sidereal day will involve no rise and fall at the free surface when the depth of the ocean is uniform. This theorem will be rigorously applicable to the luni-solar diurnal constituent usually designated by the initial  $K_1$ , while it may also be supposed to apply with a fair degree of accuracy to each of the solar diurnal constituents since the motion of the sun in his orbit is sufficiently slow. There will however be an important lunar diurnal constituent for which the speed is  $0.92700\omega$ , in dealing with which we propose to take into account the difference between the period and a sidereal day. The method of computation is exactly similar to that used for the lunar semi-diurnal tides, and thus we find when  $hg/4\omega^2a^2 = \frac{1}{40}$ ,

$$\zeta = \mathfrak{C}_2^1[-0.07638P_2^1 + 0.03543P_4^1 - 0.00845P_6^1 + 0.001207P_8^1 \\ - 0.000114P_{10}^1 + 0.000008P_{12}^1 - \dots];$$

when  $hg/4\omega^2a^2 = \frac{1}{20}$ ,

$$\zeta = \mathfrak{C}_2^1[-0.1691P_2^1 + 0.04738P_4^1 - 0.00628P_6^1 + 0.000480P_8^1 \\ - 0.000024P_{10}^1 + 0.000001P_{12}^1 - \dots];$$

when  $hg/4\omega^2a^2 = \frac{1}{10}$ ,

$$\zeta = \mathfrak{C}_2^1[-0.4145P_2^1 + 0.06576P_4^1 - 0.00464P_6^1 + 0.000184P_8^1 - 0.000005P_{10}^1 + \dots];$$

and when  $hg/4\omega^2a^2 = \frac{1}{5}$ ,

$$\zeta = \mathfrak{C}_2^1[-1.4428P_2^1 + 0.1231P_4^1 - 0.00449P_6^1 + 0.00009P_8^1 - 0.000001P_{10}^1 + \dots].$$

It appears, then, that these tides will increase with the depth, and that they will be in the main inverted. For small depths the rise and fall will be small, but with a depth as great as 58,080 feet the tide will be in excess of the equilibrium-tide. The type will tend to approximate more and more closely to that represented by a second order harmonic alone as the depth increases.

So long as the depth is uniform the tidal constituents whose periods are rigorously equal to a sidereal day will never tend to become infinite, and consequently no period of free oscillation of the type of the diurnal tides can coincide exactly with one day. As  $\omega$  diminishes however the shortest period of the second class, which for the depths under consideration is longer than the period of the lunar diurnal



constituents, approximates to one day, and attains this as a limiting value when  $\omega = 0$ . Hence, as  $\omega$  diminishes, or  $h$  increases, the largest root of the second class must pass through the value 0.92700, thus rendering one of the lunar diurnal constituents infinite. This accounts for the rapid increase in the coefficients in the series given above for these tides as  $h$  increases.

The roots of the first class must however all be greater than unity, no matter how great the depth may be. Since they all decrease with the depth, they must approach finite limiting values greater than unity as the depth diminishes to zero.

The tides of rigorously diurnal period will become infinite when the depth is variable if  $l$  assumes the value

$$\frac{1}{2} \frac{\omega^2 a^2}{g_2},$$

so that with this value of  $l$  we may anticipate that there will be a period of free oscillation exactly a sidereal day in duration. Now the above value of  $l$  will require that the surface of the solid earth should be rigorously spherical in order that the free surface of the ocean may be an equipotential surface under gravity and centrifugal force. It is easy to see why in this case a free oscillation of rigorously diurnal period must exist. For if the water be set in rotation as a solid body about an axis not rigorously coincident with the rotation-axis of the solid earth, and the form of the free surface be adjusted for equilibrium under centrifugal force about the new axis of rotation, there will be no forces acting which tend to modify this state of motion, and it will continue permanently, provided the system be free from friction. The motion of the water will be steady in space, but it will be oscillatory with a period of one day relatively to the solid earth.

It is easy to verify that in the forced oscillations of rigorously diurnal period the motion of the water is of like character, involving no relative motion of the parts and being steady in space. In this case the axis about which the rotation of the water takes place lies in the plane containing the earth's polar axis and the disturbing body.



VI. *Electrification of Air, of Vapour of Water, and of Other Gases.*

By Lord KELVIN, G.C.V.O., F.R.S., MAGNUS MACLEAN, D.Sc., F.R.S.E., and  
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§ 1. IN this paper we describe a long series of experiments on the electrification of air and other gases, with which we have been occupied from May, 1894, up to the present time (June, 1897). Some results of our earlier experiments, and of preliminary efforts to find convenient methods of investigation, have from time to time been communicated to the Royal Society, the British Association, and the Glasgow Philosophical Society.\*

§ 2. The method for testing the electrification of air, which we used in our earliest experiments, was an application of the water-dropper† (long well-known in the ordinary observation of atmospheric electricity). Its use by MACLEAN and GOTO,‡ in 1890, led to an interesting discovery that air in an enclosed vessel, previously non-electrified, becomes electrified by a jet of water falling through it. An investigation of properties of matter concerned in this effect, related as it is to the "development of electricity in the breaking up of a liquid into drops," which had been discovered by HOLMGREN§ as early as 1873, and to the later investigations and discoveries

\* "On the Electrification of Air," 'Proc. Roy. Soc.,' May 31, 1894, and 'Phil. Mag.,' August, 1894. "Preliminary Experiments to find if Subtraction of Water from Air Electrifies It," 'Brit. Assoc. Report,' 1894, p. 554. "Electrification of Air and other Gases by Bubbling through Water and other Liquids," 'Proc. Roy. Soc.,' February, 1895. "On the Diselectrification of Air," 'Proc. Roy. Soc.,' March, 1895. "On the Electrification of Air," 'Proc. Glasgow Phil. Soc.,' March, 1895. "On the Electrification and Diselectrification of Air and other Gases," 'Brit. Assoc. Report,' 1895, p. 630.

† KELVIN and MACLEAN, 'Proc. Roy. Soc.,' 1894, and KELVIN, MACLEAN, and GALT, 'Proc. Roy. Soc.,' February, 1895. "Electrostatics and Magnetism," § 262 (from 'Proc. Lit. and Phil. Soc. of Manchester,' October 18, 1859).

‡ "Electrification of Air by Water-jet." By MAGNUS MACLEAN and MAKITA GOTO, 'Phil. Mag.,' August, 1890.

§ "Sur le développement de l'électricité à l'occasion de la dissolution en gouttes des liquides." 'Kongl. Svensk. Vet. Ak. Handl.,' 1 c., vol. 11, No. 8, pp. 14-43 (pour l'an 1873), -

described by LENARD,\* in his paper on the "Electricity of Waterfalls," forms the subject of §§ 25-37 of the present communication.

§ 3. The electrification of air by drops of water, breaking from a jet in it, or falling through it, or striking on the ground, or on water, or on metal below it, produces absolutely no practical disturbance of the electric potential measured by the water-dropper in its use for the observation of open-air atmospheric electricity, but constitutes a serious objection to its application for investigating atmospheric electricity within doors, unless in a very large room or hall, and renders it altogether unsuitable for the experimental investigations with which we are now concerned.

§ 4. We were, therefore, early led to abandon it; and, for testing the electrification of air, we have used three different methods, one or other of which we have found convenient in different cases.

Method (1). Observation of electrification of the substance receiving the electricity equal and opposite to that taken by air in any case of electrification of air.

Method (2). Observation of the electricity of a hollow metal vessel into which electrified air is introduced, or from which electrified air is removed.

Method (3). Observation of the electricity taken out of air by the electric filter (§ 9).

§ 5. Method (1) was used in the experiments described in our communication to the Royal Society of February, 1895, from which we concluded that air, and several other gases tried, became electrified by blowing them in bubbles through water, and through solutions of various salts, acids, and alkalies in water. We verified this conclusion, for the case of common air and pure water, by collecting into a large reservoir over water, air which had been bubbled through pure water in a U-tube. We tested the electrification of the air thus collected by a water-dropper taking the same potential as the air at the centre of the reservoir. We thus proved that the electrification of the air was negative, as was to be expected from the positive electrification which we had found on insulated vessels containing water through which air had been bubbled.

§ 6. Method (2) was used in the first experiments described in the present paper (§§ 16-24), which were undertaken for the purpose of determining approximately in absolute measure the total quantity of electricity in a given mass of electrified air, and particularly for finding the greatest electrification which we could communicate to a large quantity of air by needle points supplied with electricity from an electric machine. The result thus found in § 23 below,  $3.7 \times 10^{-4}$  C.G.S. electrostatic, is the greatest electric density (quantity of electricity per cubic centim.) which we have been able to communicate to air by electrified needle points. But, by an electrified hydrogen flame a density of  $22 \times 10^{-4}$  C.G.S. electrostatic unit was obtained (§ 65).

§ 7. In all the experiments described in our paper after § 24, method (3) was used; but, probably, we must return to method (2) if, in future, we undertake

\* "Ueber die Electricitat der Wasserfälle." By P. LENARD, 'Annalen der Physik und Chemie,' 1892,



further experiments to find the greatest electric density which we can measure in air or other gases.

§ 8. LENARD'S important discovery of very strong electric effects produced by drops of water falling on a hard surface, gave us a very convenient method for obtaining a steady and strong negative electrification of air, which we used in §§ 30-32 for preliminary efforts for finding a good and convenient form of electric filter to be used in further investigations on the electrification and diselectrification of air.

§ 9. In testing the efficiencies of the electric filters used in method (3), we at first used the filter described in our paper on "Diselectrification of Air," 'Proc. Roy. Soc.,' vol. 57, and which consisted of twelve discs of brass wire cloth, fixed in a short metal pipe, supported in a paraffin tunnel. This filter was joined to the insulated quadrants of a quadrant electrometer, and electrified air was sucked through it (§ 25) till a convenient deflection was obtained. Then the filter to be tested was connected to the sheath of the electrometer, and so placed that the electrified air passed through the tested filter before it passed through the filter attached to the electrometer (§ 68). In this way, by drawing equal quantities of electrified air, as nearly as may be equally electrified, through the different tested filters, a comparison of their relative diselectrifying powers was obtained. For example,  $d$  being the deflection obtained when no tested filter was used, and  $d_1, d_2 \dots d_n$  when the tested filters were successively used, then the relative diselectrifying powers of the filters would be  $\frac{d - d_1}{d}, \frac{d - d_2}{d}, \dots \frac{d - d_n}{d}$ , if the primary electrifications were equal.

§ 10. In other sets of experiments (§§ 32, 69) we successively joined each separate tested filter to the insulated quadrants of the electrometer, and sucked approximately equal quantities of electrified air through them. The diselectrifying efficiencies of the filters were now approximately in simple proportion to the final readings on the electrometer.

§ 11. But none of these methods gave us a means of determining the absolute diselectrifying power of any filter without realizing an equality of primary electrification of the air in different experiments. We therefore, a long time later, used two insulated filters and two electrometers, as described in § 55 and fig. 7. Let the filters be called AB and A'B' and their diselectrifying powers  $n$  and  $n'$ . In a first experiment the electrified air was sucked through AB and A'B' in immediate succession, and in a second experiment the electrified air was sucked through A'B' and AB.

§ 12. On the assumption that the two filters took out the same proportions (respectively  $n$  or  $n'$ ) of the electricity of the electrified air entering them in the two experiments, we get the following equations:

In the first experiment

Let  $Q$  = total quantity of electricity in air entering,

$q_1$  = total quantity of electricity taken out by filter AB,

$q_2$  = total quantity of electricity taken out by filter A'B'.



§ 15. With such a filter as this it is possible to determine the quality of the natural electricity of the atmosphere, and it may be desirable that it should be used for that purpose in meteorological observatories.

GREATEST ELECTRIFICATION WHICH WE COULD COMMUNICATE TO A LARGE QUANTITY OF AIR BY ONE OR MORE ELECTRIFIED NEEDLE POINTS (§§ 16–24).

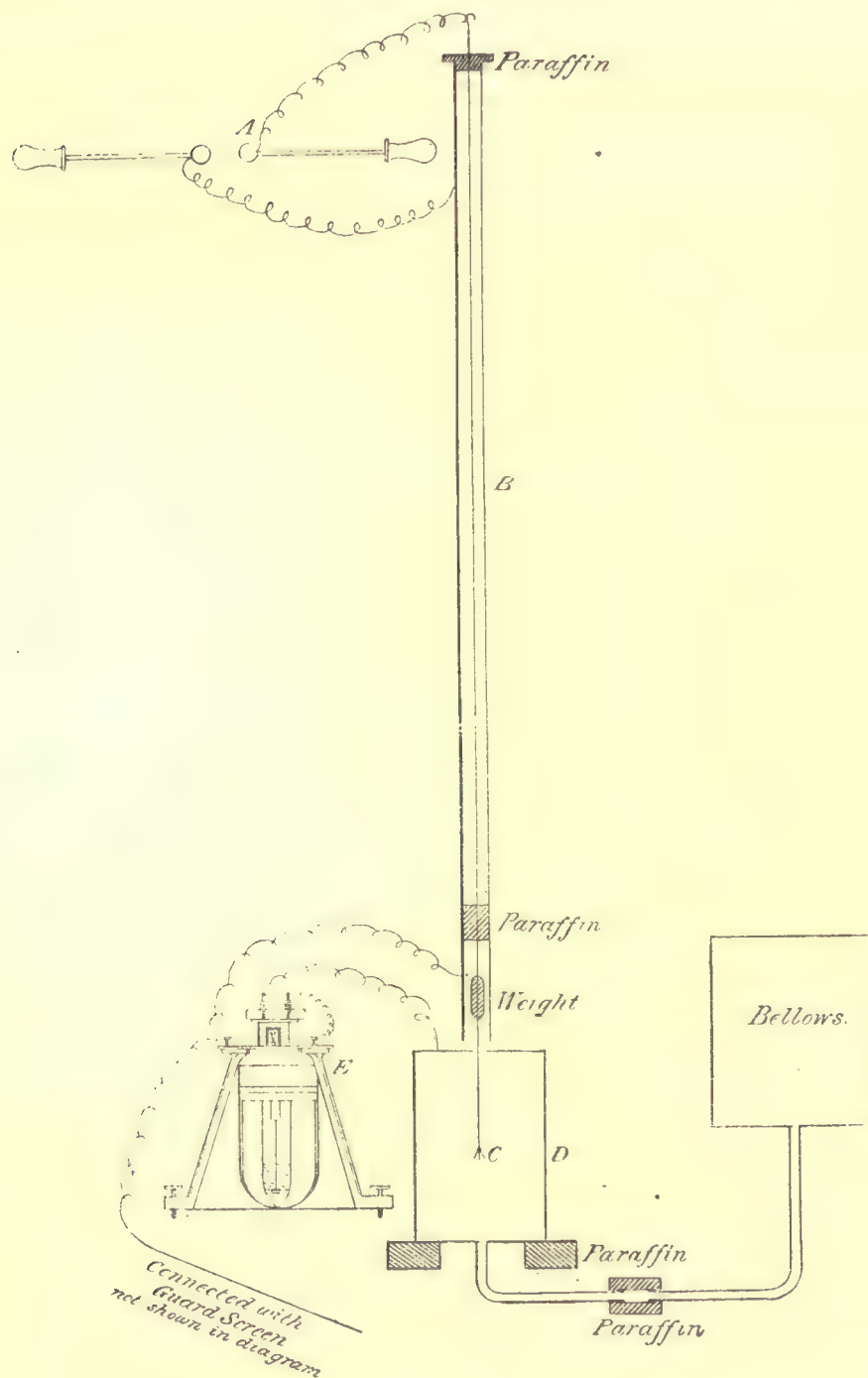
§ 16. The first apparatus used is shown in fig. 1. It consisted of a metal can, D, 48 centims. high and 21 centims. in diameter, supported by paraffin blocks, and connected to one pair of quadrants of a quadrant electrometer. It had a hole at the top to admit the electrifying wire, which was 531 centims. long, hanging vertically within a metallic guard-tube, B. This guard-tube was always metallically connected to the other pair of quadrants of the electrometer, and to its sheath and to a metallic screen surrounding it, which is not shown in the diagram. This prevented any external influences from sensibly affecting the electrometer, such as the working of the electric machine, A, which stood on a shelf five metres above it.

§ 17. The experiment is conducted as follows :—One terminal of the electric machine is connected with the guard-tube and the other with the electrifying wire, which is tipped with needle points or tinsel, and which is let down to place the point or points nearly in the centre of the can. The can is temporarily connected to the sheath of the electrometer. The electric machine is then worked for some minutes, so as to electrify the air in the can. As soon as the machine is stopped, the electrifying wire is lifted clear out of the can. The can and the quadrants in metallic connection with it are disconnected from the sheath of the electrometer, and the electrified air is very rapidly drawn away from the can by a blowpipe bellows, arranged to suck. This releases the opposite kind of electricity from the inside of the can, and allows it to place itself in equilibrium on the outside of the can and on the insulated quadrants of the electrometer in metallic connection with it.

§ 18. We tried different lengths of time of electrification and different numbers of needles and tinsel, but we found that one needle and four minutes' electrification gave as great electrification as we could get. The greatest deflection observed was 936 scale divisions on a half millim. scale put up in the usual way, with lamp at a distance of about a metre from the electrometer. To find from this reading the electric density of the air in the can, we took a metallic disc of 2 centims. radius, attached to a long varnished glass rod and placed at a distance of 1.45 centims. from another and larger metallic disc. This small air condenser was charged from the electric light conductors in the laboratory to a difference of potential amounting to 100 volts, or  $\frac{1}{3}$  of an electrostatic unit. The insulated disc thus charged was removed and laid upon the roof of the large insulated can. This addition to the metal in connection with it does not sensibly influence its electrostatic capacity.



Fig. 1.



The deflection obtained was 122 scale divisions. The capacity of the condenser is approximately—

$$\frac{\pi \times 2^2}{4\pi \times 1.45} = \frac{1}{1.45} \text{ C.G.S. electrostatic unit.}$$



The quantity of electricity with which it was charged was therefore

$$\frac{1}{1.45} \times \frac{1}{3} = \frac{1}{4.35} \text{ C.G.S. electrostatic unit.}$$

Hence, the quantity on the can and connected metal to give 936 scale divisions was

$$\frac{1}{4.35} \times \frac{936}{122} = 1.7637 \text{ C.G.S. electrostatic units.}$$

The capacity of the can was 16,632 cub. centims., which gives, for the quantity of electricity per cub. centim.

$$\frac{1.7637}{16632} = 1.06 \times 10^{-4} \text{ of the C.G.S. electrostatic unit.}$$

§ 19. This is about four times the electric density which we roughly estimated as about the greatest given to the air in the inside of a large vat, electrified by a needle point and then left to itself, and tested by the potential of a water-dropper with its nozzle in the centre of the vat, in experiments made more than three years ago and described in a communication to the Royal Society of date May, 1894.\*

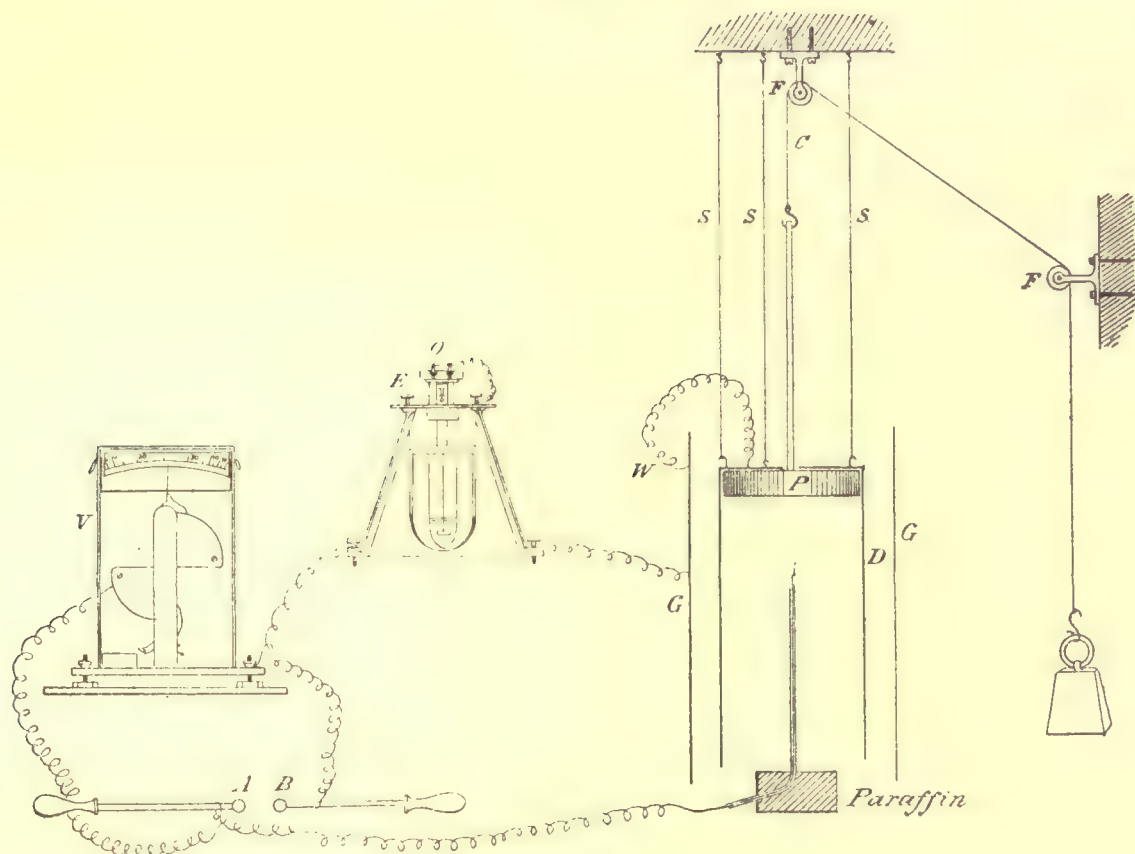
§ 20. To enable us to remove the electrified air quickly from the can, the following modification was adopted:—The can was suspended vertically by three stout silk threads (S, S, S, fig. 2) which had been previously soaked in melted paraffin; it was quite open at the bottom but closed at the top, with the exception of a central aperture for the piston-rod of a piston, P. The piston was of wood encased in lead, and was free to move up and down in the can by the movement of the paraffined silk cord, C, over the pulleys, F, F. The can and the piston and piston-rod were connected metallically by spiral springs of fine wire. The can was surrounded by a metallic guard-screen, G, kept in connection with the sheath of the quadrant electrometer, and with the sheath of a vertical electrostatic voltmeter, and with one terminal, B, of an electric machine. The other terminal of the machine was connected to an insulated needle-point inside the can and to the insulated terminal of the voltmeter.

§ 21. By working the machine the needle was kept charged positively or negatively at 12,000 volts for four minutes. The air inside the can became charged similarly by the brush discharge from the needle point. As soon as possible after stopping the machine, the needle was removed and A and B were joined. The wire, W, from the can was disconnected from the guard screen, G, and then attached to the electrometer terminal, O, after which this terminal was insulated and the downward

\* "On the Electrification of Air," by Lord KELVIN and MAGNUS MACLEAN.

movement of the piston begun. Usually about thirteen seconds were required to make these changes.

Fig. 2.



§ 22. When the electrified air inside the can was expelled by dropping the piston to the bottom, the reading of the electrometer went off the scale, and a shorter drop had to be used to get a convenient deflection.

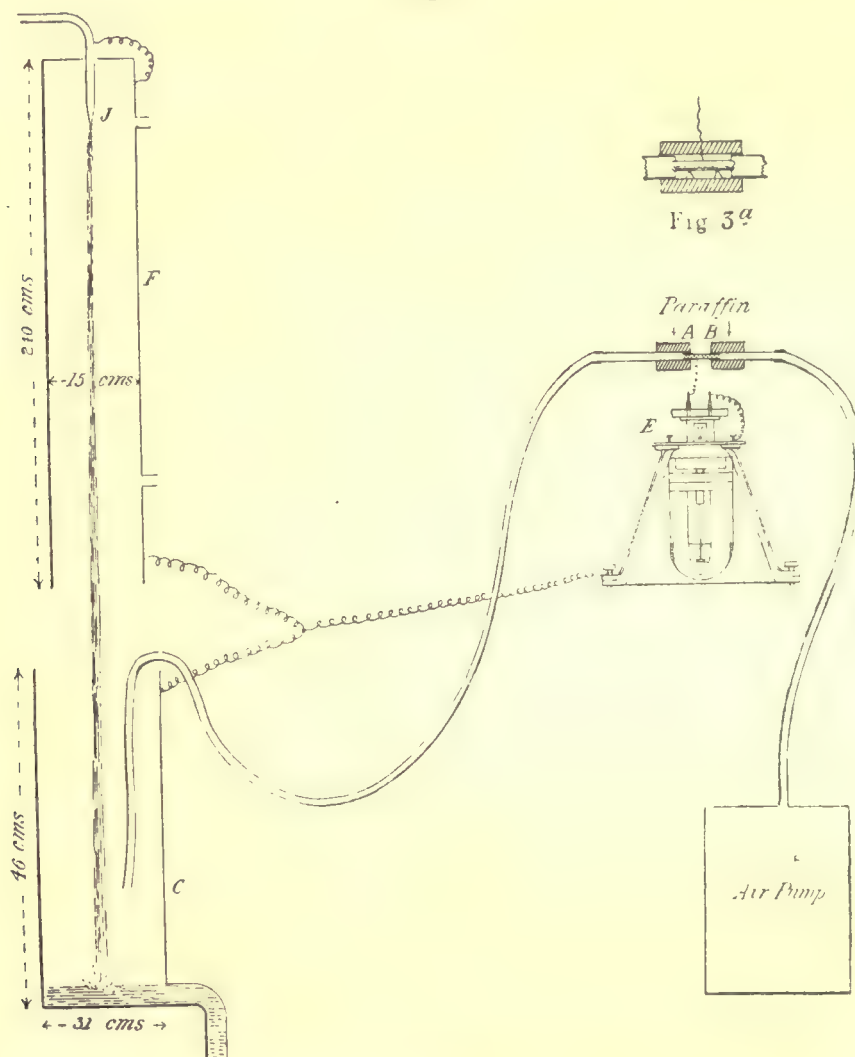
§ 23. A drop of 11.5 centims., by which 3979 cub. centims. of air were expelled, gave a deflection of 1060 scale divisions. The quantity of electricity on the can and connecting wire and insulated pair of quadrants of the electrometer which gives this deflection, was (by the method of § 18) found to be 1.47 C.G.S. electrostatic. This, therefore, was the quantity of electricity of the opposite kind in the 3979 cub. centims. of air expelled from the open bottom of the can; and the electric density of this air was therefore  $3.7 \times 10^{-4}$  C.G.S. electrostatic per cubic centimetre.

§ 24. In preliminary experiments before electrifying the air inside the can by needle point, it was found that the dropping of the piston produced no deflection on the electrometer.

## ELECTRIFICATION OF AIR BY WATER JET (§§ 25-37).

§ 25. In previous experiments\* we found air to be negatively electrified by water falling in drops through it, and to pursue the investigation further the arrangement shown in fig. 3 was put up. Loch Katrine water, under full pressure, issues from a

Fig. 3.



jet, J, fixed in the lid of a funnel, F, and falls down the funnel centrally into a metal can, C, below. By means of an air-pump some of the air is withdrawn from the can or funnel through metallic tubing, as shown in the diagram, to a metallic filter, AB, containing 50 fine brass wire gauzes. In each experiment 200 strokes of the pump are taken.

\* "Electrification of Air by Water Jet," by MACLEAN and GOTO. 'Phil. Mag.,' August, 1890.  
 "Electrification of Air," by Lord KELVIN and MAGNUS MACLEAN. 'Proc. Roy. Soc.,' May, 1894.



§ 26. The greatest effect, 4·5 volts negative, was obtained, as was to be expected from LENARD'S discovery, when the air was withdrawn from a point well down in the can, the can being close to the funnel, and the falling water rattling on the bottom of the can. Decreasing effects were observed (1) when the air was withdrawn from the can at increasing distances from the bottom; (2) when several inches of water were kept in the can;\* (3) when air was drawn from the funnel, and the distance between the can and the funnel was gradually increased.

§ 27. A filter of 100 wire gauzes gave at the rate of 26 volts, and a filter of 2 gauzes, with a loose plug of cotton wool between them, gave 6·3 volts, in the same time and under the same conditions as the filter of 50 wire gauzes gave (§ 26) 4·5 volts.

§ 28. A *sloping metallic plate* was next fitted to the bottom of the funnel in such a way that the falling water on striking the plate passed out by the aperture between the funnel and the lower edge of the plate. In each experiment 120 strokes of the pump were taken at the rate of 40 strokes per minute. Drawing the air from the aperture near the bottom of the funnel gave, in 4 experiments, results averaging about 8 volts. These results were given by a filter containing 2 brass gauzes with cotton wool between them.

§ 29. Two simple brass tubes of different bores with no wire gauze or cotton wool were now tried as filters. Each was 10 centims. long and 1 centim. external diameter. The following results were obtained :—

Air drawn from side aperture of funnel :—				
Internal diameter of brass filter.	Near the top.		Near the bottom.	
	Aperture near the bottom :—		Aperture near the top :—	
	Shut.	Open.	Shut.	Open.
	centims.	volts neg.	volts neg.	volts neg.
0·3		1·5	3·2	3·0
0·18		3·0	4·5	4·3

Using the brass filter of 0·18 centim. bore, drawing from the aperture near the bottom, and varying the water pressure, we found mean results as follows :—

Full pressure . . . . .	3·4 volts.
Diminished pressure . . . . .	2·6 „
Very low pressure (200 drops per minute) . . . . .	0·17 „

§ 30. In the long metallic tube between the funnel and the testing filter we placed, in successive experiments, an increasing number of brass gauzes and plugs of wool.

\* See a Paper by LENARD on "Electricity of Waterfalls," 'Wied. Ann.,' 1892, vol. 46, pp. 584-636.



The air had to pass through these and the long length of tubing before reaching the testing filter at the electrometer. The testing filter consisted of a block-tin tube with two wire gauzes and one plug of cotton wool. Twenty experiments were made on air drawn from the aperture of the funnel near its lower end, and with jet from full pressure of water. The following are mean results, for 120 strokes of the pump :—

No gauze between the funnel and the testing filter.	2 gauzes and 1 wool plug.	3 gauzes and 2 wool plugs.	4 gauzes and 3 wool plugs.	5 gauzes and 4 wool plugs.	6 gauzes and 5 wool plugs.	7 gauzes and 6 wool plugs.
volts. 7	volts. 2·4	volts. 1·25	volt. 0·4	volt. 0·3	volt. 0·1	volt. 0·04

These results show that a large proportion of the electricity was taken, by the 7 gauzes and 6 plugs, from the air before it reached the testing filter.

§ 31. Extracting the air with no water falling gave no perceptible electrification.

§ 32. With the water again at full pressure, and falling on the sloping plate fixed into the bottom of the funnel, the negatively electrified air was drawn from the bottom aperture through different filters at different speeds. Two experiments at each speed were usually made, and whenever possible the deflection for 120 strokes of the pump was noted. In some cases, however, the reading exceeded 8 volts, and went off the scale with much fewer strokes ; but to preserve uniformity the tabular results given below are all calculated for 120 strokes.

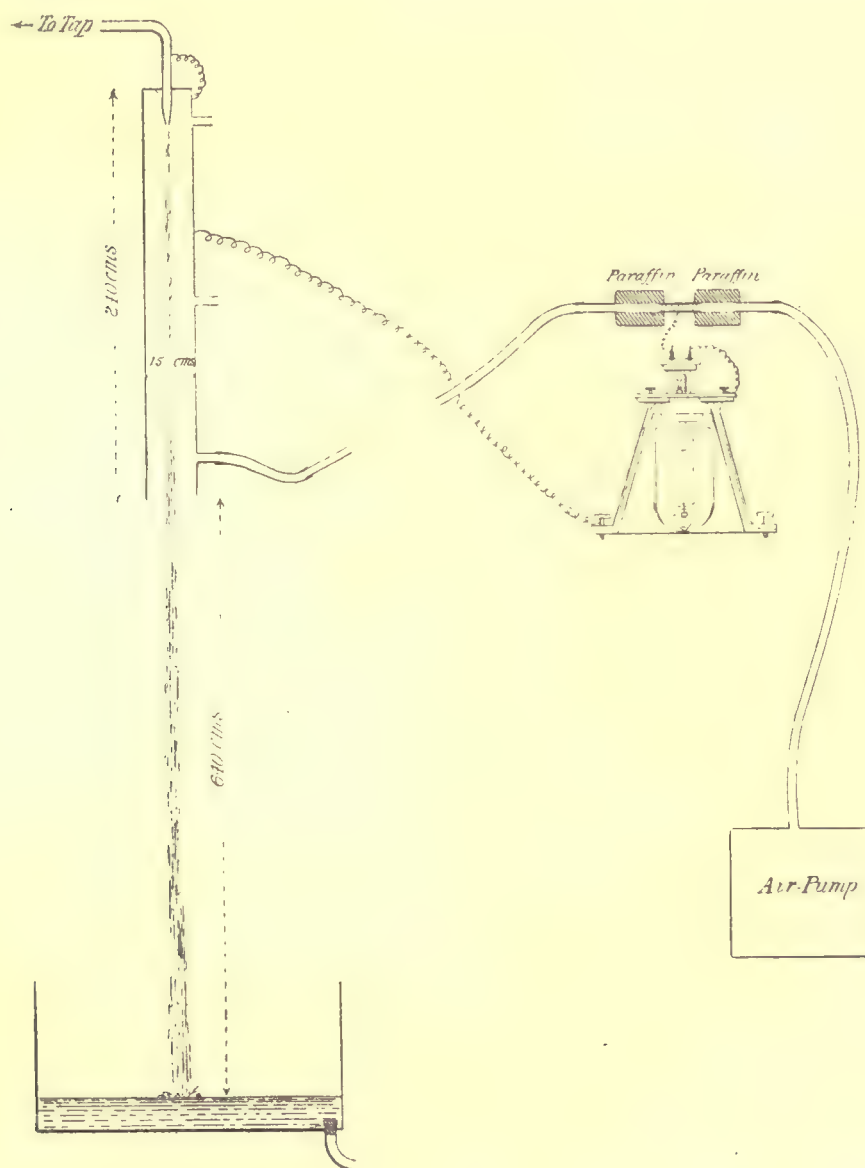
Results 2 and 4 are in accordance with Results 2 and 3 of § 68.

Filter used.	Duration of each stroke of the pump in seconds.					
	0·5	1	2	3	4	5
	volts neg.	volts neg.	volts neg.	volts neg.	volts neg.	volts neg.
1. Platinum tube 2·4 centims. long, 0·1 centim. bore	..	4·9	9·9	14·5	18·7	
2. Brass tube 4·1 centims. long, 0·2 centim. bore	7·8	12·5	21·8	25·8	26	25
3. Brass tube 4·1 centims. long, 0·34 centim. bore	4	7·2	12	13·2	12·7	
4. Solid brass cylinder with rounded ends, 8 centims. long and 1·8 centim. diameter, insulated within a paraffin tunnel (fig. 3a)	20·1	34·1	52·7	59	62·1	

§ 33. A metallic water-dropper was now fixed into the lid of the funnel, and the metallic plate at the bottom removed. A strong solution of common salt was placed in the dropper, and allowed to fall down the centre of the funnel into a basin below.

On drawing the air from the funnel by the side aperture near its lower end, and testing it by the brass tubular filter, 0.18 centim. diameter (4 of § 32), a mean of four experiments showed 2.5 volts positive.

Fig. 4.



§ 34. Arrangements were now made to test the effect of falling water upon air only, uninfluenced by impact of drops on any hard solid. The metallic plate at the bottom of the funnel was removed, and the funnel, 240 centims. long, was placed vertically in a position giving a clear fall of 640 centims. from the lower end of the funnel to a water-trough below (see fig. 4). A new aperture was made in the funnel near the centre. The filter used had 12 wire gauzes and 11 plugs of cotton wool.

The following results were obtained from 120 strokes of the pump :—

Water pressure.	Air drawn from aperture in funnel near the—			Remarks.
	Top.	Middle.	Bottom.	
	volts neg.	volts neg.	volts neg.	
Full . . . . .	0·5	0·7	0·5	
Reduced . . . . .	0·16	0·23	0·33	
Full . . . . .	21·0	37·0	21·0	Sloping plate inserted in the funnel at the bottom.
Full . . . . .	6·7	7·4	4·8	Plate removed. Funnel tilted so that the water struck against the side.

§ 35. The lower half of the funnel was now removed and the upper half used. The water now fell clear through the whole length of the funnel, and the extracted air gave, by 120 strokes of the pump as usual,  $\frac{1}{4}$  to  $\frac{1}{2}$  volt negative. No electrification in the extracted air could be detected if no water was falling.

§ 36. Putting the water-dropper (§ 33) in the top of the shortened funnel and allowing a strong solution of salt water to fall down from the dropper,  $\frac{1}{4}$  volt *positive* was got from the extracted air. Placing pure water in the dropper and testing again,  $\frac{1}{4}$  volt *negative* was found.

#### ELECTRIFICATION OF AIR BY AN INSULATED WATER-DROPPER AT DIFFERENT POTENTIALS (§ 37).

§ 37. The water-dropper was now insulated and connected with the positive terminal of 1 or more, up to 12, cells of a secondary battery, the negative terminal of which was connected with the funnel, and *vice versa*. On letting water fall from the dropper, and testing the electrification of the air in the funnel by drawing it through a testing filter, the results were not sensibly affected by substituting metallic connections for the connection of the battery terminals with the dropper and funnel. Hence, the large positive and negative electrifications thus given to the drops as they fell from the nozzle did not sensibly diminish or increase the negative electrification which they produced in the air through which they fell.

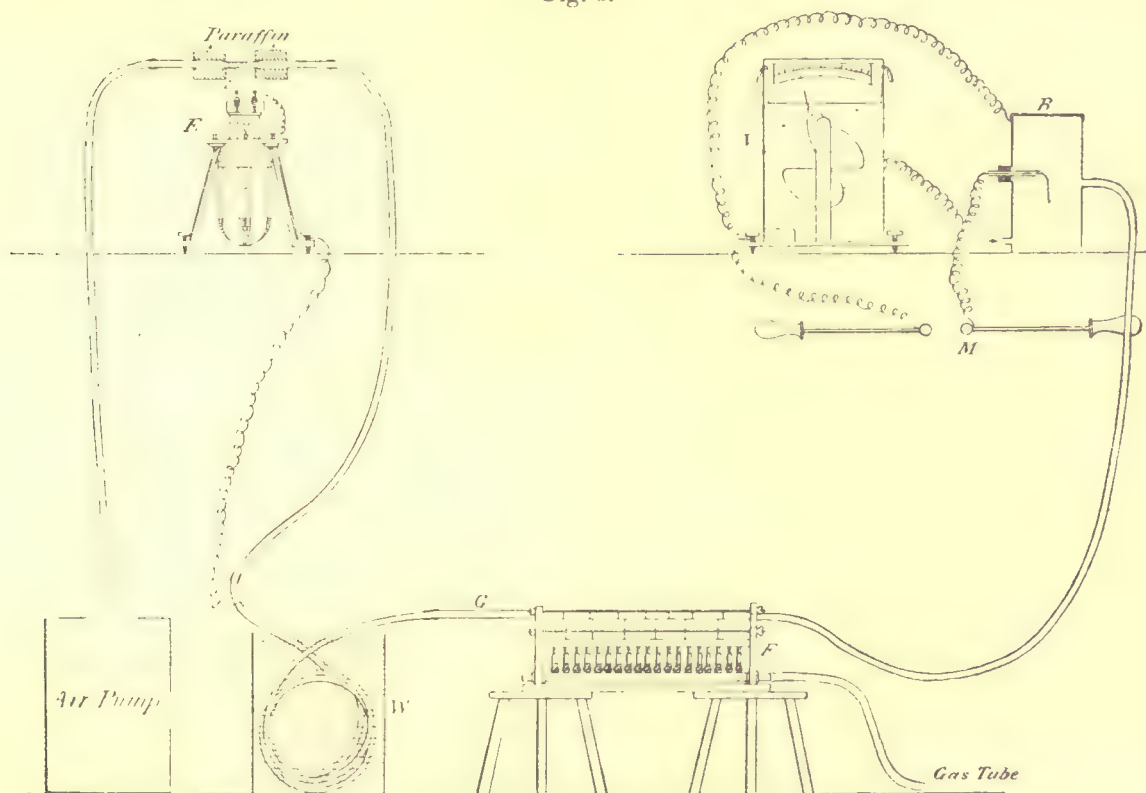
#### EFFECT OF HEAT ON ELECTRIFIED AND NON-ELECTRIFIED AIR (§§ 38, 39).

§ 38. The apparatus shown in fig. 5 was designed and used for the purpose of trying to diselectrify air by heat. Air is admitted into a tin plate biscuit canister, B, near the bottom. Two metallic tubes are fixed into it at the middle opposite each other. One of these two is plugged with paraffin through which passes a wire,



ending in a needle point inside, and connected outside with the insulated terminal of an electric machine, M. By means of an air-pump air is drawn into the canister, where it is electrified by the needle. It passes thence through a few metres of indiarubber pipe, to a 2-metre length of glass combustion tubing, G, 2 centims. internal diameter, heated to a high temperature in a gas furnace, F. The hot air passes on through a length of  $3\frac{3}{4}$  metres of block-tin piping coiled in a large vessel of cold water, W. The air thus cooled passes through two paraffin tunnels between which is the insulated filter consisting of block-tin pipe with two wire gauzes and a

Fig. 5.



plug of cotton wool. There were altogether  $10\frac{1}{2}$  metres of tubing between the canister and the filter. The air in the canister is kept electrified by an electrified needle point during an experiment.

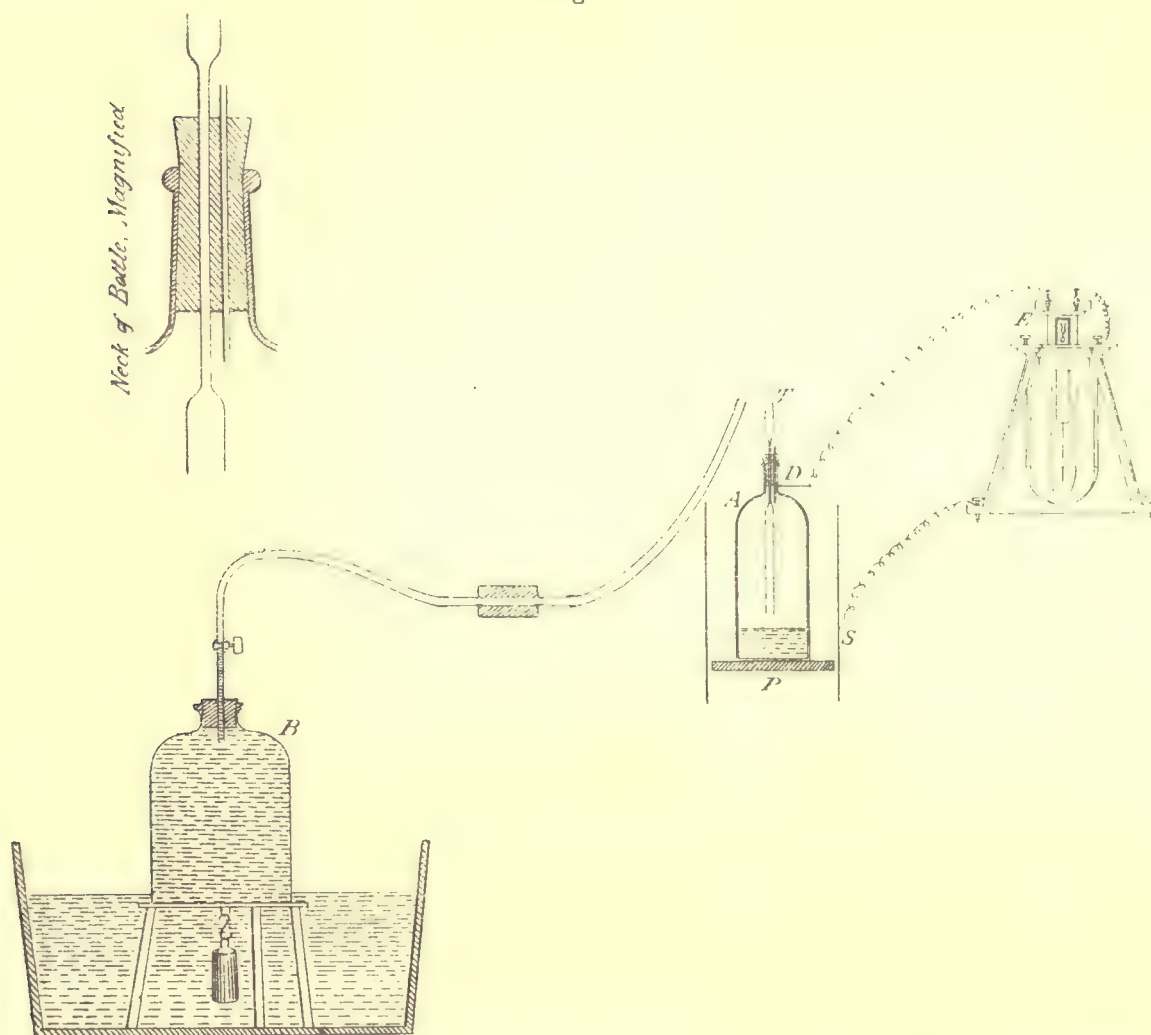
§ 39. Beginning with the glass tube cold, the air gave an electrification at the rate of 14 volts positive for 200 strokes of the pump. On gradually increasing the temperature of the tube the electrification correspondingly diminished to less than 3 volts. In cooling, the electrification, now become negative by an accidental reversal in the inductive machine used, increased to 4.5 volts negative. On another occasion 5 volts negative were got with the tube cold, decreasing to 2 volts as the temperature was raised, increasing again to 5 volts as the tube cooled. Occasionally irregular results were noted, especially with positively charged air.



## NON-ELECTRIFIED AIR PASSED OVER HOT COPPER AND HOT CHARCOAL (§ 40).

§ 40. Passing air through the apparatus without first electrifying it, but keeping the glass tube at a high temperature, we found no deflection on the electrometer. But on repeating this experiment with copper foil in the tube an electrification of 9 volts *positive* for 200 strokes was observed. Replacing the copper foil by charcoal, with temperature high enough to keep the charcoal visibly burning, we found a *negative* electrification of 7 volts for 160 strokes.

Fig. 6.

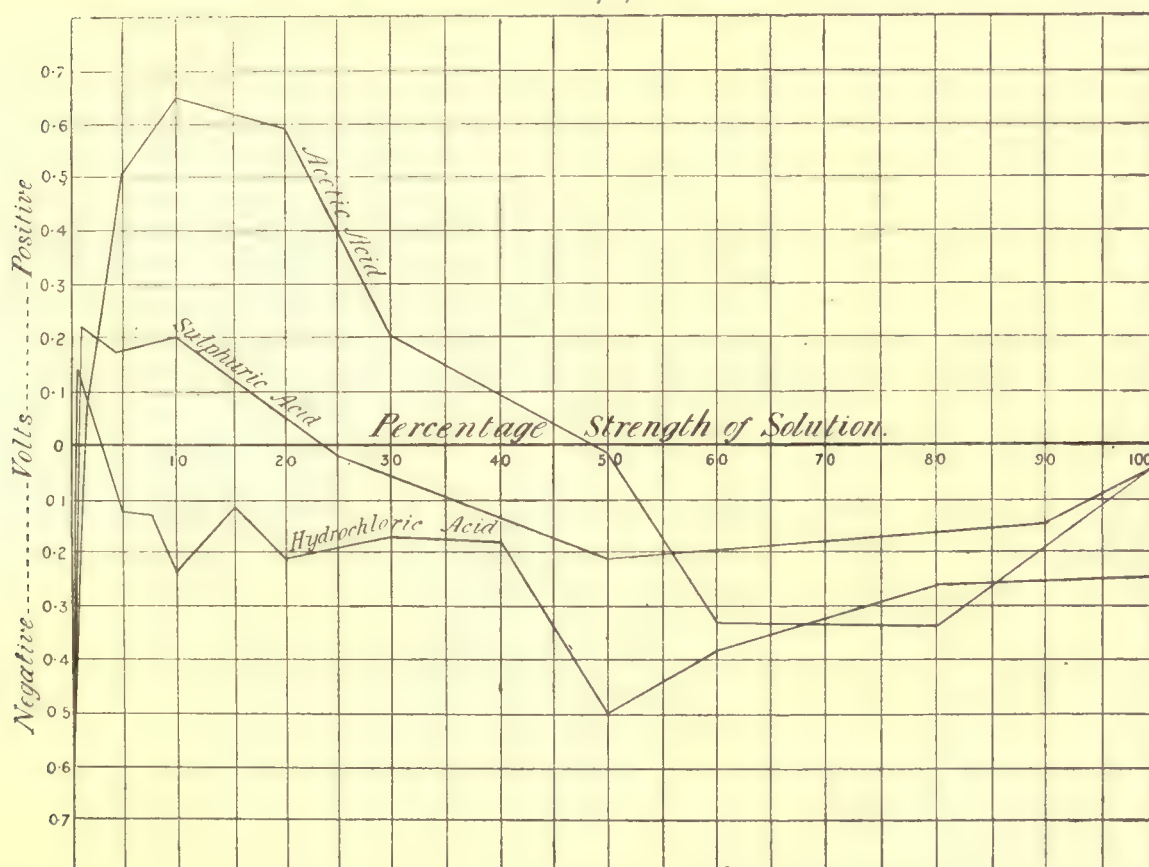


## ELECTRIFICATION PRODUCED BY SHAKING AIR AND OTHER GASES WITH WATER AND WITH SOLUTIONS OF DIFFERENT SUBSTANCES (§§ 41-46).

§ 41. An ordinary Winchester glass bottle, A, fig. 6, of capacity 2500 cub. centims., has two broad strips of tin foil cemented on its outer surface on opposite sides, from

top to bottom. To the foil at the shoulder is attached a metallic disc, D, of 5 centims. diameter, and having a small hole to allow a connecting wire from the quadrant electrometer to be quickly hooked or removed. 500 cub. centims. of Loch Katrine water having been put into the bottle, its mouth is stopped by hand, and the bottle vigorously shaken for 5 seconds, thoroughly and violently mixing the enclosed 2000 cub. centims. of air and 500 of water. It is now immediately placed on a block of paraffin, P, and the disc, D, is connected to the electrometer. A bent metallic tube, T, supported by an insulating paraffin stopper, is placed in the bottle, and a foot

Curves 1, 2, 3.

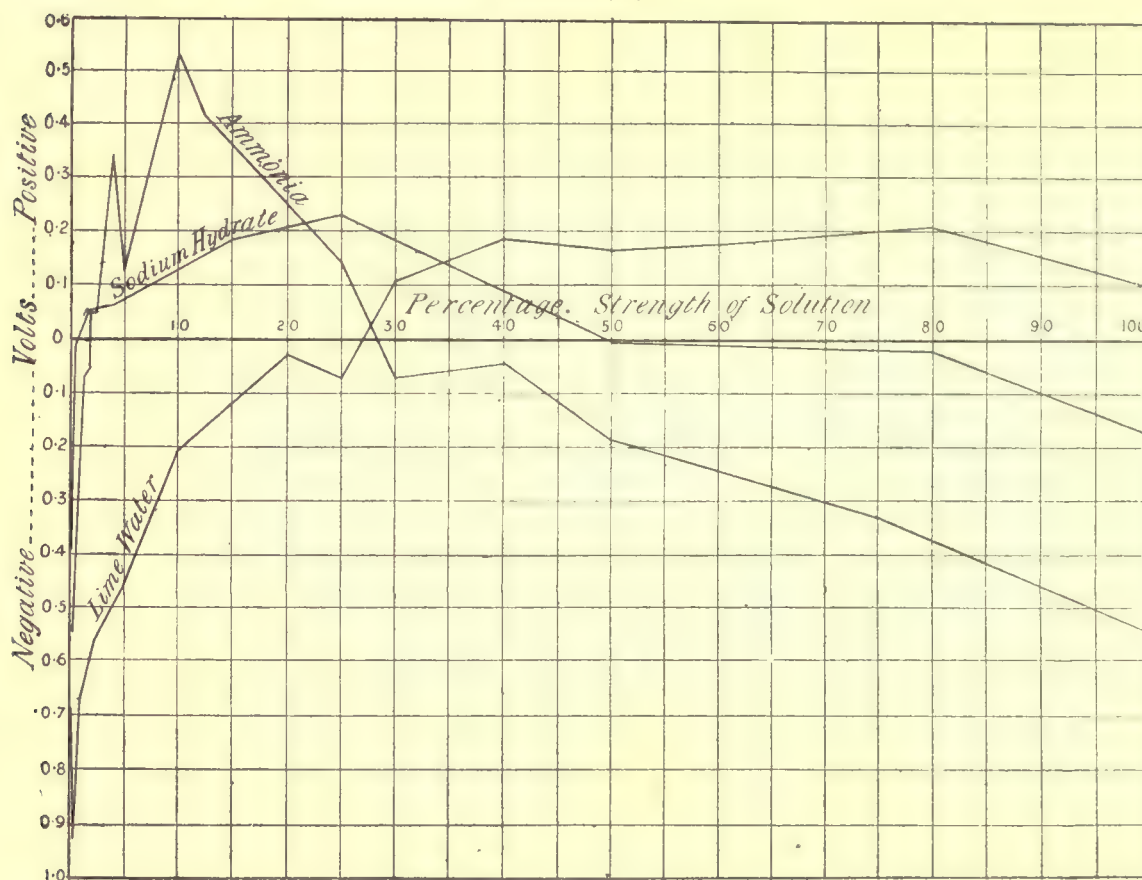


length of rubber tube connects it to one end of a paraffin tunnel, from the other end of which a rubber tube, 2 metres long, passes to an aspirator consisting of a large bell jar, B, of capacity 8500 cub. centims., filled with water, and resting on supports near the water surface. The metallic guard-screen, S, which surrounds the bottle, is always connected to the sheath of the electrometer. The outer surface of the bottle is always wet. Less than half a minute is required after shaking the bottle to make the necessary arrangements and connections. The electrometer terminal connected with the bottle is now insulated, and then, by opening the stop-cock of the aspirator, air is drawn rapidly out of the insulated bottle, its place being taken by air flowing in

through a small vertical slit in the paraffin stopper. The electrometer shows positive electricity, which proves the withdrawn air to have been negatively electrified.

§ 42. Very many experiments were made to test the effect of shaking up the air in the bottle with solutions of different substances, the solutions being varied from saturated (100 per cent.) down to practically pure water. For this purpose three acids (sulphuric, hydrochloric, and acetic), three alkalies (sodium hydrate, lime, and ammonia), and three salts (sodium chloride, sodium carbonate, and zinc sulphate), were used.

Curves 4, 5, 6.

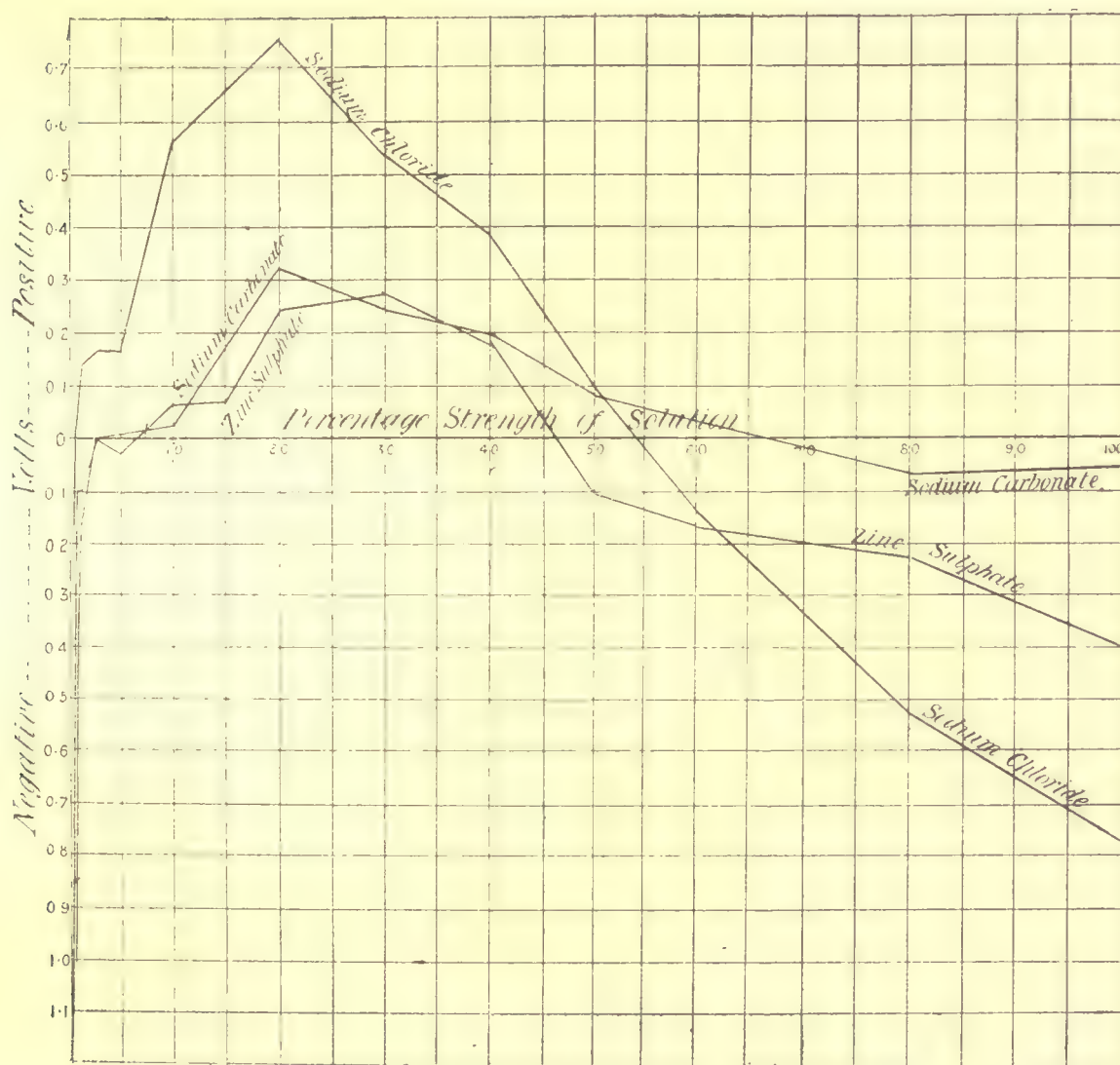


§ 43. Curves 1 to 9 show the results obtained. Generally, each test was repeated a large number of times, and the curves are drawn for mean values. It will be noticed that, with the exception of lime water, all the acids, alkalies, and salts, when added to the water in the bottle in very minute quantities, showed a rapid diminution in the negative electrification of the air produced by shaking it up with the liquid. In some cases a single drop of a saturated solution of the substance added to the water and shaken up with the air was almost enough to entirely neutralize the negative electrification of the air which is obtained by shaking up with pure water. On gradually increasing the strength of any of the solutions named, the zero line is



crossed and the electrification of the air now becomes positive, increasing to a maximum and then diminishing till the zero line is again reached, the air becoming negative again. The air continues to receive negative electrification, on shaking it with the solution, for all further increase of strength of the solution up to saturation. All the acids, all the alkalies except lime water, and all the salts tested, show two zero points.

Curves 7, 8, 9.



§ 44. Experiments were now tried in which different liquids (500 cub. centims.) and gases (2000 cub. centims.) were shaken up in the bottle, and the electrification of the gas in each case was tested. The following is a tabular summary of the results. In all the experiments, except No. 1, the gas was got into the bottle by displacement of water. The bottle was first filled with water and inverted in a large trough of water and 2000 cub. centims. of the gas admitted.



	Substances shaken up together in the bottle.		Deflection on electrometer. 1 volt = 69 divisions.	Quantity of electricity found in the gas per cub. centim., C.G.S. electrostatic unit.
	Liquid.	Gas.		
1	Water	Coal gas from gas mains . . . . .	37 pos.	$0.17 \times 10^{-4}$ neg.
2	"	Coal gas from pressure cylinder . . . .	89 "	$0.42 \times 10^{-4}$ "
3	"	Oxygen from cylinder . . . . .	52 "	$0.24 \times 10^{-4}$ "
4	"	Carbonic acid gas from cylinder . . . .	50 "	$0.23 \times 10^{-4}$ "
5	"	Carbonic acid gas from marble and hydrochloric acid, the gas being passed direct from generator to bottle . . . .	34 "	$0.16 \times 10^{-4}$ ..
6	"	Carbonic acid gas, as in 5, but allowed to pass into a gasholder first . . . . .	30 "	$0.14 \times 10^{-4}$ ..
7	"	Hydrogen from zinc and dilute sulphuric acid, direct from generator. . . . .	33 "	$0.15 \times 10^{-4}$ ..
8	"	Hydrogen, as in 7, but allowed to pass into gasholder first . . . . .	20 "	$0.09 \times 10^{-4}$ ..
9	"	Nitrogen from air, the oxygen being removed by burning phosphorus . . . .	71 "	$0.33 \times 10^{-4}$ ..
10	Water impreg- nated with car- bonic acid from a gazogen or siphon . . . .			
11	Strong solution of common salt	Carbonic acid gas, as in 4 . . . . .	40 ..	$0.19 \times 10^{-4}$ ..
12	"	Carbonic acid gas, as in 4 . . . . .	68 neg.	$0.32 \times 10^{-4}$ pos.
13	"	Coal gas from mains . . . . .	71 "	$0.33 \times 10^{-4}$ ..
	"	Oxygen, as in 3. . . . .	43 "	$0.20 \times 10^{-4}$ ..

§ 45. Some experiments were made to ascertain how long the air, on being shaken up with Loch Katrine water, retains its negative electrification.

Time between stopping the shaking of the bottle and beginning to draw out the air.	Deflection of the electrometer. 1 volt = 69 divisions.
	Divisions.
25 secs.	52 pos.
27 "	50 "
1 min.	26 "
2 mins.	28 "
5 "	17 "
15 "	7 "
15 "	10 "
30 "	1 "

§ 46. A strong solution of ammonia (500 cub. centims.) was placed in the bottle. Without shaking the bottle, the mixed air and ammonia evaporating from the surface of the liquid were aspirated. They were found to be negatively electrified to a fraction of a volt.



bent as shown. A gas burner, G, was fixed in a brass tube which was connected by sealing wax to a bent glass tube varnished with shellac leading in from the gas supply.

§ 49. The method of experimenting was as follows:—The gas was lit at the burner, G, and the machine started. The machine was worked for 4 minutes at potentials from 3000 to 10,000 volts in different experiments. At the instant of stopping the machine the gas supply was stopped, and an indiarubber cork put into the tube, U, which was used to keep the gas in the cylinder at atmospheric pressure during the electrification. Ten strokes of the pump were taken in every experiment, immediately or after the lapse of certain intervals of time from stopping the machine.

The results of a large number of experiments performed in May, 1895, are briefly summarised in the following table:—

ELECTRIFICATION OF AIR BY ELECTRIFIED COAL-GAS FLAME.

Interval between stopping electric machine and starting pump.	Number of experiments.	Average deflection by 10 strokes of the pump. (47 divs. per volt.)	Remarks.
minutes.		divs.	
0	3	+210	Potential of electrifica- tion $\pm$ 3000 volts.
0	3	—234	
1	2	+108	
1	2	—110	
2	2	+76	
2	2	—94	
3	3	+100	
3	3	—82	
5	2	+110	
5	2	—107	
30	1	+40	
30	1	—36	
60	1	+53	
0	3	+360	Potential of electrifica- tion $\pm$ 4000 volts.
0	2	—395	
1	1	—90	
5	1	—52	
10	1	—45	
15	1	—36	

§ 50. We now electrified the air in the glass cylinder by six needle points without flame. The method of experimenting was in other respects exactly the same as before. A very wonderful result was noticed, namely, that short times of electrification at 5000 volts gave greater results as indicated by electrometer and filter than longer times of electrification.



Time of electrification.	Number of experiments.	Average deflection for 10 strokes of pump.
30 secs.	6	30.6
1 min.	6	22.1
5 "	6	0

§ 51. A conjecture occurred to us that this very surprising result might possibly be due to the formation of lines of conductance from the electrifying needle to the inside of the containing cylinder, so that all the electricity from the needle might be passing to the glass and tinfoil strips without electrifying the intervening air, and that if means were provided for continually breaking these conjectural lines of conductance the results would be different. Accordingly a hexagonal ring of sheet tin an inch broad was constructed to serve as stirrer. Its outer diameter was slightly less than the diameter of the cylinder. Two rods which passed through two air-tight holes in the metallic top of the glass cylinder were fixed at the ends of a diameter of this stirrer. Several experiments were tried (1) with the stirrer not moved but resting near the surface of the water at the bottom of the cylinder, (2) with the stirrer kept moving up and down during the electrification, which was by needle points at 5000 volts. in each experiment. The results obtained disprove our conjecture. They are summarised as follows :—

Time of electrification.	Number of experiments.	Average deflection per 10 strokes of the pump.	Remarks.
30 secs.	5	16.0 neg.	No stirring.
30 "	6	16.3 "	Stirring.
1 min.	6	11.6 "	No stirring.
1 "	6	13.3 pos.	"
1 "	5	9.3 neg.	Stirring.
1 "	6	10.4 pos.	"
5 "	No deflection, stirring or no stirring!		

#### ELECTRIFICATION OF CARBONIC ACID GAS FROM PRESSURE CYLINDER.

§ 52. One of the carbonic acid cylinders of the Scottish and Irish Oxygen Company was taken and laid on its side near the glass cylinder, C (fig. 7). Carbonic acid gas from it was let in by a tube into the glass cylinder, and electrified by six needle points. To start with, the stop-cock of the carbonic acid gas cylinder was kept shut, and ten strokes of the pump caused the water in the cylinder to rise. The stop-cock was then opened and carbonic acid was allowed into the cylinder, the average time of letting it in being one minute. This was performed five or six times,

and then the experiments were done exactly as already described in § 49, except that carbonic acid gas was let in instead of air. The machine, in the two sets of experiments summarised below kept the needle points at a potential of 8000 volts positive.

## ELECTRIFICATION OF CARBONIC ACID GAS BY ELECTRIFIED NEEDLE POINTS.

Time of electrification.	Interval between stopping machine and starting pump.	Deflection for 10 strokes.
min.	min.	
1	0	463 pos.
1	2	303 „
1	5	158 „
1	10	97 „
1	30	43 „
0	..	4 pos.*
1	0	414 „
5	0	106 „
2	0	309 „
10	0	2 „
1	0	554 „
		in 9 strokes.
0	..	128 neg.†
1	0	554 pos.
		in 7 strokes.
1	1	338 pos.
1	2	254 „
1	5	160 „
1	10	89 „
1	30	37 „
1	60	6 „

§ 53. The nearly perfect annulment of the electrification by 10 minutes of the electrified needle point in carbonic acid, and by 5 minutes (§§ 50, 51) in common air, is very wonderful. So far as we can see at present, its explanation seems to be a conductive quality, such as that first discovered, we believe, by SCHUSTER, produced throughout the carbonic acid, and throughout the air, by continued electric disruptive action.

§ 54. It was noticed in some of the experiments with the carbonic acid cylinder that, when it was lying on its side on the table beside the glass vessel, C, and when the gas issued in a jerky manner, the electrification found was invariably negative, even when the needle points were positive. This was, of course, due to the boiling

\* This first experiment was made with the carbonic acid gas which had been in the glass cylinder overnight.

† For explanation of this negative electricity in the gas with no electrification of the needle points, see § 54.

and freezing, and ultimate evaporation from the liquid carbonic acid in the cylinder. Various experiments were tried, with the cylinder (1) lying on its side, (2) standing, straight up: in both cases without any electrification of the needle points. Very *slight positive electrification* was found with the cylinder straight up; and *very large negative electrification* when the cylinder was lying on its side. In the latter case, the more rapid the rush of the carbonic acid gas from its cylinder, and the shorter the time it was left in the glass vessel before observations were taken, the greater was the electrification observed.

TESTING EFFICIENCY OF DIFFERENT FILTERS BY USING TWO AT A TIME IN SERIES  
AND TWO ELECTROMETERS (§§ 55–60, with reference to §§ 11, 12 above).

§ 55. For the purpose of testing the efficiencies of various filters, two electrometers were fitted up side by side, as shown in fig. 7, each with a filter of block-tin pipe, 4 centims. long and 1 centim. bore, containing six wire gauzes and five plugs of cotton wool. These filters were put in series, with a paraffin tunnel between them to insulate the one from the other. Thus the electrified air passed the second receiver immediately after it passed the first. Call the one filter AB and the other filter A'B'; then, in the first experiment, the electrified gas passed in the direction AB A'B', and in the second experiment it passed in the direction B'A' BA. Each filter is kept metallically connected always to the same electrometer. The sensitiveness of electrometer, E, with the filter, AB, was 52·3 divisions per volt, and 100 divisions per 0·154 electrostatic unit of electricity. The sensitiveness of electrometer, E', with the filter, A'B', was 158·3 divisions per volt, and 100 divisions per 0·071 electrostatic unit of electricity.

§ 56. Air was electrified negatively by six needle points in the glass cylinder and the pump worked at the same time with the U-tube open, so that the water did not rise inside. 100 strokes of the pump gave a deflection of 160 divisions on electrometer, E, and 32 divisions on electrometer, E'. This gives

$$\begin{aligned} q_1 &= 0\cdot246 \text{ electrostatic unit,} \\ q_2 &= 0\cdot023 \quad \text{,,} \quad \text{,,} \end{aligned}$$

Reversing the direction of the current of air through the filters, 100 strokes of the pump gave a deflection of 156 divisions on electrometer, E', and 123 divisions on electrometer, E. This gives

$$\begin{aligned} q_1' &= 0\cdot111 \text{ electrostatic unit,} \\ q_2' &= 0\cdot189 \quad \text{,,} \quad \text{,,} \end{aligned}$$

Hence (§ 12)

and

$$n = 0\cdot77 \text{ (filter AB),}$$

$$n' = 0\cdot31 \text{ (filter A'B').}$$



Similar numbers were got when the air was electrified positively.

For carbonic acid gas, electrified positively and negatively, the same filters gave

$$n = 0.82 \text{ (filter AB),}$$

$$n' = 0.42 \text{ (filter A'B').}$$

§ 57. We now used the filter ( $n = 0.77$ ) for determining electric density of electrified air or gas. A known volume of the electrified gas (§ 47) was sucked through the filter in connection with an electrometer whose constant was determined as in § 18 (0.154 C.G.S. electrostatic quantity per 100 divisions deflection). We thus found—

(1.) For air electrified, positively or negatively, by six needles at a potential of 5000 volts, an electric density of  $0.92 \times 10^{-4}$  C.G.S. per cub. centim.

(2.) For air electrified, positively or negatively, by electrified gas flame at a potential of 5000 volts, an electric density of  $1.98 \times 10^{-4}$  C.G.S.

(3.) For carbonic acid gas, electrified negatively by gas out of a cylinder lying on its side (§ 54), or positively by six needle points at a potential of 5000 volts, an electric density of  $2.4 \times 10^{-4}$  C.G.S.

§ 58. We now set about to definitely determine the relative efficiencies of various forms of filters. A standard filter of block-tin pipe, 4 centims. long and 1 centim. bore, with 6 brass gauzes and 5 plugs of cotton wool was used, and it was permanently kept in metallic connection with electrometer, E'. The filter to be tested was joined to electrometer, E. Air electrified positively or negatively was sucked through in one direction, passing through the tested filter first, and then through the standard filter, the diselectrifying power of which was 0.77 for electrified air. Hence it is possible to determine the diselectrifying power of the tested filter by § 14. Thus, from the numbers in the following section, we get for the diselectrifying power, for positive electricity, of the 7 millim. brass tube—the least effective of those mentioned— $n = 1 \div (1 + 4/0.77) = 0.16$ ; and for the block-tin pipe, 90 centims. long, and coiled into a spiral  $n = 1 \div (1 + 0.63/0.77) = 0.55$ .

§ 59. Let  $q_2$  = quantity of electricity taken out by the standard filter of block-tin pipe with six brass gauzes and five plugs of cotton wool; and

$q_1$  = the quantity of electricity taken out by the tested filter from the air before passing the standard filter. A long series of experiments with no wire gauze or cotton wool in the tested filters is summarised in the following tables. The potential of the machine was in each experiment 10,000 volts.

Bore of brass tube in millims. (length = 4 centims.).	$\frac{q_2}{q_1}$ for positive.	$\frac{q_2'}{q_1}$ for negative.
2.0	3.0	2.75
3.4	2.1	3.3
4.2	3.5	3.1
6.0	3.9	3.1
7.0	4.0	5.5

Thus the 3.4 millims. filter is the most effective and the 7.0 millims. filter is the least effective of these five filters of equal length. The following table shows results for different lengths and different bores :—

Length of tube.		Bore of tube in millims.	$\frac{q_2}{q_1}$ positive.	$\frac{q_2'}{q_1}$ negative.
1. Brass,	9.9 centims. . . . .	2.0	2.96	3.05
2. "	10.0 " . . . . .	4.5	2.85	4.99
3. "	9.9 " . . . . .	8.0	4.0	6.45
4. Block tin,	2.0 " . . . . .	6.0	3.1	1.6
5. "	10.0 " . . . . .	6.0	3.26	7.5
6. "	90.0 " (coiled in spiral)	6.0	0.63	1.15
7.*	4.0 " . . . . .	10.0	1.37	1.31
8. "	4.0 " . . . . .	10.0	2.96	2.8

§ 60. A glass tube filter, 4 centims. long and 3.3 millims. in bore, covered outside with strips of tinfoil along its length, was similarly compared with the standard filter. When newly put up, and as long as the glass was dry, it took out very little electricity from the air; but as the experiment proceeded, and the glass became less dry by taking up moisture on its inner surface, the quantity of electricity taken out by the glass tube became greater and greater. Thus in a first experiment  $q_2/q_1 = 28.5$ ; but after working the pump for one hour  $q_2/q_1 = 3.8$ .

§ 61. Up to this time (December, 1895), we had not been able to find a filter which could take all the electricity from the air, and we now proceeded to search for a filter which would be able to practically do so. The first filters tried with this object were tubes filled with very small pieces of fine copper wire, and closed at each end by a plug of cotton wool and a disc of brass gauze. The diameter of the wire was 0.00296 centim. The containing tube was in one case block-tin pipe 10 centims. long and 1 centim. diameter: and in another it was a glass tube of the same length and diameter, coated both inside and outside with longitudinal strips of tin-foil. The diselectrifying power of each was calculated from the observations by the

\* Tube No. 7 had twelve wire gauzes inside it, and, as the table shows, its filtering efficiency was more than that of the equal and similar tube No. 8, which was clear inside.



formulae in § 12. That of the block tin was thus found to be 0.93 for negative electricity and 0.9 for positive, and that of the glass nearly the same, 0.9 for negative and 0.84 for positive. The block-tin filter contained 18.927 grms. of wire.

§ 62. The next filter tried was a block-tin pipe, 10 centims. in length and 1 centim. in diameter, containing 8.33 grms. of clean, very fine brass filings, enclosed at each end by a plug of cotton-wool and a piece of brass gauze. These brass filings, which were got from a brass-finishing workshop, were poured into a glass tube about 4 feet long and about three-quarters full of water. The filings and water were well shaken up, and the tube was then allowed to stand for several hours, so as to give the filings time to settle. After washing the filings three times in this manner, the top portion was taken off and dried before a fire, and used for filling the filter. It was found that when the electrified air passed through this filter of brass filings, before it passed through the copper-wire filter attached to electrometer, E, no deflection was obtained on the latter. This showed that the brass-filings filter deprived the air of practically all its electricity. We tried also a filter with sawdust instead of brass filings, but its efficiency was comparatively low.

#### FILTER OF BRASS FILINGS USED TO FIND EFFECTS OF SPIRIT FLAME, COAL GAS FLAME, AND HYDROGEN FLAME IN ELECTRIFYING AIR (§ 63-66).

§ 63. Having found the brass-filings filter thoroughly satisfactory, we used it to investigate the effect of various kinds of flames in electrifying air. First of all, we electrified the air of the laboratory by means of an insulated spirit flame, joined to the insulated positive terminal of a Voss electric machine. The machine was worked for 40 minutes, and then, 2 minutes after it was stopped, the electrification of the air in the vicinity was tested by drawing some of it through a tube leading to the brass-filings filter, joined to electrometer, E. The pump was worked at the rate of 1 stroke per 4 seconds, and after 200 strokes the electrometer read 174 divisions, or about 3.3 volts positive. After the lapse of half-an-hour, the air of the laboratory was found to be still strongly charged with positive electricity.

§ 64. Removing the water vessel from below the glass cylinder, C, in fig. 7, and substituting for it a metal plate kept in metallic connection with the sheath of the electrometer and the disinsulated terminal of the machine, we kept a coal-gas flame burning within the glass cylinder, while the machine and pump were worked and observations taken by electrometer, E, and filter, AB. With the machine at 10,000 volts, we found that, after two or three strokes of the pump, the deflection was about 500 scale divisions, positive or negative, according as the machine was positive or negative. This gives (§ 13) for the electric density, per cub. centim.,  $11 \times 10^{-4}$  C.G.S. electrostatic, which is much greater than any of our previous results (§§ 23, 57).

§ 65. To burn hydrogen, the burner, G of fig. 7, was made of rolled platinum foil with a fine nozzle, which was kept in metallic connection with the insulated terminal of the electric machine. The hydrogen gas was generated from zinc and dilute



sulphuric acid in a Woulff bottle. The electrification which we obtained in this way, with the machine at 10,000 volts, was very large, the greatest deflection being 500 divisions in one stroke of the pump. This indicates an electric density in the air of the glass cylinder, of  $22 \times 10^{-4}$  C.G.S. electrostatic, which is about six times as great as that obtained by electrified needle points (§ 23). This electric density was got for both positive and negative electrification.

§ 66. We next tried the effect of the insulated hydrogen flame alone, without working the electric machine, and we found that when the height of the liquid rising in the long, open glass tube of the Woulff bottle was not more than about 10 centims. above the level of the liquid in the bottle, there was a small negative electrification. When the liquid rose to a greater height than 10 centims. in the tube (indicating that the gas was issuing at a greater pressure to feed the flame), the electrification was positive. On one occasion, the positive electrification produced by the flame was  $0.84 \times 10^{-4}$  C.G.S. electrostatic unit per cub. centim. of the air which carried the electricity to the filter. This was the greatest effect obtained from the flame without electrification by the machine, and the height of the liquid in the tube of the Woulff bottle was 14.5 centims.

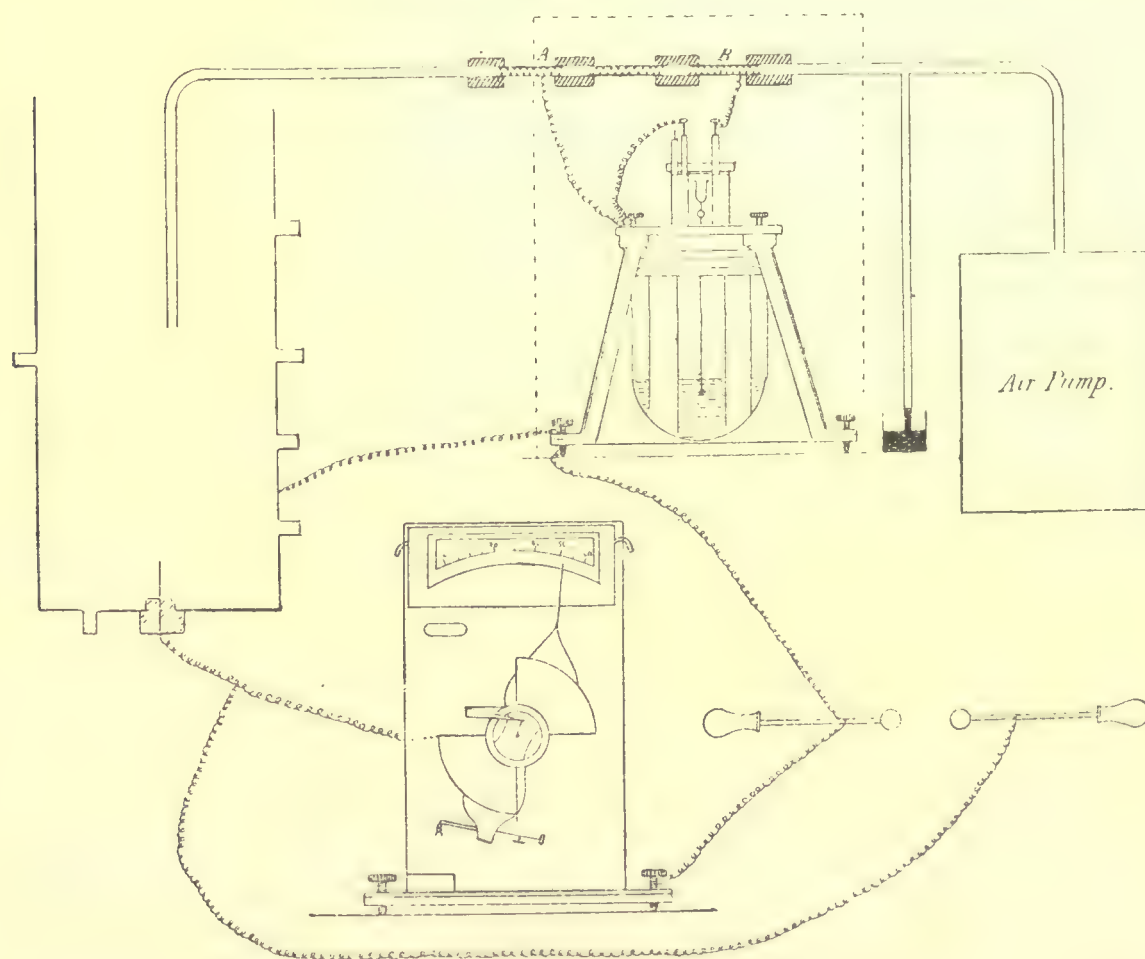
The hydrogen gas, when not burning, gave no electrification at any pressure up to 26 centims. of water.

#### PLATINUM TUBE HEATED EITHER BY A GAS FLAME OR BY AN ELECTRIC CURRENT (§ 67).

§ 67. Through the kindness of Mr. E. MATTHEY, we have been able to experiment with a platinum tube 96 centims. long and 1 millim. bore. It was put in between the glass cylinder, C, and the filter, AB, in the apparatus of fig. 7. The other filter, A'B', was not used in these experiments. The platinum tube was heated either by a gas flame or an electric current. When the tube was cold, and non-electrified air drawn through it, we found no sign of electrification by our filter and electrometer. But when the tube was made red or white hot, either by gas burner applied externally or by an electric current through the metal of the tube, the previously non-electrified air drawn through it was found to be electrified strongly positive. To get complete command of the temperature, we passed a measured electric current through 20 centims. of the platinum tube. On increasing the current till the tube began to be at a scarcely visible dull red heat we found but little electrification of the air. When the tube was a little warmer, so as to be quite visibly red hot, large electrification became manifest. Thus 60 strokes of the air-pump gave 45 scale divisions on the electrometer (0.86 of a volt) when the tube was dull red, and 395 scale divisions (7.5 volts) when it was a bright red (produced by a current of 36 amperes). With stronger currents raising the tube to white-hot temperature the electrification seemed to be considerably less. The following summary may be taken as a specimen of several experiments. It is a copy of our notes of an experiment made on 20th July, 1895:—

Time of starting.	Deflection after 60 strokes. 52.3 divs. per volt.	Time of 60 strokes.	Current in amperes.	Remarks.
hrs. mins. secs.	Divisions.	mins. secs.		
11 6 0	1.0 pos.	5 15	0	Air left overnight in glass cylinder drawn through platinum tube
11 17 0	3.5 neg.	4 50	19.7	Tube hot, but not visibly red
11 27 0	1.0 "	5 30	26.4	Tube beginning to be dull red
11 34 15	395 pos.	5 18	33.5	Tube very bright red
11 42 30	45 "	5 40	28.6	One end dull red. End next ingress of air always duller
11 50 15	0	5 10	26.4	
11 57 30	9 pos.	5 30	27.5	Tube red before using pump
12 5 30	37.5 "	5 30	28.6	One end perceptibly red.
12 14 10	190 "	5 47	30.5	One-third next ingress moderately red ; other two-thirds (14 centims.) dull red
12 22 0	174 "	5 58	34.5	Nearly whole tube (20 centims.) bright red
12 29 30	86 "	5 51	39.0	White hot
12 46 15	58 "	5 1	46.6	White hot
12 58 30	0	3 57	0	

Fig. 8.



§ 68. *The Diselectrifying Power of Various Filters* was farther tested as follows :—

Air, electrified in a metal vessel by a needle point kept electrified by a Voss electric machine, was drawn through 340 centims. of block-tin pipe (0·91 centim. bore), to one or other of the experimental filters which was connected to sheath (fig. 8). After passing through A, the air was drawn through an insulated testing filter, B, connected to the insulated terminal of the electrometer. B was a standard filter of block-tin pipe, 5 centims. long, 0·66 centim. bore, and filled with fine brass filings. Tests were occasionally made with A removed, to ascertain the diselectrifying power of the standard filter (B) alone ; then A being inserted, the effect on B was again noticed.

Filter A.	Deflection for 6 strokes of pump.		Percentage extracted by A.
	A. removed.	A. in position.	
	divisions.	divisions.	
(1.) Standard filter, similar to B. . . . .	220	35	84
(2.) Brass tube, 5 centims. long, and 0·18 centim. bore	192	158	18
(3.) Solid brass cylinder, rounded ends, 8·1 centims. long, and 1·8 centim. bore	222	132	40
(4.) Block-tin tube, 5 centims. long, and 0·91 centim. bore	160	158	1·25

§ 69. The standard filter, B, was now removed, and the various tubes used in the last experiments as tested filters (A's), were now tried separately as testing filters (that is, insulated filters, B), connected to the insulated terminal of the electrometer. The following are the particulars and deflections noted for 6 strokes of the pump in 1 minute :—

Filter used.	Divisions of deflection.	
	Positive.	Negative.
BLOCK TIN { 5 centims. long and 0·91 centim. bore . . . . .	none	none
10 " " 0·91 " . . . . .	"	"
10 " " 0·66 " . . . . .	4	4
4·1 " " 0·18 " . . . . .	5	none
BRASS . { 10 " " 0·18 " . . . . .	5	5
10 " " 0·50 " . . . . .	5	11
17·5 " " 0·90 " . . . . .	6	8
Solid brass (rounded ends) 8·1 centims. long and 1·8 centims. diameter	45	
Standard filter, block tin, 5 centims. long and 0·66 centim. bore, filled with brass filings	90	100



§ 70. We then tried as filters tubes of different materials, but all of the same length (10 centims.), and bore (0·91 centim.). Glass, brass, block tin, copper, and zinc were used: the glass tube was covered externally with tin-foil, and also a little way inside at each end. As usual, the air in the can was charged from the insulated needle point at 4000 volts positive, and 3200 volts negative (§ 74), and drawn off through 340 centims. of block-tin pipe of 0·91 centim. bore, extending from the centre of the can to the insulated filter, which was either glass, brass, or block tin. Before testing the copper and zinc tubes, the can was brought nearer to the electrometer, so that the length of the block-tin pipe conveying the electrified air to the filter was reduced from 340 centims. to 100 centims. The mean of a large number of tests gave the following deflections for 6 strokes of the pump in 1 minute, the mean result in every case being very similar to the individual results.

Filter used.	Deflection per 6 strokes of pump.			
	Positive.		Negative.	
Glass (covered outside with tin-foil) . . .	1·1	mean of 3 experiments	7·0	mean of 3 experiments
Brass . . . . .	1·4	„ 2 „	3·8	„ 2 „
Block tin . . . . .	3·3	„ 2 „	3·1	„ 2 „
Copper . . . . .	6·3	„ 4 „	8·0	„ 3 „
Zinc . . . . .	8·4	„ 6 „	2·3	„ 6 „

§ 71. The zinc filter was the only one which showed a distinctly greater deflection for positive than for negative electricity, a result which is opposite to one obtained by RUTHERFORD,\* experimenting with air which had been electrified from an electrified body under the influence of RÖNTGEN rays. The previous experiments having shown that, on drawing electrified air over the insulated and non-electrified solid brass cylinder with rounded ends, the cylinder extracted from the air a large proportion of its charge, we now arranged to charge the cylinder and draw non-electrified air over it to the standard insulated filter connected with the electrometer, to see if the air would take up from the cylinder a part of its charge. The arrangements were as shown in fig. 9. The cylinder was charged positively and negatively at potentials varying from 1000 volts to 15,000 volts, *but no trace of electrification of the air after passing over the brass cylinder could be detected.*

*The Effects of the Uranium “rays,” discovered by BECQUEREL, and of a Candle Flame, on Electrified Air.*

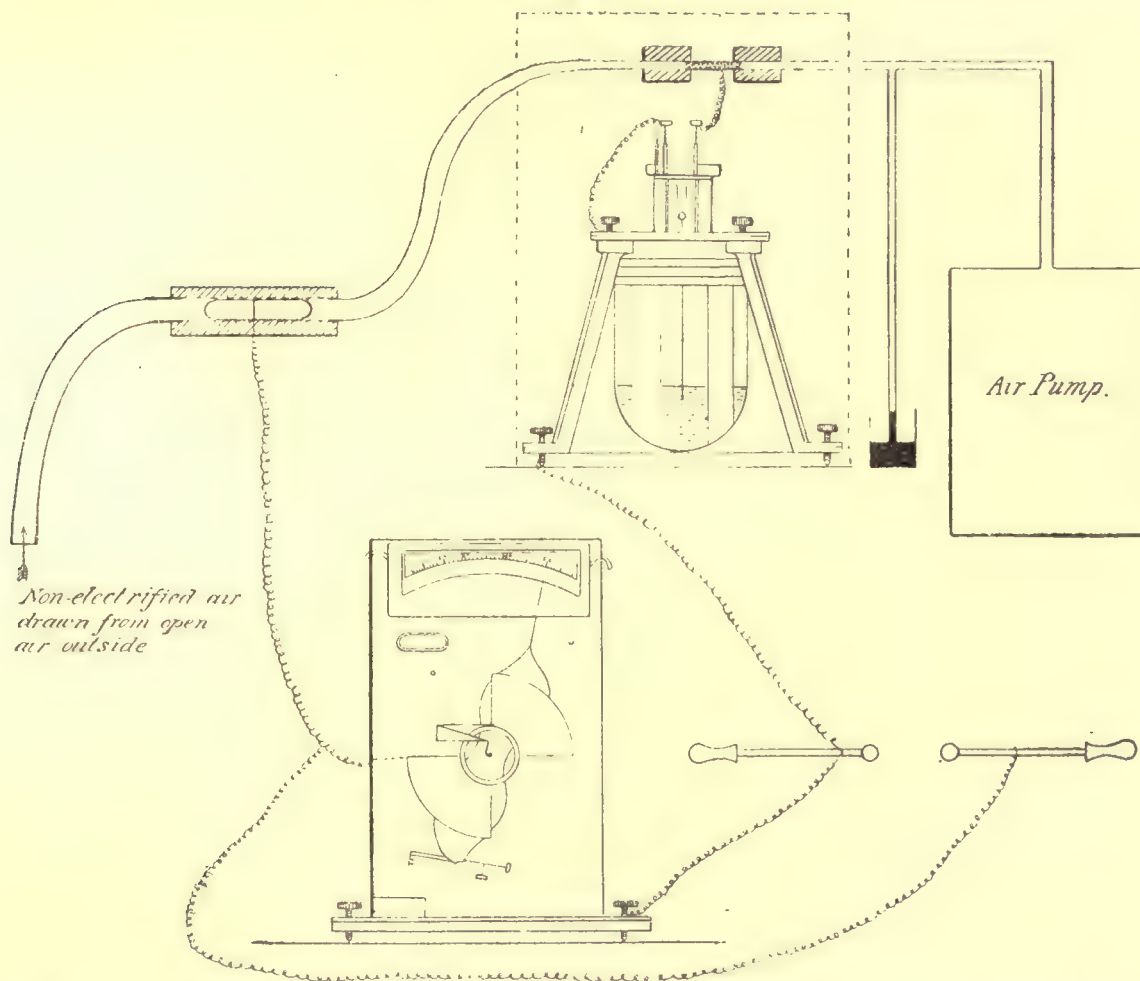
§ 72. The air in the can (fig. 8) was charged in the usual way by a needlepoint at 3200 volts negative for 1 minute, and the electric machine was then stopped; the

\* See ‘Phil. Mag.’ April, 1897, p. 246.

electrified air was drawn from a point half-way down the can to the insulated standard filter connected to the electrometer, and the deflection noted. The experiment was repeated with a uranium plate suspended three-quarters of the way down in the can by wires metallically connected with the can. The following results were noted :—

Without uranium, 50 strokes of the pump were required before the electrometer ceased to give a deflection, the total deflection being 271 divisions.

Fig. 9.



The uranium was now placed in position in the can, and the air was then charged for one minute. It was kept in position till all the electrified air was drawn off to the filter, the total deflection being 61 divisions. When the uranium was inserted after the electric machine was stopped, and before the air in the can was drawn to the filter, little more than 10 strokes were required before the electrometer ceased to give a deflection, and the deflection was now 121 divisions.

Using a very small lighted candle instead of the uranium plate, we found the following results :—

Without the candle, 195 divisions negative for 60 strokes were noted.

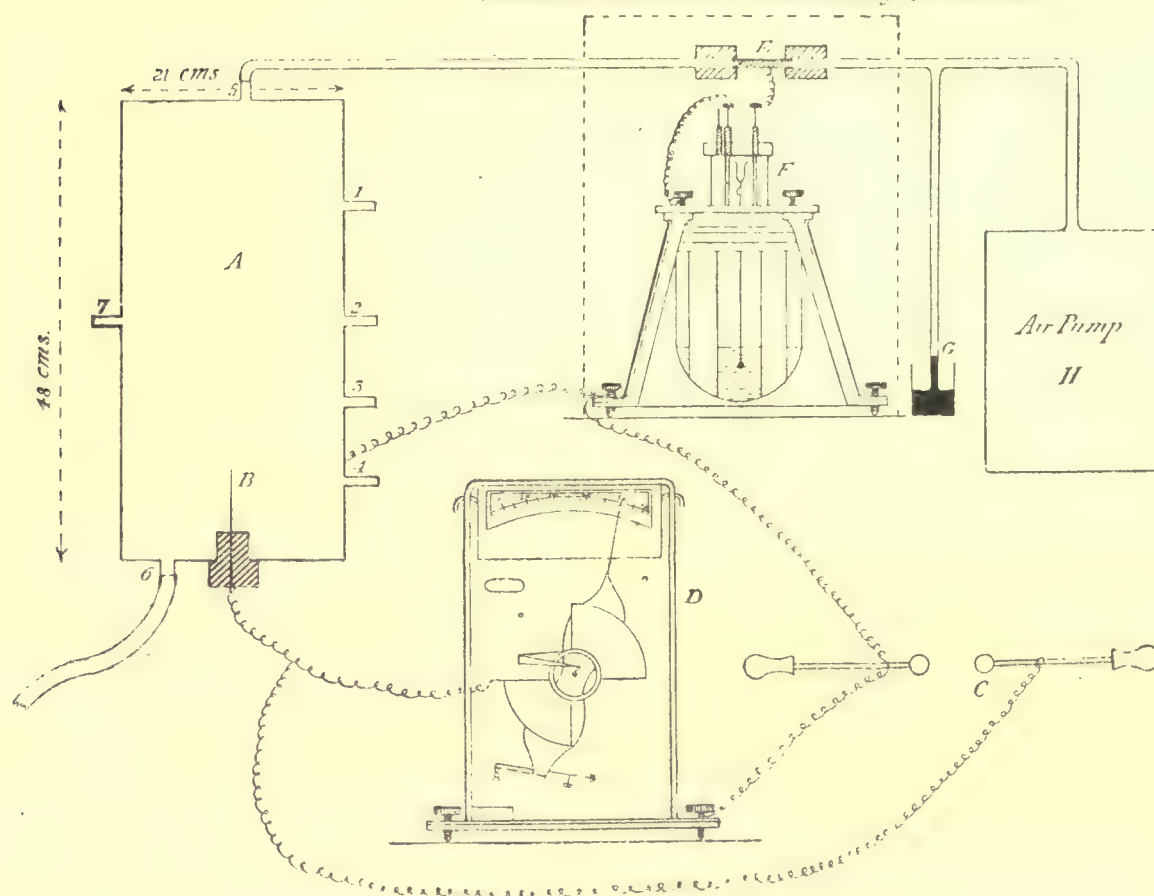
With the candle inserted when about to withdraw the electrified air to the filter, the deflection was, for 40 strokes, 81 divisions.

*Best Charging Potential for Air and for Carbonic Acid Gas, in a Cylinder of Metal, 48 centims. long and 21 centims. diameter. Greatest Electric Density Obtained, and Loss of the Electrification.*

§ 73. Numerous experiments were made to find the charging potential which would give the greatest electric density to air drawn off from a metallic can, A, fig. 10. An insulated needle-point, B, was fixed by a paraffin stopper in the bottom

Fig. 10.

*Metallic Gauze Screen surrounding Electrometer.*



of the can, and was connected with the insulated terminal of a Voss electric machine, C, and of a vertical electrostatic voltmeter, D. A pipe passed from aperture No. 5 in the top of the can to a standard electric filter, E, insulated on



two paraffin blocks. The filter was of block-tin tube, 5 centims. long and 0·6 centim. bore, and was filled with fine brass filings. It was connected by a short platinum wire to the insulated terminal of a quadrant electrometer, F, and beyond the filter were tubes passing to a mercury gauge, G, and air-pump, H. The can was connected to the uninsulated terminal of the electric machine and to the sheaths of the voltmeter and electrometer.

§ 74. The experiment was conducted as follows :—Apertures Nos. 1, 2, 3, 4, and 7 in the can were closed, and the electric machine started, the air in the can being charged by a brush discharge from the needle point. The electrometer terminal joined to the filter was insulated, and the pump worked for some time, fresh air filtered through cotton-wool entering the can by a pipe attached to aperture No. 6. The tests were made at potentials ranging from 2500 up to 12,000 volts, and it was found that 3200 volts negative and 4000 volts positive gave about the best results. The speed of the pump was kept constant, and the cubic contents of the cylinders of the pump and the electric capacity of the insulated filter and quadrants of the electrometer being known, the quantity of electricity in absolute measure taken from each cub. centim. of air by the filter could be determined. In experimenting with carbonic acid gas, the procedure adopted was almost exactly the same as that for air, the only difference being that instead of admitting air by the pipe attached to aperture No. 6 the same pipe was attached to the nozzle of an upright pressure cylinder containing carbonic acid gas. The gas was admitted to the can under very slight pressure. For carbonic acid gas, the charging potentials which gave the best results were found to be about 4000 volts negative and 5000 volts positive.

In order to find out the electric density of the electrified air or carbonic acid gas when left in the can for some time after charging had ceased, the electrification was stopped after the machine and pump had been worked for several minutes. The charging wire was removed from the needle and the apertures in the can blocked. The enclosed electrified air or carbonic acid gas was left to itself for different times in different experiments, generally just  $1\frac{1}{4}$  hours. The gas in the can was then drawn from No. 5 aperture through the insulated filter to the pump (aperture No. 6 being opened to admit fresh air), and the pump was stopped when all signs of electrification ceased.

The following results were obtained :—

## ELECTRIC Densities in C.G.S. Electrostatic Units.

Gas.	Greatest electric density, while working electric machine and pump.	Percentage loss in stated time.
Air	$0.877 \times 10^{-4}$ negative	73 per cent. in 90 minutes
"	$0.370 \times 10^{-4}$ positive	92.7 " " 120 "
Carbonic acid	$1.17 \times 10^{-4}$ "	93.1 " " $1\frac{1}{4}$ hours
"	$0.833 \times 10^{-4}$ "	96.1 " " $1\frac{1}{4}$ "
"	$0.63 \times 10^{-4}$ negative	98.8 " " $1\frac{1}{4}$ "

*Diffusion of Electricity from Carbonic Acid Gas into Air.*

§ 75. We next tried a series of experiments to find if an electric charge given to carbonic acid gas diffused from the gas into air. The method of experimenting finally adopted was as follows :—

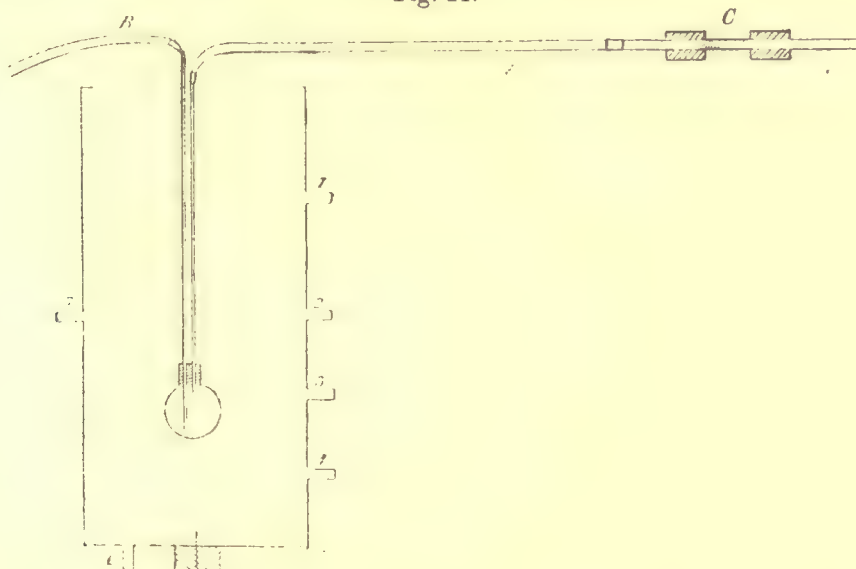
Carbonic acid gas was slowly passed from an upright pressure cylinder into the metallic can, A, by aperture No. 6; atmospheric air was freely admitted through aperture No. 7; while 12,000 cub. centims. of mixed carbonic acid and air were drawn out per minute from aperture No. 2, and 6100 cub. centims. of air from the top aperture (No. 5); the other openings in the can being kept closed.

In these conditions, it was found that air entered abundantly through No. 7, showing that the volume drawn off from (2) and (5) was much greater than that of the carbonic acid gas entering by (6). Hence there must have been nearly pure carbonic acid gas below the level of aperture No. 7, separated by a very thin transitional stratum from nearly pure air, above the level of No. 7. Nos. 7 and 2 were on the same level. The air drawn off from No. 5 passed through the insulated standard filter, E (already described), which was connected to the insulated terminal of the quadrant electrometer, F. The Voss electric machine, C, was worked to give a brush discharge from the needle point, B, the charging potential being indicated by the vertical electrostatic voltmeter, D. Within 15 seconds after starting the machine, a decided electrical effect was observed, the reading of the quadrant electrometer almost immediately rising to a maximum rate of deflection of 55 divisions per minute when the needle was charged positively, and 50 divisions per minute when it was charged negatively. The electrification observed was not sensibly affected by stopping the supply of carbonic acid gas. But when the working of the electric machine was stopped and the charging wire to the needle removed, and whether the supply of carbonic acid gas was continued or stopped, the electrical effects noticed on the electrometer rapidly fell, and 3 minutes after the electric machine was stopped, no further electrification was detected. The sensibility of the electrometer was 117 divisions per volt.

In order to test if the carbonic acid gas in the lower half of the can still retained any electrical charge, the connection to the filter and pump was removed from aperture No. 5, at the top of the can, to No. 6 at the bottom, and the gas drawn through the filter, but no electrification could be detected. We were surprised with the results, and we do not see how to explain it: we expected that the stagnant carbonic acid gas in the bottom of the vessel would have retained electricity as in experiments of § 52 and § 74.

§ 76. Further experiments on diffusion of electricity were tried with a porous ball (fig. 11). The mouth of the ball was tightly closed, and through the cork passed two glass tubes, one (B) projecting nearly to the bottom, the other being just through the

Fig. 11.



cork. The ball was suspended in the metallic can, which was filled with carbonic acid gas or air. The gas in the can was electrified from the insulated needle point at the bottom. Meantime, a strong blast of non-electrified air from a large bellows passed into the ball by the tube, B, and out again by the other tube to the insulated standard filter of block-tin tubing, 5 centims. long and 0.66 centim. bore, filled with fine brass filings, the filter being connected to the insulated terminal of the quadrant electrometer.

There was thus an air pressure from the inside of the ball towards its outside surface, and under these conditions there was no evidence on the electrometer that any part of the electric charge in the carbonic acid gas or air surrounding the ball had made its way against the outward pressure of air, from outside the ball into the interior, and thence to the filter. This experiment was varied by removing the bellows, blocking the tube, B, partially or completely, and attaching an air-pump to the insulated filter at C. On working the air-pump, some of the electrified carbonic acid gas or air surrounding the ball must have been drawn inside, and thence to the



insulated standard filter connected with the electrometer; but, again, no evidence of electrification on the filter could be detected on the electrometer. It thus appears as if the porous ball itself had withdrawn the electric charge from the gas which passed through the ball.

#### COMMUNICATION OF ELECTRICITY FROM ELECTRIFIED STEAM TO AIR (§§ 77-81).

§ 77. Steam was generated in a kettle, A, and electrified by brush discharge from a needle point, B, attached to the lower end of a long copper rod, CB. The rod was kept central and insulated in the brass tube, D, by two rubber corks. The upper end of the rod was connected to the insulated terminal of a voltmeter, E, and of a Voss electric machine, F (fig. 12).

§ 78. To preserve the insulating properties of the corks during the experiments it was found necessary to keep a current of air passing in the tube between the corks, and to surround the lower part of the tube with a jacket of oil kept at a temperature of 235° F.

§ 79. The electrified steam from the kettle passed up into a vertical tube, G, where it mixed with air drawn, by an air-pump shown on the right-hand side of the drawing, from a bottle into which the lower end of the tube was fitted. Air to take the place of that drawn from the bottle entered by a long pipe from outside a window of the laboratory. The mixed steam and air passed from G into a LIEBIG'S condenser, H, where the steam was condensed. The water thus formed dropped into a WOLFF'S bottle, and the air was drawn from another neck of the bottle through a drying tube, K, containing sulphuric pumice. From this it passed direct through an electric filter, L, insulated by two paraffin tunnels, and thence to the air-pump. The filter was connected to the insulated terminal of a quadrant electrometer, whose constant was 117 divisions per volt.

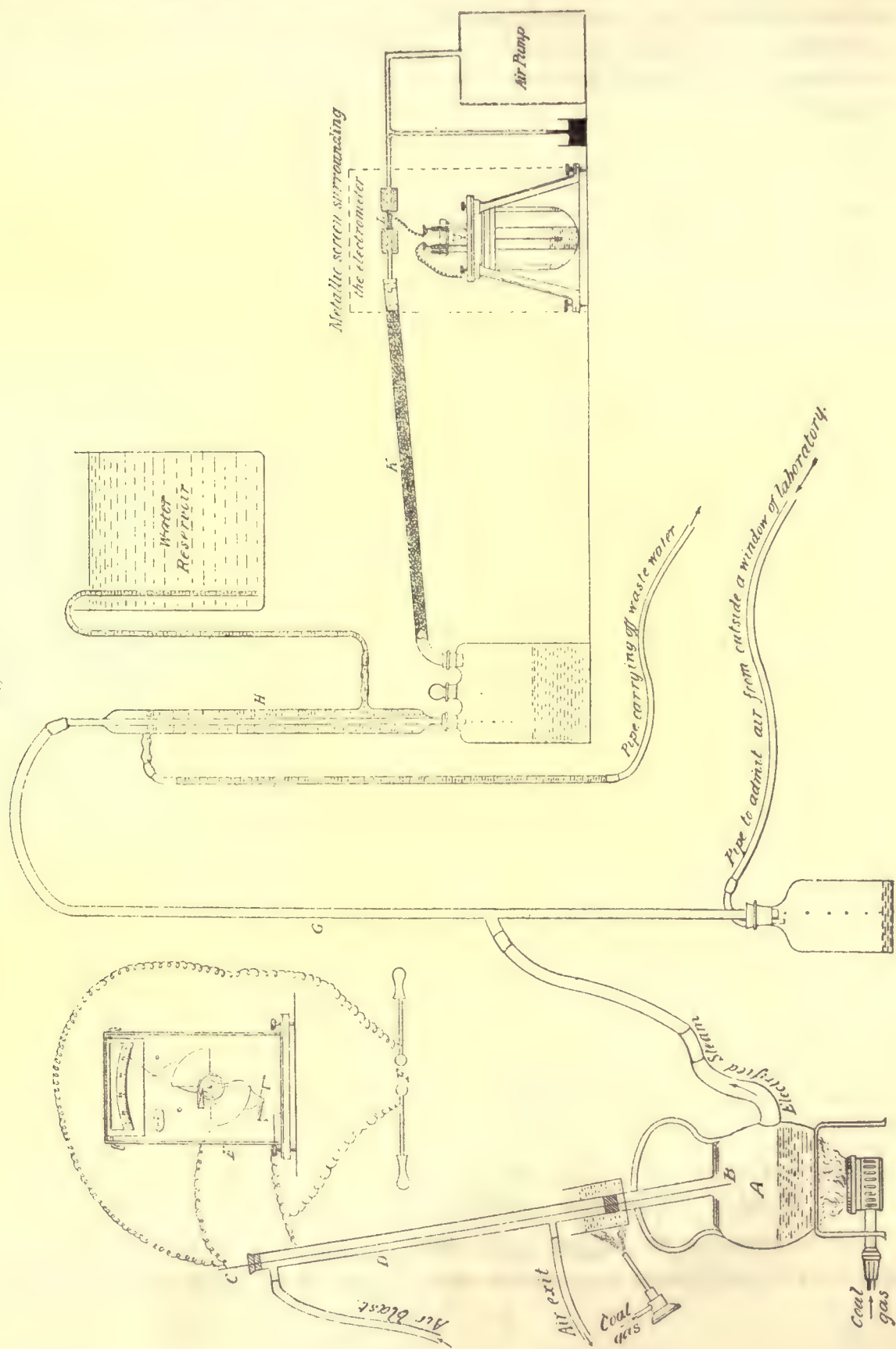
§ 80. When the air-pump and the electric machine were worked, with the kettle cold, the electrometer showed no electrification. It also showed no electrification when the kettle was boiling and the air-pump worked, but the electric machine stopped.

§ 81. When the kettle was kept boiling, and the electric machine and air-pump both worked, strong electrification, positive or negative, according as the machine was positive or negative, was observed, 52 divisions per minute being our largest value. This, with the known capacity of the electrometer, corresponds to 0.11 C.G.S. electrostatic unit taken per minute from the air by the filter.

#### ELECTRIFICATION OF AIR AT DIFFERENT AIR-PRESSURES AND AT DIFFERENT ELECTRIC POTENTIALS. MEASUREMENT OF CURRENT (§§ 82-88).

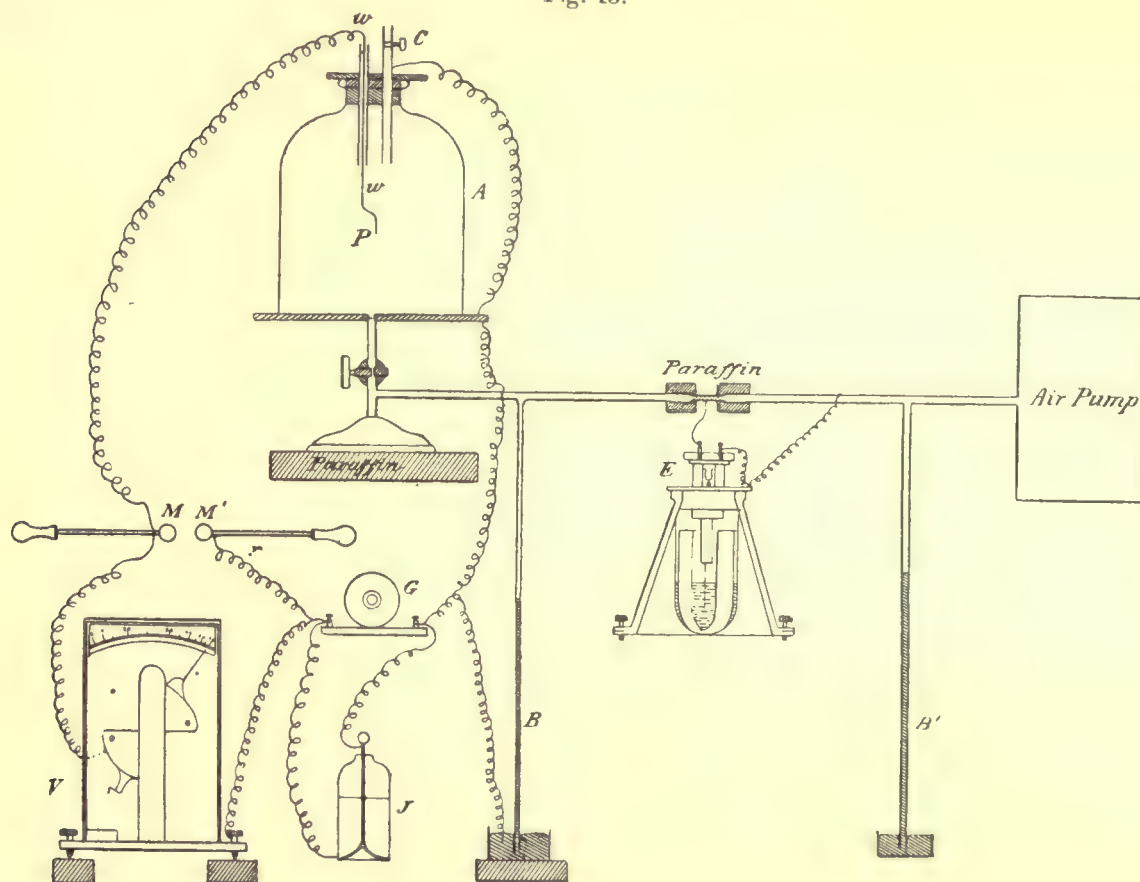
§ 82. In February, March, and April, 1896, we made experiments on the electrification of air at different air-pressures, using for this purpose the apparatus repre-

Fig. 12



sented in fig. 13, and electrifying by one needle point joined to the insulated terminal of the Voss electric machine. The air was contained in a glass bell-jar, A, which was coated inside with strips of tin-foil kept in metallic connection with one terminal, G, of a high-resistance mirror galvanometer, the other terminal of the galvanometer being joined to the sheath of the voltmeter, V, and to the disinsulated terminal, M', of the electric machine. The stand of the bell-jar rested on a piece of paraffin. The pressure of the air in the bell-jar was measured by noting the height to which mercury rose in the tube, B. The vessel containing the mercury was insulated by a paraffin block.

Fig. 13.



The pressure of the air after it had passed the electric filter (block-tin pipe with fine brass filings) was also measured by means of the rise of mercury in tube, B'. A barometer tube not shown in the diagram gave us the atmospheric pressure. The differences of the heights of the mercury in tubes B, B', and the height of the mercury in the barometer, gave the pressures of the air in the bell-jar and on the exit side of the filter respectively.

All the tubing through which the air passed was block tin.

Throughout each experiment the pressure of the air in the bell-jar was kept



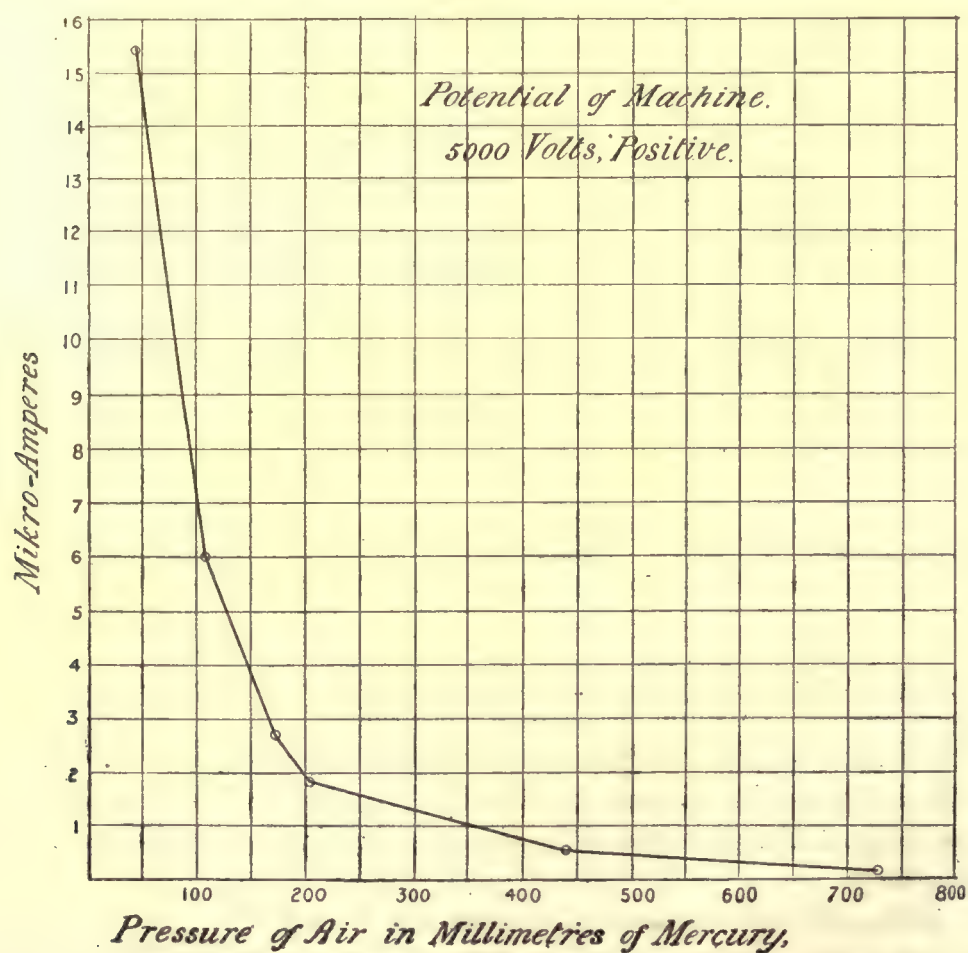
constant by regulating the stop-cock, C, so that the abstraction of air by the pump was exactly compensated by the gain through C.

The galvanometer measures only that part of the electricity entering the bell-jar by the wire, *ww*, which leaves it by its metal base. This part is *very nearly* the whole. The observed results show that it is enormously great in comparison with the very small part which is carried away in the current of air to the electric filter.

The galvanometer was shunted by a battery of LEYDEN jars, J, to give steady deflections. Its sensitiveness was  $1/22.1$  of a mikro-ampere per scale division.

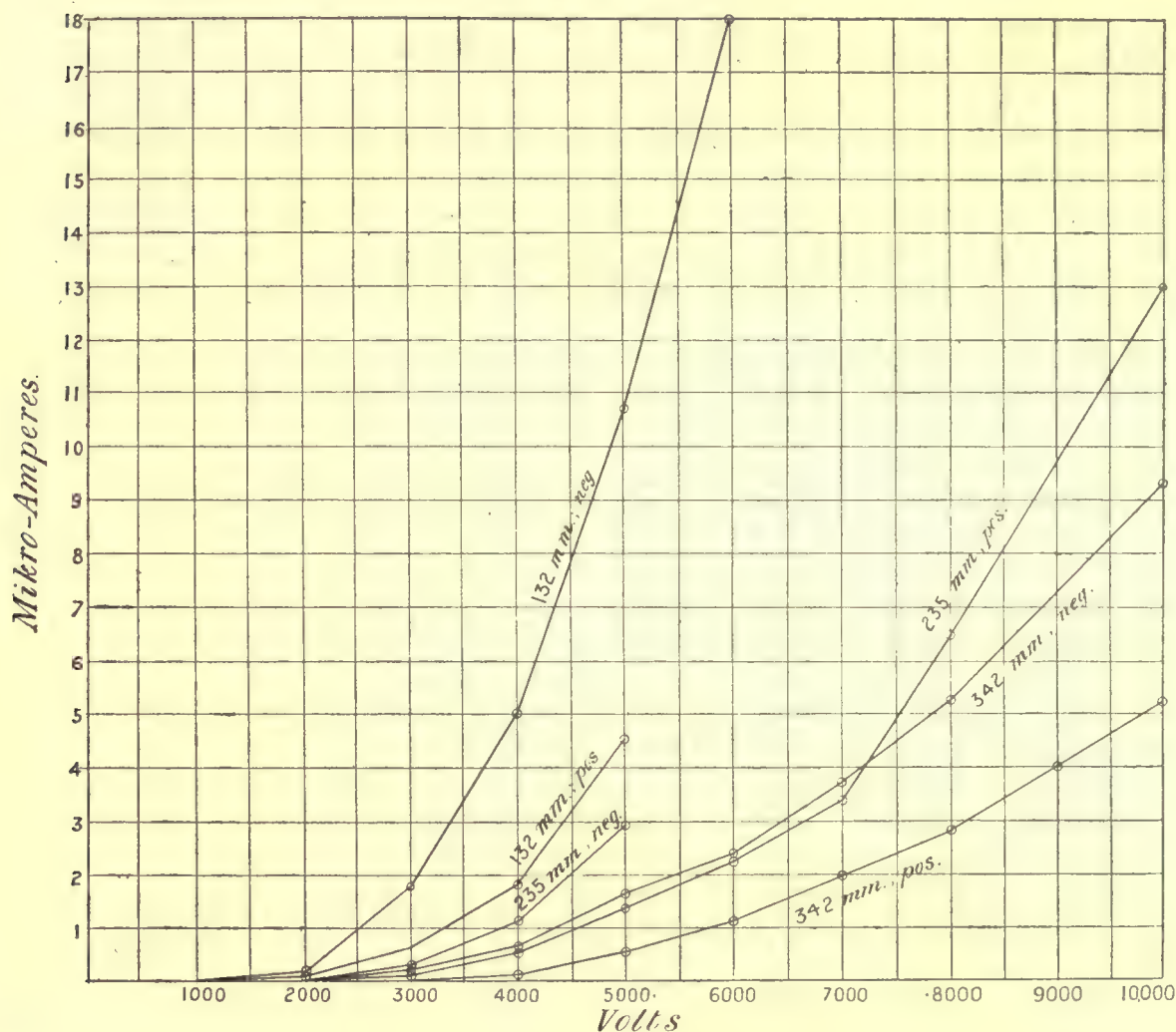
§ 83. First, we kept the potential of the needle constant throughout a set of experiments made at different air-pressures, and in this way we found that the current through the air to the metal of the jar became greater as the pressure of the air in the bell-jar became less, down to the lowest pressure to which we went, which was 40 millims. Curve 10 shows the relation between the current and the air-pressure at a potential of 5000 volts. Similar curves were got for other electric potentials.

Curve 10.



§ 84. We found also that as the air became rarer it was not so much electrified. This was shown by the electric filter and electrometer. Thus the electrometer deflection for a pressure of 360 millims. was only about one-sixth of that for 760 millims. with the same number of strokes of the pump, and the same potential of the electric machine.

Curves 11 to 16.



§ 85. We next kept the pressure of the air in the bell-jar constant and varied the electric potential of the electrifying needle. In this manner we found how much the current through the air in the jar was increased by increasing the potential. The curves 11 to 16 represent the relation between the potential of the needle and the current from the needle-point through the air to the metal of the bell-jar, for certain definite air-pressures, and for positive and negative electrification. It will be noticed that for the same air-pressure the current is greater for negative than for positive electrification.

§ 86. At each air-pressure the electrifying needle must be raised to a certain potential before the galvanometer shows any current. Thus for a pressure of 342 millims. no current was shown by the galvanometer at a potential of 2000 volts negative ; and for a pressure of 235 millims. no current was shown at a potential of 1000 volts negative or 2000 volts positive.

§ 87. We found also that air at a given pressure was electrified to a greater extent by a certain potential of the needle than by other potentials higher or lower. Thus air at a pressure of 470 millims. was electrified by the needle at 10,000 volts 1·6 times more than at any other potential we tried ; while air at a pressure of 340 millims. received maximum electrification, whether positive or negative, when the potential of the needle was 3000 volts.

§ 88. We are indebted for valuable help and co-operation in the carrying out of these experiments to WALTER STEWART, M.A., B.Sc., to VINCENT J. BLYTH, to JOHN F. HENDERSON, to WM. CRAIG HENDERSON, M.A., B.Sc., and to W. S. TEMPLETON, M.A., B.Sc.



VII. *Mathematical Contributions to the Theory of Evolution.*—IV. *On the Probable Errors of Frequency Constants and on the Influence of Random Selection on Variation and Correlation.*

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I. INTRODUCTORY.—GENERAL THEOREM.

(1) In earlier memoirs by one of the present authors, methods have been discussed for the calculation of the constants ( $a$ ) of variation, normal or skew,\* ( $b$ ) of correla-

\* “Mathematical Contributions to the Theory of Evolution.—II. On Skew Variation,” ‘Phil. Trans.’ A, vol. 186, pp. 343–414.

tion, when normal.\* The subject of skew correlation would now naturally present itself, but although several important conclusions with regard to skew correlation have been worked out, there are still difficulties which impede the completion of the memoir on that topic. Meanwhile Mr. G. U. YULE has shown that the constants of normal correlation are significant, if not completely descriptive, even in the case of skew correlation.† It seems desirable to take, somewhat out of its natural order, the subject of the present memoir, partly because the formulæ involved have been once or twice cited and several times used in memoirs by one of the present writers, and partly because the need of such formulæ seems to have been disregarded by various authors in somewhat too readily drawing conclusions from statistical data. Differences in the constants of variation or of correlation have been not infrequently asserted to be significant or non-significant of class or of type, or of race differences, without a due investigation of whether those differences are, from the standpoint of mathematical statistics, greater or less than the probable errors of the differences. Notwithstanding that every artificial or even random selection of a group out of a community changes not only the amount of variation, but the amount of correlation of the organs of its members as compared with those of the primitive group,‡ it has been supposed that correlation might be a racial constant, and the approximate constancy of coefficients of correlation of the same organs in allied species has been used as a valid argument. In the like manner differences in variation have been used as an argument for the activity of natural selection without a discussion of the probable errors of those differences.

In dealing with variation and correlation we find the distribution described by certain curves or surfaces fully determined when certain constants are known. These are the so-called constants of variation and correlation, the number of which may run up from two to a very considerable figure in the case of a complex of organs. If we deal with a complex of organs in two groups containing, say,  $n$  and  $n'$  individuals, we can only ascertain whether there is a significant or insignificant difference between those groups by measuring the extent to which the differences of corresponding constants exceed the probable errors of those differences. The probable error of a difference can at once be found by taking the square root of the sum of the squares of the probable errors of the quantities forming the difference. Hence the first step towards determining the significance of a group difference—*i.e.*, towards ascertaining whether it is really a class, race, or type difference—is to calculate the probable errors of the constants of variation and correlation of the individual groups. This will be the object of our first general theorem.

\* "Mathematical Contributions to the Theory of Evolution.—III. Regression, Heredity, and Panmixia," 'Phil. Trans.,' A, vol. 187, pp. 253–318.

† "On the Significance of BRAVAIS' Formulæ for Skew Correlation," 'Roy. Soc. Proc.,' vol. 60, pp. 477–489.

‡ This will be sufficiently indicated in the latter part of the present memoir, but has been more fully dealt with in a paper on the "Influence of Selection on Correlation," written, but not yet published.

## II. ON THE PROBABLE ERRORS OF A SYSTEM OF FREQUENCY CONSTANTS.— GENERAL THEOREM.

(2) Let there be a group of  $n$  individuals, for each of whom a complex of  $m$  organs is measured, and let  $z \delta x_1 \delta x_2 \dots \delta x_m$  be the frequency with which individuals having a complex of organs lying between  $x_1, x_1 + \delta x_1; x_2, x_2 + \delta x_2; \dots x_m, x_m + \delta x_m$ , occur in the total group of  $n$ . Here  $x$  shall measure the deviation of any organ from the mean of all like organs in the group. Let  $h_1, h_2, h_3, \dots h_m$ , be the mean measurements on the organs, so that  $h_1 + x_1, h_2 + x_2, \dots h_m + x_m$ , is the system giving the actual measurements on any individual. Then  $h_1, h_2, h_3, \dots h_m$ , are the first set of constants of the frequency; they determine the "origin" of the frequency surface.

Let this frequency surface be given by

$$z = f(x_1, x_2, x_3, \dots x_m; c_1, c_2, c_3, \dots c_p),$$

where  $c_1, c_2, c_3, \dots c_p$ , are  $p$  frequency constants, which define the form as distinguished from the position of the frequency surface, and which will be functions of standard deviations, moments, skewnesses, coefficients of correlation, &c., &c., of individual organs, and of pairs of organs in the complex.

The problem before us is to find the probable errors of the  $h$ 's and the  $c$ 's, which constants fully determine the position and shape of the frequency surface. Let  $\sigma_{h_1}, \sigma_{h_2}, \sigma_{h_3}, \dots \sigma_{h_m}, \sigma_{c_1}, \sigma_{c_2}, \dots \sigma_{c_p}$ , be the standard deviations of the quantities  $h_1, h_2, \dots h_m$ , and  $c_1, c_2, \dots c_p$ . Then a knowledge of these standard deviations will give us at once the probable errors of the frequency constants, for we have only to multiply the former by the numerical factor .6745 to obtain the latter.

Let us now suppose that the value of the frequency constants had been  $h_1 + \Delta h_1, h_2 + \Delta h_2, \dots h_m + \Delta h_m, c_1 + \Delta c_1, c_2 + \Delta c_2, \dots c_p + \Delta c_p$ , instead of the observed values.

Then the frequency of any observed individual would have varied as

$$f(x_1 + \Delta h_1, x_2 + \Delta h_2, \dots x_m + \Delta h_m; c_1 + \Delta c_1, c_2 + \Delta c_2, \dots c_p + \Delta c_p)$$

instead of as

$$f(x_1, x_2, x_3, \dots x_m; c_1, c_2, c_3, \dots c_p).$$

Hence on this hypothesis the probability,  $P_\Delta$ , of the set of individuals observed in the group actually occurring is to the probability,  $P_0$ , of the set occurring when the constants are  $h_1, h_2, \dots h_m; c_1, c_2, \dots c_p$ , in the ratio of the product of all quantities like  $f(x_1 + \Delta h_1, x_2 + \Delta h_2, \dots x_m + \Delta h_m; c_1 + \Delta c_1, c_2 + \Delta c_2, \dots c_p + \Delta c_p)$  for all values of  $x_1, x_2, \dots x_m$ , to the like product of all quantities like  $f(x_1, x_2, \dots x_m; c_1, c_2, \dots c_p)$ , or

$$\frac{P_\Delta}{P_0} = \frac{\prod f(x_1 + \Delta h_1, x_2 + \Delta h_2, \dots x_m + \Delta h_m; c_1 + \Delta c_1, c_2 + \Delta c_2, \dots c_p + \Delta c_p)}{\prod f(x_1, x_2, \dots x_m; c_1, c_2, \dots c_p)}.$$



Taking logarithms, the products  $\Pi$  become sums  $S$ , or

$$\log (P_{\Delta}/P_0) = S \log f(x_1 + \Delta h_1, x_2 + \Delta h_2, \dots x_m + \Delta h_m; c_1 + \Delta c_1, c_2 + \Delta c_2, \dots c_p + \Delta c_p) \\ - S \log f(x_1, x_2, \dots x_m; c_1, c_2, \dots c_p).$$

Let the first summation now be expanded by TAYLOR'S theorem, and typical terms up to the second order be written down. Then we have

$$\log (P_{\Delta}/P_0) = \Delta h_r S \frac{d}{dx_r} (\log f) + \frac{1}{2} (\Delta h_r)^2 S \frac{d^2}{dx_r^2} (\log f) + \Delta h_r \Delta h_{r'} S \frac{d^2}{dx_r dx_{r'}} (\log f) \\ + \Delta c_s S \frac{d}{dc_s} (\log f) + \frac{1}{2} (\Delta c_s)^2 S \frac{d^2}{dc_s^2} (\log f) + \Delta c_s \Delta c_{s'} S \frac{d^2}{dc_s dc_{s'}} (\log f) \\ + \Delta h_r \Delta c_s S \frac{d^2}{dh_r dc_s} (\log f) + \dots + \text{cubic terms in } \Delta h \text{ and } \Delta c + \&c.,$$

where  $f$  stands for  $f(x_1, x_2, x_3, \dots x_m, c_1, c_2, \dots c_p)$ .

Here  $r$  is to be given every value from 1 to  $m$ , and  $s$  to be given every value from 1 to  $p$ , but  $r'$  and  $s'$  in the third and sixth sums are only to be given values from 1 to  $m$  and 1 to  $p$  other than  $r$  and  $s$  respectively. In the above formula we may replace the sums by integrals, if we remember that the frequency of the system  $x_1, x_2, \dots x_m$ , is simply  $f \delta x_1 \delta x_2 \dots \delta x_m$ .

Writing

$$\log (P_{\Delta}/P_0) = A_r \Delta h_r - \frac{1}{2} B_r (\Delta h_r)^2 + C_{r,r'} \Delta h_r \Delta h_{r'} + D_s \Delta c_s - \frac{1}{2} E_s (\Delta c_s)^2 + F_{ss'} \Delta c_s \Delta c_{s'} \\ + G_{rs} \Delta h_r \Delta c_s + \&c. \dots,$$

we will investigate the values of the constants separately.

First,

$$A_r = \iiint \dots f \frac{d \log f}{dx_r} dx_1 dx_2 \dots dx_m, \\ = \iiint \dots \frac{df}{dx_r} dx_1 dx_2 \dots dx_m, \\ = \iiint \dots [f] dx_1 dx_2 \dots dx_{r-1} dx_{r+1} \dots dx_m,$$

the integrals now not including one with regard to  $dx_r$  and  $[f]$  denoting that  $f$  is to be taken between the extreme limits of  $f$  for  $x_r$ . Now in most cases of frequency the frequencies for extreme values of any organ are zero.\* Hence  $[f]$  equals nothing. Thus we have  $A_r = 0$ .

\* In most cases, but not *invariably*, as, for example, in the case of some florets and petals. In the cases, however, in which  $A_r$  does not vanish, the conclusions finally reached will be the same, as  $A_r$  only marks a change of origin for the constant frequency distribution.

Secondly,

$$\begin{aligned} D_s &= \iiint \dots f \frac{d \log f}{dc_s} dx_1 dx_2 \dots dx_m, \\ &= \iiint \dots \frac{df}{dc_s} dx_1 dx_2 \dots dx_m, \\ &= \frac{d}{dc_s} \iiint \dots f dx_1 dx_2 dx_m \\ &= dn/dc_s, \end{aligned}$$

where  $n$  is the total number of individuals measured, which is independent of  $c_s$ . Therefore  $D_s = 0$ .

Thirdly,

$$C_{rs} = \iiint \dots f \frac{d^2 \log f}{dx_r dx_s} dx_1 dx_2 \dots dx_m \dots \dots \dots (i.).$$

This will not as a rule vanish. If the frequency be normal, it will still not as a rule vanish. It will vanish either if the frequency be symmetrical about  $x_r = 0$ , and  $x_s = 0$ , or if there be no correlation between the  $x_r$  and  $x_s$  organs, *i.e.*, if  $f$  be of the form  $f_1(x_r^2) \times f_2(x_s^2)$ .

Fourthly,

$$G_{rs} = \iiint \dots f \frac{d^2 \log f}{dx_r^2 dx_s} dx_1 dx_2 \dots dx_m \dots \dots \dots (ii.).$$

This will not as a rule vanish. It will vanish, however, if the frequency of  $x_r$  be symmetrical about its mean.

Fifthly,

$$B_r = - \iiint \dots f \frac{d^2 \log f}{dx_r^2} dx_1 dx_2 \dots dx_m \dots \dots \dots (iii.),$$

$$E_s = - \iiint \dots f \frac{d^2 \log f}{dx_s^2} dx_1 dx_2 \dots dx_m \dots \dots \dots (iv.),$$

$$F_{rs} = \iiint \dots f \frac{d^2 \log f}{dx_r dx_s} dx_1 dx_2 \dots dx_m \dots \dots \dots (v.),$$

all of which will generally be finite, but admit, like  $C_{rs}$  and  $G_{rs}$ , of calculation when the form of the frequency  $f$  is given. Hence

$$\begin{aligned} P_\Delta &= P_0 \text{ expt.} - \frac{1}{2} \{ B_r (\Delta h_r)^2 - 2C_{rs} \Delta h_r \Delta h_s - 2G_{rs} \Delta h_r \Delta c_s \\ &\quad + E_s (\Delta c_s)^2 - 2F_{rs} \Delta c_s \Delta c_s + \&c. \dots \}, \\ &\quad \times \text{ expt. (terms in cubic and higher orders of the } \Delta \text{'s)} \dots (vi.). \end{aligned}$$

This represents the probability of the observed unit, *i.e.*, the individuals ( $x_1, x_2, \dots x_m$ , for all sets), occurring, on the assumption that errors  $\Delta h_1, \Delta h_2, \dots$

$\Delta h_m, \Delta c_1, \Delta c_2, \dots \Delta c_p$  have been made in the determination of the frequency constants. In other words, we have here the frequency distribution for errors in the values of the frequency constants.

(3) *Conclusions to be drawn from the form of (vi.).*

( $\alpha$ ) The distribution of the errors of frequency constants, if treated exactly, will generally be skew, for the cubic and higher terms in the  $\Delta$ 's do not vanish. If, however, the cubic terms are small as compared with the square terms, the frequency distribution of errors will approximate closely to a normal correlation surface.

It would be impossible to evaluate the remainder after the second power terms in TAYLOR'S series for any general expression  $f(x_1, x_2, \dots x_m; c_1, c_2, \dots c_p)$  for frequency. In special cases we have found that terms of the third order amount in the most unfavourable circumstances to 4 per cent. of the terms of the second order, generally to a good deal less.\* Probably the series in most cases converges with considerable rapidity. The fact, however, that we are dealing with the first terms of a series should be borne in mind. It does not seem to have been sufficiently emphasised when the probable error of the standard deviation is taken to be  $67.449/\sqrt{2n}$  per cent. of the standard deviation. The usual proof of this result, however, involves the same assumption as to the smallness of the cubic terms.

( $\beta$ ) Supposing the errors so small that we may neglect the cubic terms, we conclude that the errors made in calculating the constants of any frequency distribution are—

- (i.) Themselves distributed according to the normal law of errors,
- (ii.) Correlated among themselves.

Both these conclusions are of the utmost importance. The first enables us to obtain the probable errors of the frequency constants; the second depends upon the fact that  $C_{rs}, G_{rs}$ , and  $F_{ss}$  are in general not zero. The standard deviations of, and the correlations between, the frequency constant errors can now be calculated by the ordinary theory of normal correlation.

Before, however, proceeding to these calculations, we may draw one or two other conclusions of considerable generality and wide significance.

( $\gamma$ ) Consider a race fully defined by the variations  $\sigma_1, \sigma_2, \sigma_3, \dots$ , &c. of the organs of its members and their correlations  $r_{12}, r_{23}, r_{13}, \dots$ . Now let a *random* small selection be made of this race, defined by

$$\sigma_1 + \delta\sigma_1, \sigma_2 + \delta\sigma_2, \sigma_3 + \delta\sigma_3, \dots r_{12} + \delta r_{12}, r_{23} + \delta r_{23}, r_{13} + \delta r_{13}, \dots,$$

where the magnitudes of  $\delta\sigma_1, \delta\sigma_2, \delta\sigma_3, \dots \delta r_{12}, \delta r_{23}, \delta r_{13}, \dots$ , are quantities depending

\* See, for the relative order of two terms of the second and third orders, "Regression, Heredity, and Panmixia," 'Phil. Trans.' A, vol. 187, p. 266.



on the magnitude of the probable errors of the variation constants, and therefore on the size of the selection. Then the system  $\delta\sigma_1, \delta\sigma_2, \delta\sigma_3, \dots, \delta r_{12}, \delta r_{23}, \delta r_{13}, \dots$ , is not a system of independent variations, but the changes in variation and correlation are correlated together, since the terms  $F_{ss'}$  do not generally vanish. We therefore conclude that if a random selection be made out of a population with regard to one organ  $\sigma_1$ , there will be tendencies for the variation of all other organs and the correlation between all organs to change also in certain directions, which can be definitely indicated so soon as the general population has been measured, and the effect of the random selection on one organ has been ascertained. What is proved here of random selection will be shown to be true still more intensively for artificial selection,\* *i.e.*, every selection of one organ modifies in a correlated manner the variation and correlation of all other organs. It is impossible to alter one organ without altering all other organs and their relation to each other.

( $\delta$ ) The remarks in ( $\gamma$ ) are not only true for a random selection of variation, but equally well apply to selection of size. This follows because the terms  $C_{rs'}$  and  $G_{rs'}$  do not as a rule vanish. If a random selection of 100 or 1,000 individuals be made out of a general population, then the mean size of any organ in this sub-group will probably differ from that of the general population. The result is that the size of all other organs, their amounts of variation and correlation, will probably have values differing in definite directions from those of the general population.

( $\epsilon$ ) Take two random selections out of a general population; the probability is that they will have different means for any one organ, and a result will be that they will have correlated systems of changes in the sizes, variations, and correlations of all other organs. In other words, random selection produces a differentiation of all characters, which differentiation will be the more marked the smaller the random selection.†

( $\zeta$ ) This principle—that a random selection gives a system of correlated changes in the deviations of all the characters of a species—seems, to some extent, capable of explaining the small but systematic differences to be found occasionally between closely allied species. It is not necessary to suppose them due to a long process of natural selection acting on a variety of organs; a small random selection, or possibly a natural selection of one organ, might suffice to produce the systematic differences of character in all organs.

How far a succession of random selections would give an evolution biassed by the first random selection requires further consideration; but it seems impossible for the characters of a race to remain fixed under the influence of a *heavy* but non-selective death-rate. They will vary from year to year, although this systematic change of

\* "Memoir on the Influence of Selection on Correlation." Selection not only modifies correlation, but the selection of one organ can create correlation between organs previously uncorrelated.

† Continuous artificial selection of an organ produces a still more marked differentiation of all other characters, but this is treated of in another memoir.

characters be not always in the same direction. Systematic change of characters produced by random selection may be spoken of as *random evolution*. Random evolution is theoretically a possible cause of systematic change; experiment only can determine how great is its effectiveness in differentiating local races.

( $\eta$ ) In the case of a normal distribution of variation defined by the mean  $h$  and the standard deviation  $\sigma$ , it has been usual to suppose that the error made in the mean is independent of the error made in the variation. In other words, it has been assumed that  $G_{rs}$  vanishes, although no proof has been given, or possibly it has not been realised that a proof was necessary. In this case  $f$  is of the form

$\frac{1}{\sqrt{(2\pi)}\sigma} \text{expt.} - \{x^2/2\sigma^2\}$  and  $\frac{d^2(\log f)}{d\sigma d\sigma} = \frac{2x}{\sigma^3}$ , whence

$$G = 2 \int_{-\infty}^{+\infty} \frac{x}{\sigma^3} \text{expt.} - \{x^2/2\sigma^2\} \frac{n}{\sqrt{(2\pi)}\sigma} dx = 0.$$

Thus there is no correlation between error in the mean and error in the standard deviation. This assumes that we stop at the square terms in (vi.). If, however, we include the cubic terms, &c., product terms in  $\Delta h$  and  $\Delta\sigma$  do arise, and we cannot state straight off that no correlation exists, although it may be very small. In the case of all skew variation, such as is so frequent among plants and animals, a correlation will always be found between deviation or error in the mean and the like in the standard deviation. In other words, to alter the mean by selection (artificial or random) is to alter the variation of an organ.

With the exception of the statements in this paragraph ( $\eta$ ), the whole of our general conclusions in this section are independent of any particular law of frequency.

(4) *On the Determination of the Probable Errors and the Error Correlations of the Frequency Constants.*

Let  $\eta_1, \eta_2, \eta_3, \dots$ , be the frequency constants, whether they be the means, standard deviations, or correlations of a complex of organs. Then if we neglect cubic and higher terms in the deviations  $\Delta\eta_1, \Delta\eta_2, \Delta\eta_3, \dots$ , the frequency surface giving the distribution of the variations in the deviations is

$$P_{\Delta} = P_0 \text{ expt.} - \frac{1}{2} [S \{a_{rr} (\Delta\eta_r)^2\} - 2S \{a_{rs} \Delta\eta_r \Delta\eta_s\}],$$

where

$$\begin{aligned} a_r &= - \iiint \dots f \frac{d^2(\log f)}{d\eta_r^2} dx_1 dx_2 \dots dx_m, \\ a_{rs} &= \iiint \dots f \frac{d^2(\log f)}{d\eta_r d\eta_s} dx_1 dx_2 \dots dx_m \dots \dots \dots \text{(vii.).} \end{aligned}$$

It is required to find  $\Sigma_r$ , the standard deviation of  $\Delta\eta_r$ , and  $R_{rs}$ , the coefficient of correlation between  $\Delta\eta_r$  and  $\Delta\eta_s$ .

Let  $\Delta$  be the discriminant :

$$\begin{vmatrix} a_{11} & -a_{12} & -a_{13} & \dots \\ -a_{21} & a_{22} & -a_{23} & \dots \\ -a_{31} & -a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

where  $a_{rs} = a_{sr}$ . Further, let  $A_{rs}$  be the minor of the  $r^{\text{th}}$  row and  $s^{\text{th}}$  column ; then

$$\Sigma^2 = A_{rr}/\Delta \quad \text{and} \quad R_{rs} = A_{rs}/(\Delta \Sigma_r \Sigma_s) \dots \dots \dots \text{(viii.)}$$

give the required values of  $\Sigma_r$  and  $R_{rs}$ .\*

Further, the standard deviation of  $\Delta\eta_r$  for *selected* values of all the other  $\Delta\eta$ 's is  $1/\sqrt{a_{rr}}$ .

This value can often be of service. Thus suppose a considerable number of skeletons found, but that only in a comparatively few cases is it possible to pair together the femur and tibia of the same individuals. Then the variations of femur and tibia in the race will be known with great exactness, but the probable error of the correlation between femur and tibia will be given with close approximation by a form like  $\cdot 67449 \frac{1}{\sqrt{a_{rr}}}$ . In fact, whenever we have obtained from a large number of observations the values of the frequency constants of a race with great exactness, then, using these values to obtain an additional variation or correlation constant from a few observations, the probable error will be of the form just indicated, and not of the form  $\cdot 67449 A_{rr}/\Delta$ . It is needless, perhaps, to remark that the former is far easier to calculate than the latter.

### III. ON THE PROBABLE ERRORS AND THE COEFFICIENTS OF CORRELATION BETWEEN ERRORS MADE IN THE DETERMINATION OF THE CONSTANTS OF A NORMAL FREQUENCY DISTRIBUTION.

(5) In order to exhibit more clearly the method of investigation, it is desirable that a simple case be first taken. Accordingly we will start with the following problem :—

*To find the Probable Errors and Error Correlations of the Constants of a Normal Frequency Distribution for Two Organs.*

Let  $h_1, h_2$ , be the means,  $\sigma_1, \sigma_2$ , the standard deviations,  $r_{12}$  the coefficient of correlation,  $n$  the total number of pairs of observations,  $x_1, x_2$ , any pair of corresponding deviations of the organs. Then the frequency  $z dx_1 dx_2$  is given by the surface

$$z = \frac{n}{2\pi\sigma_1\sigma_2\sqrt{1-r_{12}^2}} \text{expt.} - \frac{1}{2} \left\{ \frac{x_1^2}{\sigma_1^2(1-r_{12}^2)} - \frac{2x_1x_2r_{12}}{\sigma_1\sigma_2(1-r_{12}^2)} + \frac{x_2^2}{\sigma_2^2(1-r_{12}^2)} \right\} \text{ (ix.)}$$

\* See "Regression, Panmixia, and Heredity," 'Phil. Trans.,' A, vol. 187, p. 301 (e).



We require to find

$$\begin{aligned} \Sigma_{\sigma_1}, & \text{ the standard deviation of errors in } \sigma_1, \\ \Sigma_{\sigma_2}, & \text{ " " " " } \sigma_2, \\ \Sigma_{r_{12}}, & \text{ " " " " } r_{12}, \\ \Sigma_{\rho_1}, & \text{ " " " " } \rho_1 = r_{12}\sigma_1/\sigma_2, \\ & \text{ which is the coefficient of regression of } x_1,^* \\ R_{\sigma_1\sigma_2}, & \text{ the coefficient of correlation between errors in } \sigma_1 \text{ and } \sigma_2, \\ R_{\sigma_1r_{12}}, & \text{ " " " " } \sigma_1 \text{ " " } r_{12}, \\ R_{\sigma_2r_{12}}, & \text{ " " " " } \sigma_2 \text{ " " } r_{12}, \\ \Sigma_{h_1}, & \text{ the standard deviation of errors in } h_1, \\ \Sigma_{h_2}, & \text{ " " " " } h_2, \\ R_{h_1h_2}, & \text{ the coefficient of correlation between errors in } h_1 \text{ and } h_2. \end{aligned}$$

It follows that  $R_{h_1\sigma_1}$ ,  $R_{h_2\sigma_2}$ ,  $R_{h_1\sigma_2}$ ,  $R_{h_2\sigma_1}$ ,  $R_{h_1r_{12}}$ ,  $R_{h_2r_{12}}$ , are all zero, since, by (ii.) of p. 233,  $G_{rs}$  will vanish when the distributions of  $x_1$  and  $x_2$  are symmetrical. These correlations would not, however, vanish for the skew frequency distributions, which are of most frequent occurrence in problems of heredity and fertility in man, &c.

The first stage in the investigation is to write down the second differentials of the logarithm of  $z$  for all quantities occurring in it.

We find

$$\begin{aligned} \frac{d^2(\log z)}{d\sigma_1^2} &= \frac{1}{\sigma_1^2} \left\{ 1 - \frac{3x_1^2}{\sigma_1^2(1-r_{12}^2)} + \frac{2x_1x_2r_{12}}{\sigma_1\sigma_2(1-r_{12}^2)} \right\}, \\ \frac{d^2(\log z)}{d\sigma_2^2} &= \frac{1}{\sigma_2^2} \left\{ 1 - \frac{3x_2^2}{\sigma_2^2(1-r_{12}^2)} + \frac{2x_1x_2r_{12}}{\sigma_1\sigma_2(1-r_{12}^2)} \right\}, \\ \frac{d^2(\log z)}{d\sigma_1 d\sigma_2} &= \frac{r_{12}x_1x_2}{\sigma_1^2\sigma_2^2(1-r_{12}^2)}, \\ \frac{d^2(\log z)}{d\sigma_1 dr_{12}} &= \frac{1}{\sigma_1} \left\{ \frac{2r_{12}x_1^2}{\sigma_1^2(1-r_{12}^2)^2} - \frac{x_1x_2(1+r_{12}^2)}{\sigma_1\sigma_2(1-r_{12}^2)^2} \right\}, \\ \frac{d^2(\log z)}{d\sigma_2 dr_{12}} &= \frac{1}{\sigma_2} \left\{ \frac{2r_{12}x_2^2}{\sigma_2^2(1-r_{12}^2)^2} - \frac{x_1x_2(1+r_{12}^2)}{\sigma_1\sigma_2(1-r_{12}^2)^2} \right\}, \\ \frac{d^2(\log z)}{dr_{12}^2} &= \frac{1+r_{12}^2}{(1-r_{12}^2)^2} - \frac{1+3r_{12}^2}{(1-r_{12}^2)^3} \left( \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} \right) + \frac{6r_{12}+2r_{12}^3}{(1-r_{12}^2)^3} \frac{x_1x_2}{\sigma_1\sigma_2}, \\ \frac{d^2(\log z)}{dx_1^2} &= -\frac{1}{\sigma_1^2(1-r_{12}^2)}, \\ \frac{d^2(\log z)}{dx_2^2} &= -\frac{1}{\sigma_2^2(1-r_{12}^2)}, \\ \frac{d^2(\log z)}{dx_1 dx_2} &= \frac{r_{12}}{\sigma_1\sigma_2(1-r_{12}^2)}. \end{aligned}$$

\* See the memoir on "Heredity, Regression, and Panmixia," p. 268.

Now these must be multiplied by  $z$  and integrated for  $x_1$  and  $x_2$  from  $-\infty$  to  $+\infty$ . These integrations follow at once, if we remember that

$$n\sigma_1^2 = \iint z x_1^2 dx_1 dx_2, \quad n\sigma_2^2 = \iint z x_2^2 dx_1 dx_2, \quad n\sigma_1\sigma_2r_{12} = \iint z x_1 x_2 dx_1 dx_2,$$

by definition of  $\sigma_1$ ,  $\sigma_2$ , and  $r_{12}$ .

Thus we obtain the following system :

$$\begin{aligned} a_{11} &= \frac{n}{\sigma_1^2} \frac{2 - r_{12}^2}{1 - r_{12}^2}, & a_{22} &= \frac{n}{\sigma_2^2} \frac{2 - r_{12}^2}{1 - r_{12}^2}, \\ a_{12} &= \frac{n r_{12}^2}{\sigma_1 \sigma_2 (1 - r_{12}^2)}, & a_{13} &= \frac{n r_{12}}{\sigma_1 (1 - r_{12}^2)}, \\ a_{23} &= \frac{n r_{12}}{\sigma_2 (1 - r_{12}^2)}, & a_{33} &= \frac{n (1 + r_{12}^2)}{(1 - r_{12}^2)}, \\ a_{44} &= \frac{n}{\sigma_1^2 (1 - r_{12}^2)}, & a_{55} &= \frac{n}{\sigma_2^2 (1 - r_{12}^2)}, & a_{15} &= \frac{n r_{12}}{\sigma_1 \sigma_2 (1 - r_{12}^2)} \quad \cdot \quad \cdot \quad (x.), \end{aligned}$$

where the  $a$ 's are those of Equ. (vii.), p. 236, obtained by taking the above differentials of the logarithms of  $z$  in order.

We can now write down the correlation surface, giving the frequencies of errors in the constants :

$$\begin{aligned} P_\Delta &= P_0 \text{ expt. } - \frac{1}{2} \left\{ \frac{n (\Delta h_1)^2}{\sigma_1^2 (1 - r_{12}^2)} - \frac{2n r_{12} \Delta h_1 \Delta h_2}{\sigma_1 \sigma_2 (1 - r_{12}^2)} + \frac{n (\Delta h_2)^2}{\sigma_2^2 (1 - r_{12}^2)} \right\} \\ &\times \text{expt. } - \frac{1}{2} \left\{ \frac{n (2 - r_{12}^2)}{\sigma_1^2 (1 - r_{12}^2)} (\Delta \sigma_1)^2 + \frac{n (2 - r_{12}^2)}{\sigma_2^2 (1 - r_{12}^2)} (\Delta \sigma_2)^2 + \frac{n (1 + r_{12}^2)}{(1 - r_{12}^2)^2} (\Delta r_{12})^2 \right. \\ &\left. - \frac{2n r_{12}}{\sigma_2 (1 - r_{12}^2)} \Delta \sigma_2 \Delta r_{12} - \frac{2n r_{12}}{\sigma_1 (1 - r_{12}^2)} \Delta \sigma_1 \Delta r_{12} - \frac{2n r_{12}^2}{\sigma_1 \sigma_2 (1 - r_{12}^2)} \Delta \sigma_1 \Delta \sigma_2 \right\}. \quad (xi.). \end{aligned}$$

Now several important conclusions follow at once from this result :

( $\alpha$ ) For the case of normal frequency (but in general only for this case) the errors in the means are uncorrelated with the errors in the variations and correlations. The error correlation surface breaks up into two parts, of which the first part we have written down involves only the means, and would coincide exactly with the correlation surface (ix.), with which we started, if we write in (ix.)  $\Delta h_1$  for  $x_1$ ,  $\Delta h_2$  for  $x_2$ , and  $\sigma_1/\sqrt{n}$ ,  $\sigma_2/\sqrt{n}$ , for  $\sigma_1$  and  $\sigma_2$  respectively.

It follows accordingly that the standard deviations of the errors in the means are

$$\Sigma_{h_1} = \sigma_1/\sqrt{n}, \quad \Sigma_{h_2} = \sigma_2/\sqrt{n} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (xii.),$$

and the correlation of the errors made in the means is

$$R_{h_1 h_2} = r_{12} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (xiii.).$$

Further, the standard deviations for errors in  $h_1$  when the error in  $h_2$  is known, or for errors in  $h_2$  when the error in  $h_1$  is known, are respectively

$$\frac{\sigma_1}{\sqrt{n}}(1 - r_{12}^2), \quad \frac{\sigma_2}{\sqrt{n}}(1 - r_{12}^2) \dots \dots \dots \text{(xiv.)}$$

The results (xii.) are, of course, well known; the results (xiii.) and (xiv.) are, we believe, novel and important. An illustration may be of service.

Suppose the stature and arm-length of a population to be under consideration. Let us suppose the mean stature of the population known from a great number of observations. Now let the arm-lengths be determined for a random selection of the population; then, if the stature of these individuals so selected differs  $\Delta h_1$  in excess from the mean stature of the whole population, the arm-length of the random selection will most probably differ from that of the whole population by  $\Delta h_2$ , a quantity fixed by (xiii.), or rather by the coefficient of regression

$$R_{\sigma_1\sigma_2/\Sigma h_2/\Sigma h_1}, \text{ i.e., } r_{12}\sigma_2/\sigma_1.$$

Thus

$$\Delta h_2 = r_{12}\sigma_2/\sigma_1 \cdot \Delta h_1,$$

with a probable error of  $\cdot 67449 \times \frac{\sigma_2}{\sqrt{n}}(1 - r_{12}^2)$ .

In other words, if the arm-length is to be found from a selection of the general population only, and the stature of this selection differs from that of the general population, it is most reasonable to take the arm-length of the general population to be the mean arm-length of the selected population less the quantity  $r_{12}\sigma_2/\sigma_1 \cdot \Delta h_1$ .

Or, again, if a selection from a general population show a mean organ  $\Delta h_1$  in excess of that of the general population, the whole system of correlated organs will exhibit changes of which the magnitudes are most probably given by the type  $\frac{r_{12}\sigma_{12}}{\sigma_1} \Delta h_1$ .

The bearing of this on what we have termed random evolution will be obvious.

( $\beta$ ) Turning to the second part of the error correlation surface, we note at once that, *if two organs be correlated, random selection will give a system of correlated deviations in their variations and their correlation.*

Random selection (and *a fortiori* it may be added, artificial or natural selection), which alters an organ's variability, alters the variability and correlation of all other organs. In fact, when it is once realised how two random selections from a general population will as a rule have organs of different means, of different variabilities, and of different correlations, the means among themselves and the variabilities and correlations among themselves forming *systematic* groups, it becomes obvious how any assumption of the coefficient of correlation as a constant for local races runs wide of the mark; and this, whether natural or random evolution, is to be looked upon as the source of the observed differences in character. What chiefly concerns the biologist in this matter at present is this, that *even* a random selection of one organ



will produce changes in all other organic characters, which, if small, are still sensible and capable of quantitative expression.\*

(6) Returning now to the algebra of our investigation, we have to discuss the second part of (xi.) by aid of the formulæ given in (viii.), p. 237.

We require, in the first place, to evaluate the determinant  $\Delta$ .

Now

$$\Delta = \begin{vmatrix} \frac{n}{\sigma_1^2} \frac{2 - r_{12}^2}{1 - r_{12}^2}, & -\frac{r_{12}^2}{\sigma_1 \sigma_2 (1 - r_{12}^2)}, & -\frac{n r_{12}}{\sigma_1 (1 - r_{12}^2)} \\ -\frac{n r_{12}^2}{\sigma_1 \sigma_2 (1 - r_{12}^2)}, & \frac{n}{\sigma_2^2} \frac{2 - r_{12}^2}{1 - r_{12}^2}, & -\frac{n r_{12}}{\sigma_2 (1 - r_{12}^2)} \\ -\frac{n r_{12}}{\sigma_1 (1 - r_{12}^2)}, & -\frac{n r_{12}}{\sigma_2 (1 - r_{12}^2)}, & \frac{n (1 + r_{12}^2)}{(1 - r_{12}^2)^2} \end{vmatrix}.$$

Divide the first row by  $\frac{n}{\sigma_1 (1 - r_{12}^2)}$ , the second row by  $\frac{n}{\sigma_2 (1 - r_{12}^2)}$ , the third row by  $\frac{n}{1 - r_{12}^2}$ , the first column by  $\sigma_1$ , and the second by  $\sigma_2$ . Hence

$$\Delta = \frac{n^3}{\sigma_1^2 \sigma_2^2 (1 - r_{12}^2)^3} \begin{vmatrix} 2 - r_{12}^2 & -r_{12}^2 & -r_{12} \\ -r_{12}^2 & 2 - r_{12}^2 & -r_{12} \\ -r_{12} & -r_{12} & \frac{1 + r_{12}^2}{1 - r_{12}^2} \end{vmatrix}.$$

Subtract the second column from the first, and then add the first row to the second; we have

$$\begin{aligned} \Delta &= \frac{n^3}{\sigma_1^2 \sigma_2^2 (1 - r_{12}^2)^3} \begin{vmatrix} 2 & -r_{12}^2 & -r_{12} \\ 0 & 2(1 - r_{12}^2) & -2r_{12} \\ 0 & -r_{12} & \frac{1 + r_{12}^2}{1 - r_{12}^2} \end{vmatrix}, \\ &= \frac{4n^3}{\sigma_1^2 \sigma_2^2 (1 - r_{12}^2)^3}. \end{aligned}$$

The minors are now easily found to be

$$\begin{aligned} A_{11} &= \frac{2n^2}{\sigma_2^2 (1 - r_{12}^2)^3}, & A_{22} &= \frac{2n^2}{\sigma_1^2 (1 - r_{12}^2)^3}, \\ A_{33} &= \frac{4n}{\sigma_1^2 \sigma_2^2 (1 - r_{12}^2)^3}, & A_{12} &= \frac{2n^2 r_{12}^2}{\sigma_1 \sigma_2 (1 - r_{12}^2)^3}, \\ A_{13} &= \frac{2n^2 r_{12}}{\sigma_1 \sigma_2^2 (1 - r_{12}^2)^3}, & A_{23} &= \frac{2n^2 r_{12}}{\sigma_2 \sigma_1^2 (1 - r_{12}^2)^3}. \end{aligned}$$

\* Take (xvii.) below, for example; it expresses for the first time quantitatively the important biological principle that, if a group be selected at random from the general population, and it has more variability in one character, it will be more variable than the general population in all other characters.



It will thus be seen that when  $r_{12}$  is small the absolute and partial probable errors are both large and nearly equal; that when  $r_{12}$  is large the absolute and partial probable errors differ more widely, but, as both are small in this case, their difference is not of much importance. Bearing these facts in mind, it will be found that the reasoning based on the partial error in previous memoirs remains valid, even if the partial error be replaced, as it generally should be, by the somewhat larger absolute error.\*

(γ) If we know definitely the variability of any organ, and we take a definite group of the general population to find the variability of a second correlated organ, then there will be correlation between the deviation of the variation of this group with regard to the first organ from the variation of the first organ in the general population and the deviation of the variation of the group with regard to its second organ from that of the general population's variation for the second organ. This correlation is measured by  $r_{12}^2$ , and thus, if the organs be slightly correlated, it is small; but if the organs be closely correlated, it is large. Suppose, for example, we know the variability of the tibia, and require to find that of the ulna from a comparatively few specimens. Let  $\sigma_1 + \Delta\sigma_1$  be the variability of the tibia in the specimens for which the ulna can be measured, and  $r_{12}$  the correlation observed between the ulna and tibia in these specimens; then, the variability of the ulna being observed as  $\sigma_2' = \sigma_2 + \Delta\sigma_2$ , the most probable variability of the ulna in the general population is

$$\sigma_2' = r_{12}^2 \Sigma_{\sigma_2} \Delta\sigma_1 / \Sigma_{\sigma_1}$$

or

$$\sigma_2' = \frac{r_{12}^2 \sigma_2}{\sigma_1} \Delta\sigma_1,$$

or, since the second term is small, we may write  $\sigma_2'$  for  $\sigma_2$ , and the above expression equals

$$= \sigma_2' \left( 1 - r_{12}^2 \frac{\Delta\sigma_1}{\sigma_1} \right).$$

For the long bones,  $r_{12} = .9$  roughly, and therefore we have the ratio of variability of the ulna in the general population to the variability observed in the group  $= 1 - .8 \Delta\sigma_1/\sigma_1$ .

It is clear that this expression also measures the change in the variability of the ulna due to a random selection of tibia.

(δ) Although the correlation between deviations in the variability of two organs from their mean variabilities only varies as the square of their correlation, the correlation between the deviations in the variability of an organ and in its correlation with a second organ varies as the first power of the correlation of the two organs. In other words,

\* See, for example, the reasoning as to the non-constancy of the correlation coefficient for local races in 'Phil. Trans.,' A, vol. 187, pp. 267 and 278.



while a selection of variability may produce only a small or moderate change on the variability of correlated organs, a selection of correlation or a selection of variability is likely to produce considerable changes on variability and correlation respectively. Let  $\sigma_1, \sigma_2, r_{12}$ , be the mean values of the standard deviations and the coefficient of correlation for any three organs; let  $\sigma_1 + \Delta\sigma_1, \sigma_2 + \Delta\sigma_2, r_{12} + \Delta r_{12}$ , be the like quantities for a group selected at random. Then the principle of regression tells us that most probably

$$\Delta\sigma_1 = R_{\sigma_1 r_{12}} \sum_{r_{12}} \Delta r_{12},$$

$$\Delta r_{12} = R_{r_{12} \sigma_1} \sum_{\sigma_1} \Delta\sigma_1.$$

Substituting the values given by (xv.) to (xviii.), we find

$$\left. \begin{aligned} \Delta\sigma_1 &= \frac{1}{2} \sigma_1 \frac{r_{12}^2}{1 - r_{12}^2} \Delta r_{12} \\ \Delta r_{12} &= r_{12} \left(1 - r_{12}^2\right) \frac{\Delta\sigma_1}{\sigma_1} \end{aligned} \right\} \dots \dots \dots (xx.).$$

Now these equations lead us to some important conclusions. In the first place, if the correlation be very small or very large, then a random selection of variability ( $\Delta\sigma_1$ ) makes only a small change in correlation ( $\Delta r_{12}$ ). The change in correlation for a selection of variability is greatest when  $r_{12} = 1/\sqrt{3}$ , and then is approximately  $\cdot 385 \Delta\sigma_1/\sigma_1$ , or over 6 per cent., if  $\Delta\sigma_1/\sigma_1$  were as high as  $\frac{1}{10}$ . On the other hand, the change in variability ( $\Delta\sigma_1$ ) due to a selection ( $\Delta r_{12}$ ) of correlation is small if the correlation be small, but increases rapidly if the correlation become nearly perfect. Of course, for perfect correlation the probable error of  $r_{12}$  is zero, and accordingly it is infinitely improbable that a selection can be made with  $\Delta r_{12}$  differing from zero. But if  $r_{12}$  be not unity, then a selection in which  $\Delta r_{12}$  is large, however improbable, will give very large changes in the variability, if  $r_{12}$  be very large. Our conclusion is accordingly that considerable changes in variability are likely to be produced whenever there is a correlation selection among highly correlated organs.

(7) *To find the Probable Error of the Regression Coefficient for Two Organs.*

The regression coefficient  $\rho_1$  is given by

$$\rho_1 = r_{12} \sigma_1 / \sigma_2.$$

Its standard deviation  $\Sigma_{\rho_1}$  is given by the summation equation

$$(\Sigma_{\rho_1})^2 = S (\Delta\rho_1)^2 / n.$$

To find its value we adopt a method, which we give on this first occasion at length, as it will be frequently used in the sequel.

Take logarithmic differentials

$$\frac{\Delta \rho_1}{\rho_1} = \frac{\Delta r_{12}}{r_{12}} + \frac{\Delta \sigma_1}{\sigma_1} - \frac{\Delta \sigma_2}{\sigma_2}.$$

Square and divide by  $n$  after summing

$$\frac{S(\Delta \rho_1)^2}{n \rho_1^2} = \frac{S(\Delta r_{12})^2}{n r_{12}^2} + \frac{S(\Delta \sigma_1)^2}{n \sigma_1^2} + \frac{S(\Delta \sigma_2)^2}{n \sigma_2^2} + \frac{2S(\Delta r_{12} \Delta \sigma_1)}{n r_{12} \sigma_1} - \frac{2S(\Delta r_{12} \Delta \sigma_2)}{n r_{12} \sigma_2} - \frac{2S(\Delta \sigma_1 \Delta \sigma_2)}{n \sigma_1 \sigma_2}.$$

Now, remembering the definitions of standard deviations and coefficients of correlation, this may be written

$$\frac{(\sum \rho_1)^2}{\rho_1^2} = \frac{(\sum r_{12})^2}{r_{12}^2} + \frac{(\sum \sigma_1)^2}{\sigma_1^2} + \frac{(\sum \sigma_2)^2}{\sigma_2^2} - \frac{2\sum \sigma_1 \sum \sigma_2 R_{\sigma_1 \sigma_2}}{\sigma_1 \sigma_2} + \frac{2\sum r_{12} \sum \sigma_1 R_{r_{12} \sigma_1}}{r_{12} \sigma_1} - \frac{2\sum r_{12} \sum \sigma_2 R_{r_{12} \sigma_2}}{r_{12} \sigma_2}.$$

Now all the quantities on the right have already been found in equations (xv.) to (xviii.). Hence, substituting, we have

$$\frac{(\sum \rho_1)^2}{\rho_1^2} = \frac{1 - r_{12}^2}{n r_{12}^2}.$$

Hence

$$\sum \rho_1 = \frac{\sigma_1}{\sigma_2} \sqrt{\left( \frac{1 - r_{12}^2}{n} \right)} \quad \dots \quad \text{(xxi.)}$$

Thus the probable error of a regression coefficient

$$= .67449 \frac{\sigma_1}{\sigma_2} \sqrt{\left( \frac{1 - r_{12}^2}{n} \right)}.$$

This is of fundamental importance for testing the significance of results obtained by applying the theory of regression to problems in heredity, panmixia, &c.

The probable percentage error in a regression coefficient  $= \frac{67.449}{\sqrt{n}} \frac{\sqrt{1 - r_{12}^2}}{r_{12}}$ , and hence is small if the correlation be close, and increases rapidly if the correlation be small. This again illustrates the point to which reference has been made in another memoir,\* namely, that when only a few individuals can be measured, the most reliable results for the purposes of the quantitative theory of evolution are to be found from the measurements of the most highly correlated organs.

\* PEARSON and LEE: "Correlation in Civilised and Uncivilised Races," 'Roy. Soc. Proc.,' vol. 61, p. 345.

Attention should be drawn to the fact that we have replaced errors by differentials. This is only legitimate so long as product terms in the errors are negligible as compared with linear terms. This is the assumption almost universally made by writers on the theory of errors.\* It will not lead us astray, so long as we take care in any practical applications to verify the smallness of  $\Delta r_{12}$ ,  $\Delta \sigma_1$ ,  $\Delta \sigma_2$ , as compared with  $r_{12}$ ,  $\sigma_1$ , and  $\sigma_2$  respectively.

(8) *To find the Probable Errors and Error Correlations of the Constants of a Normal Frequency Distribution for Three Organs or more.*

It will scarcely have escaped the attentive reader that our investigation hitherto, only involving two organs, has left several important problems untouched. For example, it has dealt only with the *direct* effect of random selection. But we may ask such a question as this: What is the change in the correlation of two organs when the variability of a third is randomly selected? Or again: What is the change in the correlation of two organs when the correlation between one of these and a third, or between a third and a fourth, is randomly selected? All these are important problems in the theory of evolution.

The general equation to a normal frequency-surface for  $m$  organs is:—

$$z = (2\pi)^{\frac{1}{2}m} \sigma_1 \sigma_2 \dots \sigma_m \sqrt{R} \exp \left\{ -\frac{1}{2} R \left\{ R_{11} \frac{x_1^2}{\sigma_1^2} + R_{22} \frac{x_2^2}{\sigma_2^2} + \dots + 2R_{12} \frac{x_1 x_2}{\sigma_1 \sigma_2} + \dots \right\} \right\}$$

where  $R$  is the determinant

$$\begin{vmatrix} 1 & r_{12} & r_{13} & \dots & r_{1m} \\ r_{21} & 1 & r_{23} & \dots & r_{2m} \\ r_{31} & r_{32} & 1 & \dots & r_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ r_{m1} & r_{m2} & \dots & \dots & 1 \end{vmatrix}$$

and  $R_{ss'}$  is the minor of the term in the  $s^{\text{th}}$  row and  $s'^{\text{th}}$  column.

We require first to find the quantities like (vii.) of our Art. (4).

$$\log z = \log (n/(2\pi)^{\frac{1}{2}m}) - \frac{1}{2} \log R - S_s (\log \sigma_s) - \frac{1}{2} S_s \left( \frac{R_{ss}}{R} \frac{x_s^2}{\sigma_s^2} \right) - S_{ss'} \left( \frac{R_{ss'}}{R} \frac{x_s x_{s'}}{\sigma_s \sigma_{s'}} \right),$$

$$\frac{d(\log z)}{d\sigma_1} = -\frac{1}{\sigma_1} \left\{ 1 - \frac{R_{11}}{R} - S_s \left( \frac{R_{1s}}{R} \frac{x_s}{\sigma_1 \sigma_s} \right) \right\} \dots \dots \dots \text{(xxii.)}$$

\* GAUSS, admittedly, 'Theoria Combinationis Observationum' . . . . p. 53, Problema; LAPLACE and POISSON, actually but obscurely; see 'Théorie analytique des Probabilités, Liv. II., chap. IV., and 'Recherches sur la Probabilité des Jugements,' chap. IV.; more clearly in TODHUNTER'S account, 'History of Theory of Probability,' Art. 1,002 *et seq.* Further, CROFTON, Article 'Probability,' § 48, for a like assumption.



$$\frac{d(\log z)}{dr_{12}} = -\frac{1}{2} \frac{d(\log R)}{dr_{12}} - \frac{1}{2} \frac{d}{dr_{12}} S_s \left( \frac{R_{1s} r_{1s}^2}{R \sigma_s^2} \right) - \frac{d}{dr_{12}} S_{s'} \left( \frac{R_{ss'} r_{ss'}^2}{R \sigma_s \sigma_{s'}} \right) \quad (\text{xxiii}).$$

Differentiating the first of these again with regard to  $\sigma_1$ , and summing for all possible values of  $x$ 's, we find

$$-a_{11} = \iiint \dots z \frac{d^2(\log z)}{d\sigma_1^2} dx_1 dx_2 \dots dx_m = \frac{n}{\sigma_1^2} \left( 1 - \frac{3R_{11}}{R} - \frac{2S_s(R_{1s} r_{1s})}{R} \right).$$

But

$$R = R_{11} + S_s(R_{1s} r_{1s}).$$

Hence

$$a_{11} = \frac{n}{\sigma_1^2} \left( \frac{R + R_{11}}{R} \right) \quad \dots \quad (\text{xxiv}).$$

Differentiating (xxii.) with regard to  $\sigma_2$  and summing, we have at once

$$\iiint \dots z \frac{d^2(\log z)}{d\sigma_1 d\sigma_2} dx_1 dx_2 \dots dx_m = a_{12} = -\frac{n}{\sigma_1 \sigma_2} \frac{r_{12} R_{12}}{R} \quad \dots \quad (\text{xxv}).$$

Differentiating (xxii.) with regard to  $r_{12}$  and summing, we have

$$\begin{aligned} \iiint \dots z \frac{d^2(\log z)}{d\sigma_1 dr_{12}} dx_1 dx_2 \dots dx_m &= {}_1c_{12} = \frac{n}{\sigma_1} \left\{ \frac{d}{dr_{12}} \left( \frac{R_{11}}{R} \right) + S_s r_{1s} \frac{d}{dr_{12}} \left( \frac{R_{1s}}{R} \right) \right\}, \\ &= \frac{n}{\sigma_1} \left\{ \frac{d}{dr_{12}} \left( \frac{R_{11} + S_s(r_{1s} R_{1s})}{R} \right) - \frac{R_{12}}{R} \right\}, \end{aligned}$$

or

$${}_1c_{12} = -nR_{12}/\sigma_1 R \quad \dots \quad (\text{xxvi}).$$

Differentiating (xxii.) with regard to  $r_{23}$  and summing, we find

$$\begin{aligned} {}_1c_{23} &= \frac{n}{\sigma_1} \frac{d}{dr_{23}} \frac{R_{11}}{R} + \frac{n}{\sigma_1} \frac{d}{dr_{23}} S_s \left( \frac{R_{1s} r_{1s}}{R} \right), \\ &= \frac{n}{\sigma_1} \frac{d}{dr_{23}} \left( \frac{R_{11} + S_s(R_{1s} r_{1s})}{R} \right). \end{aligned}$$

or

$${}_1c_{23} = 0 \quad \dots \quad (\text{xxvii}).$$

Our next step is to differentiate (xxiii.) with regard to  $r_{12}$  and sum. We have

$$\begin{aligned}
\iiint \dots z \frac{d^2(\log z)}{dr_{12}^2} dx_1 \dots dx_m &= - {}_{12}b_{12}, \\
&= - \frac{n}{2} \frac{d^2(\log R)}{dr_{12}^2} - \frac{n}{2} \frac{d^2}{dr_{12}^2} S_{ss} \left( \frac{R_{ss}}{R} \right) - n S_{ss'} \left( r_{ss'} \frac{d^2}{dr_{12}^2} \left( \frac{R_{ss'}}{R} \right) \right), \\
&= - \frac{n}{2} \frac{d^2(\log R)}{dr_{12}^2} - \frac{n}{2} \frac{d^2}{dr_{12}^2} S_{ss} \left( \frac{R_{ss}}{R} \right) + 2n \frac{d}{dr_{12}} \left( \frac{R_{12}}{R} \right) - n S_{ss'} \left( \frac{d^2}{dr_{12}^2} \left( \frac{r_{ss'} R_{ss'}}{R} \right) \right).
\end{aligned}$$

Now, since  $R = R_{11} + S_s(R_{1s} r_{1s})$ ,

$$mR = S_{ss}(R_{ss}) + 2S_{ss'}(R_{ss'} r_{ss'}),$$

and therefore

$$S_{ss'}(R_{ss'} r_{ss'}/R) = \frac{1}{2} m - \frac{1}{2} S_{ss}(R_{ss}/R).$$

Substituting, we find

$$- {}_{12}b_{12} = - \frac{n}{2} \frac{d^2(\log R)}{dr_{12}^2} + 2n \frac{d}{dr_{12}} \left( \frac{R_{12}}{R} \right).$$

But  $\frac{d}{dr_{12}}(\log R) = \frac{1}{R} \frac{dR}{dr_{12}}$  and  $2R_{12} = \frac{dR}{dr_{12}}$ ; hence

$$\frac{d^2(\log R)}{dr_{12}^2} = 2 \frac{d(R_{12}/R)}{dr_{12}}.$$

Thus finally

$${}_{12}b_{12} = - n \frac{d}{dr_{12}} \left( \frac{R_{12}}{R} \right) = n \frac{2R_{12}^2 - RR_{12,12}}{R^2} \dots \dots \dots (\text{xxviii.}),$$

where  $R_{12,12}$  is the second minor, found by striking out from  $R$  the first and second rows and columns.

In the next place let us differentiate (xxiii.) with regard to  $r_{13}$  and sum. We find in precisely similar manner

$$\begin{aligned}
\iiint \dots z \frac{d^2 \log z}{dr_{12} dr_{13}} dx_1 dx_2 \dots dx_m &= {}_{12}b_{13} \\
&= n \frac{d}{dr_{12}} \left( \frac{R_{13}}{R} \right), \text{ or, } = n \frac{d}{dr_{13}} \left( \frac{R_{12}}{R} \right),
\end{aligned}$$

or

$${}_{12}b_{13} = - n \frac{(2R_{12}R_{13} - RR_{12,13})}{R^2} \dots \dots \dots (\text{xxix.}).$$

Similarly, we deduce

$${}_{12}b_{34} = n \frac{d}{dr_{12}} \left( \frac{R_{34}}{R} \right) = - n \frac{(2R_{12}R_{34} - 2RR_{12,34})}{R^2} \dots \dots \dots (\text{xxx.}).$$

This completes all the possible types.

We are now in a position to write down from (vii.) of Art. (4) the complete frequency surface for errors in the constants of a normal frequency surface for  $m$  organs. It will suffice to write a type of each term. We have

$$\begin{aligned} P_{\Delta} = P_0 \times \text{exponent} - \frac{n}{2} \left\{ \frac{R + R_{11}}{R} \frac{(\Delta\sigma_1)^2}{\sigma_1^2} + \frac{2R_{12}r_{12}}{R} \frac{\Delta\sigma_1 \Delta\sigma_2}{\sigma_1\sigma_2} \right. \\ + \dots + \frac{2R_{12}^2 - RR_{12,12}}{R^2} (\Delta r_{12})^2 + \dots + \frac{2R_{12}}{R} \frac{\Delta\sigma_1}{\sigma_1} \Delta r_{12} + \dots \\ \left. + 2 \frac{2R_{12}R_{13} - RR_{12,13}}{R^2} \Delta r_{12} \Delta r_{13} + \dots + 2 \frac{2R_{12}R_{34} - 2RR_{12,34}}{R^2} \Delta r_{12} \Delta r_{34} + \dots \right\} \text{(xxxi.)} \end{aligned}$$

Now this result again seems at once to give conclusions of considerable importance. Thus:—

( $\alpha$ ) Since there are no terms of the type  $\Delta\sigma_1 \Delta r_{23}$ , we infer (i.) that the random selection of variation in one organ will most likely only vary the correlation between two other organs by terms of the second order; and that (ii.) the random selection of correlation between two organs will in all probability only change the variability of a third organ by terms of the second order.

( $\beta$ ) The selection of correlation between any two organs will most probable vary the correlation between a second pair, *i.e.*, terms exist in  $\Delta r_{12} \Delta_{34}$ , &c.

( $\gamma$ ) The selection of the variation for any organ varies the correlation between that organ and a third organ, and *vice versa* the selection of correlation between two organs changes the variability of both organs. And lastly

( $\delta$ ) The selection of correlation between two organs varies the correlation between either organ and any other organs.

We may exhibit these results more clearly by taking four special organs, say, femur, tibia, humerus, and radius. Then a group having the variability of its femur different from that of the general population, will also have, in all probability, the variability in its tibia, humerus, and radius different; the correlations femur-tibia, femur-humerus, and femur-radius different; but those of tibia-humerus, humerus-radius, and radius-tibia only slightly different. Further, a group having the correlation of its femur-tibia different from that of the general population, will also have all the other correlations, humerus-radius, femur-humerus, femur-radius, tibia-humerus, tibia-radius, different from the values for the general population. Further, the variability in femur and tibia will be changed; but in all likelihood the variability in humerus and radius only slightly changed.

These general conclusions, which seem to cast considerable light on the manner in which selection influences the variability and correlation of organs, must now be reduced to quantitative expression.



(9) *Evaluation of the Constants*  $a_{11}, a_{12}, {}_{12}b_{12}, {}_{12}b_{13}, {}_{12}b_{34}$ , &c., of the last Article for the cases of Three or Four Correlated Organs.

We will proceed first to calculate a general form for the determinant  $\Delta$  of our Article (4). Using the values (xxiv. to xxx.) we find for four organs

$$\Delta = \begin{vmatrix} \frac{n(R+R_{11})}{\sigma_1^2 R} & \frac{n {}_{12}R_{12}}{\sigma_1 \sigma_2 R} & \frac{n {}_{13}R_{13}}{\sigma_1 \sigma_3 R} & \frac{n {}_{14}R_{14}}{\sigma_1 \sigma_4 R} & \frac{n R_{12}}{\sigma_1 R} & \frac{n R_{13}}{\sigma_1 R} & \frac{n R_{14}}{\sigma_1 R} & 0 & 0 \\ \frac{n {}_{12}R_{12}}{\sigma_1 \sigma_2 R} & \frac{n(R+R_{22})}{\sigma_2^2 R} & \frac{n {}_{23}R_{23}}{\sigma_2 \sigma_3 R} & \frac{n {}_{24}R_{24}}{\sigma_2 \sigma_4 R} & \frac{n R_{12}}{\sigma_2 R} & 0 & 0 & \frac{n R_{24}}{\sigma_2 R} & 0 \\ \frac{n {}_{13}R_{13}}{\sigma_1 \sigma_3 R} & \frac{n {}_{23}R_{23}}{\sigma_2 \sigma_3 R} & \frac{n(R+R_{33})}{\sigma_3^2 R} & \frac{n {}_{34}R_{34}}{\sigma_3 \sigma_4 R} & 0 & \frac{n R_{13}}{\sigma_3 R} & 0 & 0 & \frac{n R_{34}}{\sigma_3 R} \\ \frac{n {}_{14}R_{14}}{\sigma_1 \sigma_4 R} & \frac{n {}_{24}R_{24}}{\sigma_2 \sigma_4 R} & \frac{n {}_{34}R_{34}}{\sigma_3 \sigma_4 R} & \frac{n(R+R_{44})}{\sigma_4^2 R} & 0 & \frac{n R_{14}}{\sigma_4 R} & \frac{n R_{24}}{\sigma_4 R} & \frac{n R_{34}}{\sigma_4 R} & \frac{n R_{44}}{\sigma_4 R} \\ \frac{n R_{12}}{\sigma_1 R} & \frac{n R_{12}}{\sigma_2 R} & 0 & -\frac{nd}{dr_{12}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{12}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{12}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{12}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{12}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{12}} \left( \frac{R_{34}}{R} \right) \\ \frac{n R_{13}}{\sigma_1 R} & 0 & \frac{n R_{13}}{\sigma_3 R} & 0 & -\frac{nd}{dr_{13}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{13}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{13}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{13}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{13}} \left( \frac{R_{34}}{R} \right) \\ \frac{n R_{14}}{\sigma_1 R} & 0 & 0 & \frac{n R_{14}}{\sigma_4 R} & -\frac{nd}{dr_{14}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{14}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{14}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{14}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{14}} \left( \frac{R_{34}}{R} \right) \\ 0 & \frac{n R_{23}}{\sigma_2 R} & 0 & 0 & -\frac{nd}{dr_{23}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{23}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{23}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{23}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{23}} \left( \frac{R_{34}}{R} \right) \\ 0 & \frac{n R_{24}}{\sigma_2 R} & 0 & \frac{n R_{24}}{\sigma_4 R} & -\frac{nd}{dr_{24}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{24}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{24}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{24}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{24}} \left( \frac{R_{34}}{R} \right) \\ 0 & 0 & \frac{n R_{34}}{\sigma_3 R} & \frac{n R_{34}}{\sigma_4 R} & -\frac{nd}{dr_{34}} \left( \frac{R_{12}}{R} \right) & -\frac{nd}{dr_{34}} \left( \frac{R_{13}}{R} \right) & -\frac{nd}{dr_{34}} \left( \frac{R_{14}}{R} \right) & -\frac{nd}{dr_{34}} \left( \frac{R_{24}}{R} \right) & -\frac{nd}{dr_{34}} \left( \frac{R_{34}}{R} \right) \end{vmatrix}$$

(xxxii.)

Divide each row by  $n$ , the first, second, third, and fourth rows by  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  respectively, and the first, second, third, and fourth columns by  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  respectively. Multiply the fifth column by  $r_{12}$  and subtract it from the first and second; the sixth column by  $r_{13}$  and subtract it from the first and third; the seventh column by  $r_{14}$  and subtract it from the first and fourth; the eighth column by  $r_{23}$  and subtract it from the second and third; the ninth column by  $r_{24}$  and subtract it from the second and fourth; the tenth column by  $r_{34}$  and subtract it from the third and fourth. Remembering that

$$\begin{aligned} R &= R_{11} + r_{12}R_{12} + r_{13}R_{13} + r_{14}R_{14}, \\ R &= R_{22} + r_{21}R_{12} + r_{23}R_{23} + r_{24}R_{24}, \text{ \&c.,} \end{aligned}$$

and that

$$\begin{aligned} \frac{R_{12}}{R} + r_{12} \frac{d}{dr_{12}} \left( \frac{R_{12}}{R} \right) + r_{13} \frac{d}{dr_{12}} \left( \frac{R_{13}}{R} \right) + r_{14} \frac{d}{dr_{12}} \left( \frac{R_{14}}{R} \right) \\ = \frac{d}{dr_{12}} \left( \frac{r_{12}R_{12} + r_{13}R_{13} + r_{14}R_{14}}{R} \right) = - \frac{d}{dr_{12}} \left( \frac{R_{11}}{R} \right), \end{aligned}$$

$$r_{13} \frac{d}{dr_{12}} \left( \frac{R_{13}}{R} \right) + r_{23} \frac{d}{dr_{12}} \left( \frac{R_{23}}{R} \right) + r_{34} \frac{d}{dr_{12}} \left( \frac{R_{34}}{R} \right) = - \frac{d}{dr_{12}} \left( \frac{R_{33}}{R} \right), \text{ \&c.,}$$

we find, writing  $d_{12}$  for  $d/dr_{12}$ , &c., for brevity,

$$\Delta = \frac{n^{10}}{\sigma_1^9 \sigma_2^9 \sigma_3^9 \sigma_4^9}$$

$$\begin{aligned} \times \begin{vmatrix} 2R_{11}/R, & 0, & 0, & 0, & R_{12}/R, & R_{13}/R, & R_{14}/R, & 0, & 0, & 0, \\ 0, & 2R_{22}/R, & 0, & 0, & R_{12}/R, & 0, & 0, & R_{23}/R, & R_{24}/R, & 0, \\ 0, & 0, & 2R_{33}/R, & 0, & 0, & R_{13}/R, & 0, & R_{23}/R, & 0, & R_{34}/R, \\ 0, & 0, & 0, & 2R_{44}/R, & 0, & 0, & R_{14}/R, & 0, & R_{24}/R, & R_{34}/R, \\ d_{12} \left( \frac{R_{11}}{R} \right), & d_{12} \left( \frac{R_{22}}{R} \right), & d_{12} \left( \frac{R_{33}}{R} \right), & d_{12} \left( \frac{R_{44}}{R} \right), & d_{12} \left( \frac{R_{12}}{R} \right), & d_{12} \left( \frac{R_{13}}{R} \right), & d_{12} \left( \frac{R_{14}}{R} \right), & d_{12} \left( \frac{R_{23}}{R} \right), & d_{12} \left( \frac{R_{24}}{R} \right), & d_{12} \left( \frac{R_{34}}{R} \right), \\ d_{13} \left( \frac{R_{11}}{R} \right), & d_{13} \left( \frac{R_{22}}{R} \right), & d_{13} \left( \frac{R_{33}}{R} \right), & d_{13} \left( \frac{R_{44}}{R} \right), & d_{13} \left( \frac{R_{12}}{R} \right), & d_{13} \left( \frac{R_{13}}{R} \right), & d_{13} \left( \frac{R_{14}}{R} \right), & d_{13} \left( \frac{R_{23}}{R} \right), & d_{13} \left( \frac{R_{24}}{R} \right), & d_{13} \left( \frac{R_{34}}{R} \right), \\ d_{14} \left( \frac{R_{11}}{R} \right), & d_{14} \left( \frac{R_{22}}{R} \right), & d_{14} \left( \frac{R_{33}}{R} \right), & d_{14} \left( \frac{R_{44}}{R} \right), & d_{14} \left( \frac{R_{12}}{R} \right), & d_{14} \left( \frac{R_{13}}{R} \right), & d_{14} \left( \frac{R_{14}}{R} \right), & d_{14} \left( \frac{R_{23}}{R} \right), & d_{14} \left( \frac{R_{24}}{R} \right), & d_{14} \left( \frac{R_{34}}{R} \right), \\ d_{23} \left( \frac{R_{11}}{R} \right), & d_{23} \left( \frac{R_{22}}{R} \right), & d_{23} \left( \frac{R_{33}}{R} \right), & d_{23} \left( \frac{R_{44}}{R} \right), & d_{23} \left( \frac{R_{12}}{R} \right), & d_{23} \left( \frac{R_{13}}{R} \right), & d_{23} \left( \frac{R_{14}}{R} \right), & d_{23} \left( \frac{R_{23}}{R} \right), & d_{23} \left( \frac{R_{24}}{R} \right), & d_{23} \left( \frac{R_{34}}{R} \right), \\ d_{24} \left( \frac{R_{11}}{R} \right), & d_{24} \left( \frac{R_{22}}{R} \right), & d_{24} \left( \frac{R_{33}}{R} \right), & d_{24} \left( \frac{R_{44}}{R} \right), & d_{24} \left( \frac{R_{12}}{R} \right), & d_{24} \left( \frac{R_{13}}{R} \right), & d_{24} \left( \frac{R_{14}}{R} \right), & d_{24} \left( \frac{R_{23}}{R} \right), & d_{24} \left( \frac{R_{24}}{R} \right), & d_{24} \left( \frac{R_{34}}{R} \right), \\ d_{34} \left( \frac{R_{11}}{R} \right), & d_{34} \left( \frac{R_{22}}{R} \right), & d_{34} \left( \frac{R_{33}}{R} \right), & d_{34} \left( \frac{R_{44}}{R} \right), & d_{34} \left( \frac{R_{12}}{R} \right), & d_{34} \left( \frac{R_{13}}{R} \right), & d_{34} \left( \frac{R_{14}}{R} \right), & d_{34} \left( \frac{R_{23}}{R} \right), & d_{34} \left( \frac{R_{24}}{R} \right), & d_{34} \left( \frac{R_{34}}{R} \right), \end{vmatrix} \end{aligned}$$

Here the signs of the terms in the last six rows have been changed from minus to plus.\* Now multiply the first four rows by  $R_{12}/R$  and add them to the fifth row, the first four rows by  $R_{13}/R$  and add them to the sixth row, the first four rows by  $R_{14}/R$  and add them to the seventh row, and so on. Then, remembering that

$$\begin{aligned} d_{12} \left( \frac{R_{11}}{R} \right) &= - \frac{2R_{11}R_{12}}{R^2} + \frac{1}{R} \frac{dR_{11}}{dr_{12}}, \\ d_{12} \left( \frac{R_{22}}{R} \right) &= - \frac{2R_{22}R_{12}}{R^2} + \frac{1}{R} \frac{dR_{22}}{dr_{12}}, \\ d_{12} \left( \frac{R_{34}}{R} \right) &= - \frac{2R_{34}R_{12}}{R^2} + \frac{1}{R} \frac{dR_{34}}{dr_{12}}, \text{ \&c., \&c.,} \end{aligned}$$

and, further, that  $R_{pp}$  does not contain  $r_{rs}$ , so that  $d(R_{pp})/dr_{rs} = 0$ , we have, taking a factor  $1/R$  out of each row,

$$\Delta = \frac{(-1)^{6 \cdot 10}}{\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2} \frac{1}{R^{10}} \begin{vmatrix} 2R_{11} & 0 & 0 & 0 & R_{12} & R_{13} & R_{14} & 0 & 0 & 0 \\ 0 & 2R_{22} & 0 & 0 & R_{12} & 0 & 0 & R_{23} & R_{24} & 0 \\ 0 & 0 & 2R_{33} & 0 & 0 & R_{13} & 0 & R_{23} & 0 & R_{34} \\ 0 & 0 & 0 & 2R_{44} & 0 & 0 & R_{14} & 0 & R_{24} & R_{34} \\ 0 & 0 & \frac{dR_{33}}{dr_{12}} & \frac{dR_{44}}{dr_{12}} & \frac{dR_{12}}{dr_{12}} & \frac{dR_{13}}{dr_{12}} & \frac{dR_{14}}{dr_{12}} & \frac{dR_{23}}{dr_{12}} & \frac{dR_{24}}{dr_{12}} & \frac{dR_{34}}{dr_{12}} \\ 0 & \frac{dR_{22}}{dr_{13}} & 0 & \frac{dR_{44}}{dr_{13}} & \frac{dR_{12}}{dr_{13}} & \frac{dR_{13}}{dr_{13}} & \frac{dR_{14}}{dr_{13}} & \frac{dR_{23}}{dr_{13}} & \frac{dR_{24}}{dr_{13}} & \frac{dR_{34}}{dr_{13}} \\ 0 & \frac{dR_{22}}{dr_{14}} & \frac{dR_{33}}{dr_{14}} & 0 & \frac{dR_{12}}{dr_{14}} & \frac{dR_{13}}{dr_{14}} & \frac{dR_{14}}{dr_{14}} & \frac{dR_{23}}{dr_{14}} & \frac{dR_{24}}{dr_{14}} & \frac{dR_{34}}{dr_{14}} \\ \frac{dR_{11}}{dr_{23}} & 0 & 0 & \frac{dR_{44}}{dr_{23}} & \frac{dR_{12}}{dr_{23}} & \frac{dR_{13}}{dr_{23}} & \frac{dR_{14}}{dr_{23}} & \frac{dR_{23}}{dr_{23}} & \frac{dR_{24}}{dr_{23}} & \frac{dR_{34}}{dr_{23}} \\ \frac{dR_{11}}{dr_{24}} & 0 & \frac{dR_{33}}{dr_{24}} & 0 & \frac{dR_{12}}{dr_{24}} & \frac{dR_{13}}{dr_{24}} & \frac{dR_{14}}{dr_{24}} & \frac{dR_{23}}{dr_{24}} & \frac{dR_{24}}{dr_{24}} & \frac{dR_{34}}{dr_{24}} \\ \frac{dR_{11}}{dr_{34}} & \frac{dR_{22}}{dr_{34}} & 0 & 0 & \frac{dR_{12}}{dr_{34}} & \frac{dR_{13}}{dr_{34}} & \frac{dR_{14}}{dr_{34}} & \frac{dR_{23}}{dr_{34}} & \frac{dR_{24}}{dr_{34}} & \frac{dR_{34}}{dr_{34}} \end{vmatrix} \quad (\text{xxxiii}).$$

The form of  $\Delta$  is now clear for the case of any number of correlated organs.†

(10) *Case (i)*. Let us evaluate  $\Delta$  and its minors for the case of three correlated organs.

\* The factor  $(-1)^{\frac{1}{2}p(p-1)}$  must be introduced if we deal with  $p$  organs.

† We may reduce this determinant as follows to one of the 6th order. Divide the 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, 10th columns by  $2R_{11}$ ,  $2R_{22}$ ,  $2R_{33}$ ,  $2R_{44}$ ,  $R_{12}$ ,  $R_{13}$ ,  $R_{14}$ ,  $R_{23}$ ,  $R_{24}$ ,  $R_{34}$  respectively. After this division, subtract the first column from the 5th, 6th, and 7th. The determinant reduces to one of the 9th order; subtract the first column of this new determinant from its 4th, 7th, and 8th



In this case  $R = 1 - r_{23}^2 - r_{31}^2 - r_{12}^2 + 2r_{23}r_{31}r_{12}$ , and we have

$$\begin{aligned} R_{11} &= 1 - r_{23}^2, & R_{22} &= 1 - r_{13}^2, & R_{33} &= 1 - r_{12}^2, \\ R_{23} &= r_{31}r_{12} - r_{23}^2, & R_{31} &= r_{12}r_{23} - r_{13}^2, & R_{12} &= r_{23}r_{13} - r_{12}^2, \end{aligned}$$

whence we have

$$\Delta = \frac{(-1)^3 n^6}{\sigma_1^2 \sigma_2^2 \sigma_3^2} \frac{1}{R^6} \begin{vmatrix} 2(1 - r_{23}^2), & 0, & 0, & r_{23}r_{13} - r_{12}^2, & r_{12}r_{23} - r_{13}^2, & 0, \\ 0, & 2(1 - r_{13}^2), & 0, & r_{23}r_{13} - r_{12}^2, & 0, & r_{31}r_{12} - r_{23}^2, \\ 0, & 0, & 2(1 - r_{12}^2), & 0, & r_{12}r_{23} - r_{13}^2, & r_{31}r_{12} - r_{23}^2, \\ 0, & 0, & -2r_{12}, & -1, & r_{23}, & r_{31}, \\ 0, & -2r_{13}, & 0, & r_{23}, & -1, & r_{12}, \\ -2r_{23}, & 0, & 0, & r_{31}, & r_{12}, & -1. \end{vmatrix}$$

Divide the first three columns by 2; multiply the last row by  $r_{23}$  and subtract from the first row; the fifth row by  $r_{13}$  and subtract from the second; the fourth row by  $r_{12}$  and subtract from the third; we find

column, it again reduces by one degree. Repeat the process twice more, and after a slight rearrangement, the fact that  $d(R_{pp})/dr_{pp} = 0$  being remembered, we have, if  $\xi_{uv} = R_{uv}/\sqrt{R_{uu}R_{vv}}$ ,

$$\Delta = \frac{n^{10}}{\sigma_1^2 \sigma_2^2 \sigma_3^2 \sigma_4^2} \cdot \frac{2^4 R_{11} R_{22} R_{33} R_{44}}{R^{10}} \cdot R_{12} R_{13} R_{14} R_{23} R_{24} R_{34} \times \begin{vmatrix} \frac{d}{dr_{12}} \log \xi_{12}, & \frac{d}{dr_{12}} \log \xi_{13}, & \frac{d}{dr_{12}} \log \xi_{14}, & \frac{d}{dr_{12}} \log \xi_{23}, & \dots & \dots & \dots & \dots \\ \frac{d}{dr_{13}} \log \xi_{12}, & \frac{d}{dr_{13}} \log \xi_{13}, & \frac{d}{dr_{13}} \log \xi_{14}, & \dots & \dots & \dots & \dots & \dots \\ \frac{d}{dr_{14}} \log \xi_{12}, & \frac{d}{dr_{14}} \log \xi_{13}, & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{d}{dr_{23}} \log \xi_{12}, & \frac{d}{dr_{23}} \log \xi_{13}, & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{d}{dr_{24}} \log \xi_{12}, & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{d}{dr_{34}} \log \xi_{12}, & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

the general run of terms being obvious.

In precisely similar manner the value of  $\Delta$  for  $p$  organs can be written down, its degree being  $p$  less than the form given in (xxxiii.). We have not succeeded in reducing  $\Delta$  for the general case [since writing this, Mr. ARTHUR BERRY, of King's College, Cambridge, has succeeded in reducing the determinant for  $p = 4$ , and also in showing its relation to elliptic space], but we feel fairly confident that its

value will be found to be  $-\frac{2^p n^{3p(p+1)}}{\sigma_1^2 \sigma_2^2 \dots \sigma_p^2 R^{p+1}}$ .

$$\Delta = \frac{8n^6(-1)^3}{\sigma_1^2\sigma_2^2\sigma_3^2} \frac{1}{R^6} \begin{vmatrix} 1, & 0, & 0, & -r_{12}, & -r_{13}, & r_{23}, \\ 0, & 1, & 0, & -r_{12}, & r_{13}, & -r_{23}, \\ 0, & 0, & 1, & r_{12}, & -r_{13}, & -r_{23}, \\ 0, & 0, & -r_{12}, & -1, & r_{23}, & r_{31}, \\ 0, & -r_{13}, & 0, & r_{23}, & -1, & r_{12}, \\ -r_{23}, & 0, & 0, & r_{31}, & r_{12}, & -1. \end{vmatrix}$$

Add the first column multiplied by  $r_{12}$  to the fourth; add the first multiplied by  $r_{13}$  to the fifth; and subtract the first multiplied by  $r_{23}$  from the sixth, the determinant will reduce to the minor of the first row and column. Continuing this process twice more, we ultimately deduce

$$\Delta = \frac{8n^6(-1)^3}{\sigma_1^2\sigma_2^2\sigma_3^2} \frac{1}{R^6} \begin{vmatrix} -R_{11}, & -R_{12}, & -R_{13} \\ -R_{12}, & -R_{22}, & -R_{23} \\ -R_{13}, & -R_{23}, & -R_{33} \end{vmatrix} = \frac{8n^6}{\sigma_1^2\sigma_2^2\sigma_3^2} \frac{1}{R^4} \dots \text{(xxxiv.)}$$

We will now proceed to calculate such of the minors of  $\Delta$  as will give us results beyond those obtained for two correlated organs.\*

We require the correlation of  $\sigma_1$  and  $r_{23}$ , and of  $r_{12}$  and  $r_{13}$ .

Taking  $\sigma_1$  and  $r_{23}$  we must strike out in (xxxii.), as we have taken only three organs, the 4th, 7th, 9th, and 10th rows and columns straight off, and for the required minor the 1st row and the 8th column. We have then for the minor  $M(\sigma_1, r_{23})$ ,

$$M(\sigma_1, r_{23}) = -\frac{n^5}{\sigma_1\sigma_2^2\sigma_3^2} \begin{vmatrix} \frac{r_{12}R_{12}}{R}, & \frac{R+R_{22}}{R}, & \frac{r_{23}R_{23}}{R}, & \frac{R_{12}}{R}, & 0, \\ \frac{r_{13}R_{13}}{R}, & \frac{r_{23}R_{23}}{R}, & \frac{R+R_{33}}{R}, & 0, & \frac{R_{13}}{R}, \\ \frac{R_{12}}{R}, & \frac{R_{12}}{R}, & 0, & -\frac{d}{dr_{12}}\left(\frac{R_{12}}{R}\right), & -\frac{d}{dr_{12}}\left(\frac{R_{13}}{R}\right), \\ \frac{R_{13}}{R}, & 0, & \frac{R_{13}}{R}, & -\frac{d}{dr_{13}}\left(\frac{R_{12}}{R}\right), & -\frac{d}{dr_{13}}\left(\frac{R_{13}}{R}\right), \\ 0, & \frac{R_{23}}{R}, & \frac{R_{23}}{R}, & -\frac{d}{dr_{23}}\left(\frac{R_{12}}{R}\right), & -\frac{d}{dr_{23}}\left(\frac{R_{13}}{R}\right). \end{vmatrix}$$

To reduce this expression take the third row multiplied by  $r_{12}$  from the first, and the fifth row multiplied by  $r_{23}$  from the first. Then take the fourth row multiplied by  $r_{13}$  from the second, and the fifth row multiplied by  $r_{23}$  from the second, then remembering that

\* As a matter of fact all the minors were worked out and the results of (xv.) to (xviii.) thus verified.

$$R = R_{22} + r_{12}R_{12} + r_{23}R_{23},$$

$$R = R_{33} + r_{13}R_{13} + r_{23}R_{23},$$

and that generally,

$$\frac{d(R_{s'p'}/R)}{dr_{sp}} = \frac{d}{dr_{s'p'}} \left( R_{s'p'}/R \right).$$

We find :

$$M(\sigma_1, r_{23}) = -\frac{n^5}{\sigma_1\sigma_2^2\sigma_3^2} \begin{vmatrix} 0, & \frac{2R_{22}}{R}, & 0, & -\frac{d}{dr_{12}} \left( \frac{R_{22}}{R} \right), & -\frac{d}{dr_{13}} \left( \frac{R_{22}}{R} \right), \\ 0, & 0, & \frac{2R_{33}}{R}, & -\frac{d}{dr_{12}} \left( \frac{R_{33}}{R} \right), & -\frac{d}{dr_{13}} \left( \frac{R_{33}}{R} \right), \\ \frac{R_{12}}{R}, & \frac{R_{12}}{R}, & 0, & -\frac{d}{dr_{12}} \left( \frac{R_{12}}{R} \right), & -\frac{d}{dr_{13}} \left( \frac{R_{12}}{R} \right), \\ \frac{R_{13}}{R}, & 0, & \frac{R_{13}}{R}, & -\frac{d}{dr_{13}} \left( \frac{R_{12}}{R} \right), & -\frac{d}{dr_{13}} \left( \frac{R_{13}}{R} \right), \\ 0, & \frac{R_{23}}{R}, & \frac{R_{23}}{R}, & -\frac{d}{dr_{23}} \left( \frac{R_{12}}{R} \right), & -\frac{d}{dr_{23}} \left( \frac{R_{13}}{R} \right). \end{vmatrix}$$

Multiply the first two columns by  $R_{12}/R$  and subtract their sum from the fourth ; multiply by the second two columns by  $R_{13}/R$  and subtract their sum from the fifth ; divide out by the factor  $1/R^5$ . We obtain

$$M(\sigma_1, r_{23}) = -\frac{n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \begin{vmatrix} 0, & 2R_{22}, & 0, & \frac{dR_{22}}{dr_{12}}, & \frac{dR_{22}}{dr_{13}}, \\ 0, & 0, & 2R_{33}, & \frac{dR_{33}}{dr_{12}}, & \frac{dR_{33}}{dr_{13}}, \\ R_{12}, & R_{12}, & 0, & \frac{dR_{12}}{dr_{12}}, & \frac{dR_{12}}{dr_{13}}, \\ R_{13}, & 0, & R_{13}, & \frac{dR_{12}}{dr_{13}}, & \frac{dR_{13}}{dr_{13}}, \\ 0, & R_{23}, & R_{23}, & \frac{dR_{12}}{dr_{23}}, & \frac{dR_{13}}{dr_{23}}. \end{vmatrix}$$

Now remembering that

$$R = 1 - r_{23}^2 - r_{31}^2 - r_{12}^2 + 2r_{23}r_{31}r_{12},$$

substitute the various terms, and we find :

$$M(\sigma_1, r_{23}) = -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \begin{vmatrix} 0, & 1 - r_{13}^2, & 0, & 0, & -r_{13}, \\ 0, & 0, & 1 - r_{12}^2, & -r_{12}, & 0, \\ -r_{12} + r_{23}r_{13}, & -r_{12} + r_{23}r_{13}, & 0, & -1, & r_{23}, \\ -r_{13} + r_{23}r_{12}, & 0, & -r_{13} + r_{12}r_{23}, & r_{23}, & -1, \\ 0, & -r_{23} + r_{12}r_{13}, & -r_{23} + r_{12}r_{13}, & r_{13}, & r_{12}. \end{vmatrix}$$



Subtract  $r_{13}$  times the fifth from the second, and  $r_{12}$  times the fourth from the third column :

$$M(\sigma_1, r_{23}) = -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \begin{vmatrix} 0, & 1, & 0, & 0, & -r_{13} \\ 0, & 0, & 1, & -r_{12}, & 0 \\ -r_{12} + r_{23}r_{13}, & -r_{12}, & r_{12}, & -1, & r_{23} \\ -r_{13} + r_{23}r_{12}, & r_{13}, & -r_{13}, & r_{23}, & -1 \\ 0, & -r_{23}, & -r_{23}, & r_{13}, & r_{12} \end{vmatrix}.$$

Add  $r_{13}$  times the second column to the fifth and  $r_{12}$  times the third to the fourth, we have

$$\begin{aligned} M(\sigma_1, r_{23}) &= -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \begin{vmatrix} 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ R_{12}, & -r_{12}, & r_{12}, & -R_{33}, & -R_{23} \\ R_{13}, & r_{13}, & -r_{13}, & -R_{23}, & -R_{22} \\ 0, & -r_{23}, & -r_{23}, & -R_{13}, & -R_{12} \end{vmatrix} \\ &= -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \begin{vmatrix} R_{33}, & R_{23}, & R_{21} \\ R_{23}, & R_{22}, & R_{13} \\ R_{13}, & R_{12}, & 0 \end{vmatrix} \\ &= -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \{R_{12}(R_{23}R_{13} - R_{21}R_{33}) + R_{12}(R_{21}R_{23} - R_{13}R_{22})\} \\ &= -\frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^5} \{R_{13}r_{12}R + R_{12}r_{13}R\} \\ &= \frac{4n^5}{\sigma_1\sigma_2^2\sigma_3^2} \frac{1}{R^4} \{r_{12}(r_{13} - r_{12}r_{23}) + r_{13}(r_{12} - r_{13}r_{23})\} \quad \dots \quad (\text{xxxv}). \end{aligned}$$

Now we have seen that

$$R_{\sigma_1 r_{23}} = \frac{M(\sigma_1, r_{23})}{\Delta \Sigma_{\sigma_1} \Sigma_{r_{23}}} \quad \text{by (viii.) of Art. (4),}$$

and further  $\Sigma_{\sigma_1}$  and  $\Sigma_{r_{23}}$  are given by (xv.) and (xvi.) of Art. (6). Hence

$$R_{\sigma_1 r_{23}} = \frac{r_{12}(r_{13} - r_{12}r_{23}) + r_{13}(r_{12} - r_{13}r_{23})}{\sqrt{2} \cdot (1 - r_{23}^2)} \quad \dots \quad (\text{xxxvi}).$$

In the next place we will determine  $R_{r_{12} r_{13}}$ . The minor  $M(r_{12}, r_{13})$  is given by

$$\begin{aligned}
 M(r_{12}, r_{13}) &= -\frac{n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2} \begin{vmatrix} R+R_{11} & r_{12}R_{12} & r_{13}R_{13} & R_{13} & 0 \\ r_{12}R_{12} & R+R_{22} & r_{23}R_{23} & 0 & R_{23} \\ r_{13}R_{13} & r_{23}R_{23} & R+R_{33} & R_{13} & R_{23} \\ R_{13} & R_{12} & 0 & -\frac{d}{dr_{12}}\left(\frac{R_{13}}{R}\right) & -\frac{d}{dr_{12}}\left(\frac{R_{23}}{R}\right) \\ 0 & R_{23} & R_{23} & -\frac{d}{dr_{23}}\left(\frac{R_{13}}{R}\right) & -\frac{d}{dr_{23}}\left(\frac{R_{23}}{R}\right) \end{vmatrix} \\
 &= -\frac{n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R+R_{11} & r_{12}R_{12} & r_{13}R_{13} & R_{13} & 0 \\ r_{12}R_{12} & R+R_{22} & r_{23}R_{23} & 0 & R_{23} \\ r_{13}R_{13} & r_{23}R_{23} & R+R_{33} & R_{13} & R_{23} \\ R_{12} & R_{12} & 0 & \frac{2R_{13}R_{12}}{R} - r_{23} & \frac{2R_{23}R_{12}}{R} - r_{13} \\ 0 & R_{23} & R_{23} & \frac{2R_{13}R_{23}}{R} - r_{12} & \frac{2R_{23}^2}{R} + 1 \end{vmatrix}
 \end{aligned}$$

Add the first three rows together, multiply by  $R_{12}/R$  and subtract from the fourth, and by  $R_{23}/R$  and subtract from the fifth, we find

$$\begin{aligned}
 M(r_{12}, r_{13}) &= -\frac{n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R+R_{11} & r_{12}R_{12} & r_{13}R_{13} & R_{13} & 0 \\ r_{12}R_{12} & R+R_{22} & r_{23}R_{23} & 0 & R_{23} \\ r_{13}R_{13} & r_{23}R_{23} & R+R_{33} & R_{13} & R_{23} \\ -R_{12} & -R_{12} & -2R_{12} & -r_{23} & -r_{13} \\ -2R_{23} & -R_{23} & -R_{23} & -r_{12} & 1 \end{vmatrix}
 \end{aligned}$$

Multiply the fourth column by  $r_{13}$ , and the fifth by  $r_{23}$ , and subtract from the third column

$$\begin{aligned}
 M(r_{12}, r_{13}) &= -\frac{n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R+R_{11} & r_{12}R_{12} & 0 & R_{13} & 0 \\ r_{12}R_{12} & R+R_{22} & 0 & 0 & R_{23} \\ r_{13}R_{13} & r_{23}R_{23} & 2R_{33} & R_{13} & R_{23} \\ -R_{12} & -R_{12} & 2r_{12} & -r_{23} & -r_{13} \\ -2R_{23} & -R_{23} & 0 & -r_{12} & 1 \end{vmatrix}
 \end{aligned}$$

Multiply the third row by  $r_{12}$ , and the fourth by  $1 - r_{12}^2$  or  $R_{33}$ , and subtract the latter; the determinant now reduces to one of the fourth order, and we find:—

$$M(r_{12}, r_{13}) = -\frac{2n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R + R_{11}, & r_{12}R_{12}, & R_{13}, & 0 \\ r_{12}R_{12}, & R + R_{22}, & 0, & R_{23} \\ r_{12}R_{33} + r_{13}R_{23}, & r_{12}R_{33} + r_{23}R_{13}, & R_{23}, & R_{13} \\ -2R_{23}, & -R_{23}, & -r_{12}, & 1 \end{vmatrix}.$$

Add the third column, multiplied by  $r_{13}$  to, and subtract the fourth multiplied by  $r_{23}$  from, the second, and then subtract the first

$$M(r_{12}, r_{13}) = -\frac{2n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R + R_{11}, & -2R_{11}, & R_{13}, & 0 \\ r_{12}R_{12}, & 2R_{22}, & 0, & R_{23} \\ r_{12}R_{33} + r_{13}R_{23}, & 0, & R_{23}, & R_{13} \\ -2R_{23}, & -2R_{23}, & -r_{12}, & 1 \end{vmatrix}.$$

Divide out the 2 in the second column, add it to the first, and subtract the third column multiplied by  $r_{13}$

$$M(r_{12}, r_{13}) = -\frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R - r_{13}R_{13}, & -R_{11}, & R_{13}, & 0 \\ R - r_{23}R_{23}, & R_{22}, & 0, & R_{23} \\ r_{12}R_{33}, & 0, & R_{23}, & R_{13} \\ -R_{23}, & -r_{23}, & -r_{12}, & 1 \end{vmatrix}.$$

Add  $r_{23}$  times the last row to the first

$$M(r_{12}, r_{13}) = -\frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R_{33}, & -1, & -r_{13}, & r_{23} \\ R - r_{23}R_{23}, & R_{22}, & 0, & R_{23} \\ r_{12}R_{33}, & 0, & R_{23}, & R_{13} \\ -R_{23}, & -r_{23}, & -r_{12}, & 1 \end{vmatrix}.$$

Subtract  $r_{23}$  times the first row from the last, and remember that  $-R_{23} - r_{23}R_{33} = r_{12}R_{13}$ ,

$$M(r_{12}, r_{13}) = -\frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R_{33}, & -1, & -r_{13}, & r_{23} \\ R - r_{23}R_{23}, & R_{22}, & 0, & R_{23} \\ r_{12}R_{33}, & 0, & R_{23}, & R_{13} \\ r_{12}R_{13}, & 0, & R_{12}, & R_{11} \end{vmatrix}.$$

Multiply the first row by  $R_{22}$  and add it to the second; the determinant reduces to the third order,

$$M(r_{12}, r_{13}) = -\frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 R^5} \begin{vmatrix} R - r_{23}R_{23} + R_{22}R_{33}, & -r_{13}R_{22}, & -r_{13}R_{12} \\ r_{12}R_{33}, & R_{23}, & R_{13} \\ r_{12}R_{13}, & R_{12}, & R_{11} \end{vmatrix}.$$



Expanding this determinant from its first column, remembering that  $R_{23}R_{11} - R_{13}R_{12} = -r_{23}R$ ,

$$M(r_{12}, r_{13}) = -\frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 11^5} \{ -r_{23}R(R - r_{23}R_{23} + R_{22}R_{33}) + r_{12}r_{13}(R_{22}R_{11} - R_{12}^2)R_{33} \\ - r_{12}r_{13}(R_{22}R_{13} - R_{12}R_{23})R_{13} \}.$$

Or, since  $R_{22}R_{11} - R_{12}^2 = R$ ,  $R_{12}R_{23} - R_{22}R_{13} = r_{13}R$ , and  $R - r_{23}R_{23} + R_{22}R_{33} = 2R_{22}R_{33} - r_{12}r_{13}R_{23}$ ,

$$M(r_{12}, r_{13}) = \frac{4n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 11^5} \{ 2r_{23}R_{22}R_{33} - r_{12}r_{13}(R_{33} + r_{23}R_{23} + r_{13}R_{13}) \}, \\ = \frac{8n^5}{\sigma_1^2 \sigma_2^2 \sigma_3^2 11^5} \left\{ r_{23}R_{22}R_{33} - \frac{r_{12}r_{13}}{2}R \right\}.$$

Hence, since

$$R_{r_{12}r_{13}} = \frac{M_{(r_{12}, r_{13})}}{\Delta \sum_{r_{12}} \sum_{r_{13}}},$$

we have by (xvi.) of Art. (6) and (xxxiv.) of Art. (9)

$$R_{r_{12}r_{13}} = r_{23} - \frac{1}{2} r_{12}r_{13} \frac{R}{R_{22}R_{33}} \\ = r_{23} - \frac{r_{12}r_{13} \{ 1 - r_{23}^2 - r_{13}^2 - r_{12}^2 + 2r_{23}r_{12}r_{13} \}}{2(1 - r_{13}^2)(1 - r_{12}^2)} \quad \dots \quad (\text{xxxvii.}).$$

To complete the theory for the errors made in an investigation of the constants for a system of three correlated organs, we require to determine the probable error of a regression coefficient for a partial regression of a first organ on a second, the third organ being constant. This coefficient is given by

$${}_3\rho_{12} = \frac{r_{12} - r_{23}r_{13}}{1 - r_{23}^2} \frac{\sigma_1}{\sigma_2}.$$

Take logarithmic differentials

$$\frac{\delta {}_3\rho_{12}}{{}_3\rho_{12}} = \frac{\delta r_{12}}{r_{12} - r_{23}r_{13}} - \frac{r_{23} \delta r_{13}}{r_{12} - r_{23}r_{13}} + \delta r_{23} \left\{ \frac{-r_{13}}{r_{12} - r_{23}r_{13}} + \frac{2r_{23}}{1 - r_{23}^2} \right\} + \frac{\delta \sigma_1}{\sigma_1} - \frac{\delta \sigma_2}{\sigma_2}.$$

Let this be squared and divided by  $n$ , and then the values found above for the standard deviations of the errors in  $r_{12}$ ,  $r_{13}$ ,  $r_{23}$ ,  $\sigma_1$  and  $\sigma_2$ , and for the correlations of errors in these quantities be substituted. After some lengthy algebraic reductions, which it seems unnecessary to reproduce, there results

$$\frac{1}{({}_3\rho_{12})^2} (\sum {}_3\rho_{12})^2 = \frac{1 - r_{23}^2 - r_{13}^2 - r_{12}^2 + 2r_{23}r_{13}r_{12}}{n(r_{12} - r_{13}r_{23})^2},$$

2 L 2

or

$$\Sigma_{sp_{12}} = \frac{1}{\sqrt{n}} \frac{\sigma_1}{\sigma_2} \sqrt{(1 - r_{23}^2 - r_{13}^2 - r_{12}^2 + 2r_{23}r_{13}r_{12})} \dots \dots \dots (\text{xxxviii}).$$

The percentage probable error in a partial coefficient of regression is accordingly

$$67.449 \sqrt{R} / \sqrt{n} (-R_{12}).$$

Before discussing the significance of these quantitative results for three organs, it seems desirable to complete the general case by investigating the correlation between the errors made in the correlation coefficients of a first pair of organs and a second different pair of organs.

(11.) Case (ii). *Case of Four or more Organs.*—In the case of four or more organs the only new probable error will be that of a partial regression coefficient, but this can theoretically always be found by the method of the preceding paragraph, provided we know all the error correlations. The only novel correlation among the errors will be that of  $r_{12}r'_{34}$  and this we shall now proceed to investigate. The discovery of an error correlation coefficient of this type completes the theory of the errors of normal frequency constants.

Instead of evaluating  $\Delta$  of (xxxi.), which in the case of four organs appears to be very laborious, we may proceed as follows:—

If  $\Delta$  be written in the form

$$\begin{vmatrix} \alpha_{\sigma_1\sigma_1} & \alpha_{\sigma_2\sigma_1} & \dots & \alpha_{\sigma_4\sigma_1} & \alpha_{\sigma_{1'12}} & \alpha_{\sigma_{1'13}} & \dots & \alpha_{\sigma_{1'34}} \\ \alpha_{\sigma_1\sigma_2} & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{\sigma_1\sigma_4} & \alpha_{\sigma_2\sigma_4} & \dots & \alpha_{\sigma_4\sigma_4} & \alpha_{\sigma_{4'12}} & \alpha_{\sigma_{4'13}} & \dots & \alpha_{\sigma_{4'34}} \\ \alpha_{\sigma_{1'12}} & \alpha_{\sigma_{2'12}} & \dots & \alpha_{\sigma_{4'12}} & \alpha_{\sigma_{2'13}} & \alpha_{\sigma_{1'13}} & \dots & \alpha_{\sigma_{1'2'34}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{\sigma_{1'34}} & \alpha_{\sigma_{2'34}} & \dots & \alpha_{\sigma_{4'34}} & \alpha_{\sigma_{1'2'34}} & \alpha_{\sigma_{1'2'13}} & \dots & \alpha_{\sigma_{1'34'34}} \end{vmatrix}$$

and  $M$  denote the minor of the corresponding  $\alpha$  in  $\Delta$ , we have by a well known property of the determinant,

$$\alpha_{\sigma_1\sigma_1}M_{(r_{12}, \sigma_1)} + \alpha_{\sigma_4\sigma_2}M_{(r_{12}, \sigma_2)} + \alpha_{\sigma_4\sigma_3}M_{(r_{12}, \sigma_3)} + \dots + \alpha_{\sigma_{1'34}}M_{(r_{12}, r_{34})} = 0.$$

Divide by  $\Delta$ ,

$$\alpha_{\sigma_4\sigma_1} \frac{M_{(r_{12}, \sigma_1)}}{\Delta} + \alpha_{\sigma_4\sigma_2} \frac{M_{(r_{12}, \sigma_2)}}{\Delta} + \alpha_{\sigma_4\sigma_3} \frac{M_{(r_{12}, \sigma_3)}}{\Delta} + \dots + \alpha_{\sigma_{4'34}} \frac{M_{(r_{12}, r_{34})}}{\Delta} = 0.$$

Now  $\frac{M_{(r_{12}, \sigma_1)}}{\Delta \Sigma_{r_{12}} \Sigma_{\sigma_1}}$ ,  $\frac{M_{(r_{12}, \sigma_2)}}{\Delta \Sigma_{r_{12}} \Sigma_{\sigma_2}}$ ,  $\frac{M_{(r_{12}, \sigma_3)}}{\Delta \Sigma_{r_{12}} \Sigma_{\sigma_3}}$ ,  $\dots$ ,  $\frac{M_{(r_{12}, r_{34})}}{\Delta \Sigma_{r_{12}} \Sigma_{r_{34}}}$ , are the correlations between the errors made in the various quantities  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{34}$ , every one of which is known by the previous investigations except that of  $r_{12}$  and  $r_{34}$  or  $M_{(r_{12}, r_{34})}/\Delta \Sigma_{r_{12}} \Sigma_{r_{34}}$ . Hence the above equation will suffice to find the latter quantity, since  $\Sigma_{r_{12}}$  and  $\Sigma_{r_{34}}$  are known. We have

$$\left. \begin{aligned} a_{\sigma_4 \sigma_1} &= \frac{n}{\sigma_1 \sigma_1} \frac{R_{14} r_{14}}{R}, & a_{\sigma_4 \sigma_2} &= \frac{n}{\sigma_1 \sigma_2} \frac{R_{24} r_{24}}{R}, \\ a_{\sigma_4 \sigma_3} &= \frac{n}{\sigma_1 \sigma_3} \frac{R_{34} r_{34}}{R}, & a_{\sigma_4 \sigma_4} &= \frac{n}{\sigma_1^2} \left(1 + \frac{R_{14}}{R}\right), \end{aligned} \right\} \text{by (xxiv.) and (xxv.),}$$

$$a_{\sigma_4 r_{12}} = 0, \quad a_{\sigma_4 r_{13}} = 0, \quad a_{\sigma_4 r_{23}} = 0, \quad \text{by (xxvii.),}$$

$$a_{\sigma_4 r_{14}} = \frac{n}{\sigma_1} \frac{R_{14}}{R}, \quad a_{\sigma_4 r_{24}} = \frac{n}{\sigma_1} \frac{R_{24}}{R}, \quad a_{\sigma_4 r_{34}} = \frac{n}{\sigma_1} \frac{R_{34}}{R}, \quad \text{by (xxvi.).}$$

Further,

$$\frac{M_{(r_{12}, \sigma_1)}}{\Delta} = R_{r_{12} \sigma_1} \Sigma_{r_{12}} \Sigma_{\sigma_1} = \frac{\sigma_1}{2n} r_{12} (1 - r_{12}^2),$$

$$\frac{M_{(r_{12}, \sigma_2)}}{\Delta} = \frac{\sigma_2}{2n} r_{12} (1 - r_{12}^2), \quad \text{by (xvi.) and (xviii.),}$$

$$\left. \begin{aligned} \frac{M_{(r_{12}, \sigma_3)}}{\Delta} &= \frac{\sigma_3}{2n} \{r_{23} (r_{13} - r_{12} r_{23}) + r_{13} (r_{23} - r_{12} r_{13})\} \\ \frac{M_{(r_{12}, \sigma_4)}}{\Delta} &= \frac{\sigma_4}{2n} \{r_{24} (r_{14} - r_{24} r_{12}) + r_{14} (r_{24} - r_{12} r_{14})\}, \end{aligned} \right\} \text{by (xxxvi.),}$$

$$\left. \begin{aligned} \frac{M_{(r_{12}, r_{14})}}{\Delta} &= \frac{1}{n} \{r_{24} (1 - r_{12}^2) (1 - r_{14}^2) - \frac{1}{2} r_{12} r_{14} R_{33}\}, \\ \frac{M_{(r_{12}, r_{24})}}{\Delta} &= \frac{1}{n} \{r_{14} (1 - r_{12}^2) (1 - r_{24}^2) - \frac{1}{2} r_{12} r_{24} R_{33}\}, \end{aligned} \right\} \text{by (xxxvii.),}$$

$$\frac{M_{(r_{12}, r_{34})}}{\Delta} = \frac{1}{n} R_{r_{12} r_{34}} (1 - r_{12}^2) (1 - r_{34}^2), \quad \text{by (xvi.).}$$

Now substitute and divide out by common factors, and we find

$$\begin{aligned} & (r_{14} R_{14} + r_{24} R_{24}) r_{12} (1 - r_{12}^2) + r_{34} R_{34} \{r_{23} (r_{13} - r_{12} r_{23}) + r_{13} (r_{23} - r_{12} r_{13})\} \\ & + (R + R_{44}) \{r_{24} (r_{14} - r_{24} r_{12}) + r_{14} (r_{24} - r_{12} r_{14})\} \\ & + 2R_{14} \{r_{24} (1 - r_{12}^2) (1 - r_{14}^2) - \frac{1}{2} r_{12} r_{14} R_{33}\} \\ & + 2R_{24} \{r_{14} (1 - r_{12}^2) (1 - r_{24}^2) - \frac{1}{2} r_{12} r_{24} R_{33}\} \\ & + 2R_{34} R_{r_{12} r_{34}} (1 - r_{12}^2) (1 - r_{34}^2) = 0 \quad \dots \dots \dots \text{(xxxix.).} \end{aligned}$$



Now

$$\begin{aligned} R &= R_{44} + r_{14}R_{14} + r_{24}R_{24} + r_{34}R_{34}, \\ 0 &= r_{14}R_{14} + R_{14} + r_{12}R_{24} + r_{31}R_{34}, \\ 0 &= r_{24}R_{44} + r_{12}R_{14} + R_{24} + r_{32}R_{34}. \end{aligned}$$

Multiply the third of these by  $r_{14}$  and the second by  $r_{24}$  and add, we have

$$0 = 2r_{14}r_{24}R_{44} + r_{24}R_{14} + r_{14}R_{24} + r_{12}(r_{14}R_{14} + r_{24}R_{24}) + (r_{13}r_{24} + r_{23}r_{14})R_{34}.$$

Hence, by the first,

$$r_{24}R_{14} + r_{14}R_{24} = R_{44}(r_{12} - 2r_{14}r_{24}) - Rr_{12} + R_{34}(r_{12}r_{34} - r_{13}r_{24} - r_{23}r_{14}),$$

while

$$r_{14}R_{14} + r_{24}R_{24} = R - R_{44} - r_{34}R_{34}.$$

By means of these relations, let us get rid of the terms in  $R_{14}$  and  $R_{24}$  in (xxxix.) above.

Re-arranging we have, after some reductions,

$$\begin{aligned} &- 2r_{12}(1 - r_{12}^2)R + 2r_{12}R_{33}R_{44} + R_{34}\{r_{34}r_{23}(r_{13} - r_{12}r_{23}) + r_{34}r_{13}(r_{23} - r_{12}r_{13}) \\ &+ 2(1 - r_{12}^2)(r_{12}r_{34} - r_{13}r_{24} - r_{23}r_{14}) + 2r_{14}r_{24}r_{34} - r_{12}r_{34}(r_{24}^2 + r_{14}^2)\} \\ &+ 2R_{34}R_{12'34}(1 - r_{12}^2)(1 - r_{34}^2) = 0. \end{aligned}$$

But

$$(1 - r_{12}^2)R = R_{33}R_{44} - R_{34}^2.$$

Hence we can divide out by  $R_{34}$ , and accordingly,

$$\begin{aligned} &2R_{12'34}(1 - r_{12}^2)(1 - r_{34}^2) \\ &= -\{r_{34}r_{23}(r_{13} - r_{12}r_{23}) + r_{34}r_{13}(r_{23} - r_{12}r_{13}) + 2(1 - r_{12}^2)(r_{12}r_{34} - r_{13}r_{24} - r_{23}r_{14}) \\ &+ 2r_{14}r_{24}r_{34} - r_{12}r_{34}(r_{24}^2 + r_{14}^2) + 2r_{12}R_{34}\}. \end{aligned}$$

Noting that

$$R_{34} = -r_{34}(1 - r_{12}^2) + r_{31}r_{41} + r_{32}r_{42} - r_{12}(r_{31}r_{42} + r_{32}r_{41}),$$

we have, after substitution and rearranging,

$$\begin{aligned} &2R_{12'34}(1 - r_{12}^2)(1 - r_{34}^2) = (r_{13} - r_{12}r_{23})(r_{24} - r_{23}r_{34}) + (r_{14} - r_{34}r_{13})(r_{23} - r_{12}r_{13}) \\ &+ (r_{13} - r_{14}r_{34})(r_{24} - r_{12}r_{14}) + (r_{14} - r_{12}r_{24})(r_{23} - r_{24}r_{34}). \end{aligned}$$

Or,

$$R_{12'34} = \frac{\left\{ \begin{aligned} &(r_{13} - r_{12}r_{23})(r_{24} - r_{23}r_{34}) + (r_{14} - r_{34}r_{13})(r_{23} - r_{12}r_{13}) \\ &+ (r_{13} - r_{14}r_{34})(r_{24} - r_{12}r_{14}) + (r_{14} - r_{12}r_{24})(r_{23} - r_{24}r_{34}) \end{aligned} \right\}}{2(1 - r_{12}^2)(1 - r_{34}^2)} \quad \text{. . . (xl.)}$$

If we put  $4 = 1$  in this result and remember that  $r_{11} = 1$ , we find, after some reductions,

$$R_{r_{12}r_{31}} = r_{23} - \frac{1}{2} r_{12} r_{13} \frac{1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{23}r_{31}}{(1 - r_{12}^2)(1 - r_{13}^2)},$$

which agrees with (xxxvii.), and may be taken as a verification of this result.\*

(12.) We may draw several conclusions from the results (xxxvi.), (xxxvii.), and (xl.).

( $\alpha$ .) While errors in the correlations of a first organ with a second and a third have a correlation themselves of the first order, errors in the variation of a first organ and the correlation of two others, or in the correlation of two organs and in the correlation of a second two, have only correlation of the second order. Thus a selection of the correlation between two organs modifies the variation of all organs correlated with one or other or both of the first, but only in the second degree. Again, a selection of the correlation between two organs modifies the correlation of every other pair of organs, one or both of which are correlated with one or both of the first pair; but this is only in the second degree.

( $\beta$ .) If two organs be entirely uncorrelated a random selection of the variation of a third organ correlated with both of them will tend to generate correlation between the hitherto uncorrelated organs, *i.e.*, put  $r_{23} = 0$  in (xxxvi.), and we have

$$R_{\sigma_1 r_{23}} = \sqrt{2} \cdot r_{12} r_{13}.$$

If a variation  $\Delta\sigma_1$  be made in  $\sigma_1$  the probable value of  $r_{23}$  is

$$\Delta r_{23} = R_{\sigma_1 r_{23}} \frac{\sum_{r_{23}}}{\sum_{\sigma_1}} \Delta\sigma_1 = 2r_{12}r_{13} \frac{\Delta\sigma_1}{\sigma_1},$$

which may clearly be of sensible magnitude. Thus correlation may be generated by selection of variation, and *vice versa*.

( $\gamma$ .) If two organs be each entirely uncorrelated with a third, yet a random selection, which produces a correlation between one of these organs and the third, will produce a correlation of the first order between the other of these organs and the third, *i.e.*, put  $r_{12} = r_{13} = 0$  in (xxxvii.), we have

$$R_{r_{12}r_{13}} = r_{23},$$

a correlation of the first order between the probable changes.

( $\delta$ .) Consider four organs of which the first is alone correlated with the third and the second with the fourth, the third and fourth being themselves uncorrelated. Then any random selection which produces a correlation between the first and

\* The probable error of a partial regression coefficient for  $p$  organs has not been worked owing to the labour involved, but judging by the cases on pp. 245 and 260, it may safely be taken as  $67.449 \sqrt{R/\sqrt{n}} (-R_{12})$ , where  $R$  is now the determinant of the  $p^{\text{th}}$  degree.

second will tend to produce a correlation between the third and fourth, *i.e.*, if  $r_{12} = r_{14} = r_{23} = r_{34} = 0$ , we still have from (xl.)

$$R_{r_{12}r_{34}} = r_{13}r_{24}.$$

(ε.) We may further illustrate these principles by one or two hypothetical examples drawn from actual organs.

Let the actual organs be (1) physique of father, (2) artistic sense of mother, (3) physique of offspring, (4) artistic sense of offspring. Suppose in the general population there is no correlation between physique of father and artistic sense of mother, or between physique or artistic sense of parent, and artistic sense and physique respectively of offspring. Then  $r_{12} = 0$ ,  $r_{14} = 0$ ,  $r_{23} = 0$ , and, presumably,  $r_{34} = 0$ . Hence

$$R_{r_{12}r_{34}} = r_{13}r_{24}$$

is the product of the two coefficients for inheritance of physique from father to child, and for inheritance of artistic sense from mother to child.

Now let a random selection be made out of the general population in which assortative mating between physique in the male and artistic sense in the female presents itself, *i.e.*, let  $\Delta r_{12}$  be sensible; then we have, most probably,

$$\Delta r_{34} = R_{r_{12}r_{34}} \frac{\sum_{r_{34}} \Delta r_{12}}{\sum_{r_{12}}} = r_{13}r_{24} \Delta r_{12},$$

or, a correlation between physique and artistic sense in the offspring will tend to be developed. Generally, when  $r_{34}$  and  $r_{12}$  do not start from zero, we have,

$$\Delta r_{34} = \frac{r_{13}r_{24} + \frac{1}{2}r_{12}r_{34}(r_{13}^2 + r_{24}^2)}{(1 - r_{12}^2)^2} \Delta r_{12},$$

or, any increase of sexual selection in a group tends to emphasise the correlation of the selected qualities in the offspring.

Let the three characteristics be artistic sense (1) in a man, (2) in his mother, (3) in his wife. Then

$$R_{r_{13}r_{12}} = -\frac{1}{2}r_{12}r_{13} \frac{1 - r_{12}^2 - r_{13}^2}{(1 - r_{12}^2)(1 - r_{13}^2)},$$

if we suppose  $r_{23}$  to be zero.

Hence any selected group with a higher coefficient of maternal inheritance of heredity will have a less coefficient of sexual selection than the general population, and *vice versa*. The *tendency* is, of course, independent of the magnitude of  $r$ , and really of the particular character. Supposing likeness of faculty or character to be a rough measure of "sympathy," we might conclude for any population with inheritance and sexual selection, that on the average a selected sub-group of men having greater sympathy with mothers than the general population will have less sympathy with wives, and *vice versa*.



(13.) Many like propositions may be stated with regard to the action of selection on the correlation of characters. They require but little modification to state them for artificial or natural selection, as they are here stated for what we have termed random selection. The above will, however, suffice to indicate how every form of selection of variability or correlation influences in a manner capable of quantitative expression the variability and correlation of all other directly and indirectly correlated organs. Selection cannot be of service in altering *one* organ only, it alters at the same time the whole inter-relationship of a complex of organs. Evolution by natural selection can never be the change of one organ to suit a particular environment; it is the balance of advantage and disadvantage produced by the change of all organs involved in the attempt to select one of them. The moment the intimate correlation of organs in animal or plant life has been fully realised—and this realisation owing to recent statistical investigations has become fairly easy—then the conception of natural selection as moulding any single organ to what may be fittest to its surroundings must be discarded. The selection of the “fittest” in one organ would probably mean the selection of the unfit in other organs, and a general balance of fitness in the complex of organ is all that is possible.\*

#### IV. ON THE PROBABLE ERRORS AND THE COEFFICIENTS OF CORRELATION BETWEEN ERRORS MADE IN THE DETERMINATION OF THE CONSTANTS IN THE CASE OF SKEW VARIATION.

(14.) The case of Skew Variation has been dealt with at length by one of the present authors in the second paper of this series. He has shown that in a great variety of cases it can be dealt with by a series of curves having three principal algebraical types, each defined by a certain number of constants. The probable errors of the determination of these constants were not then investigated, but it is clearly of great importance for the practical use of these curves to know how far these constants can, for any given number of observations, be depended upon to give an accurate measure of the skewness and its special features. At the same time an investigation of the probable errors of these constants leads us to a number of novel properties which are connected with the theory of evolution in the frequent case of skew variation.

\* Take, for example, result (xxxvi.); as far as terms of the second order are concerned  $R_{\sigma_1'23} = \sqrt{2} \cdot r_{12}r_{13}$ . Hence, with positive correlation between three organs, the effect of trying to get a group very stable in one organ, *i.e.*, with a negative  $\Delta\sigma_1$ , is to reduce the correlation between every other pair of organs! In other words, we have to reduce variation at the expense of correlation, increased stability of one organ is gained at the expense of decreased stability in the inter-relationship of other organs. This may possibly be illustrated by the long bones of the French, where the lesser variability of the male relative to the female connotes also a lesser correlation. See LEE and PEARSON: “On the Relative Variation and Correlation in Civilised and Uncivilised Races,” ‘R. S. Proc.’ vol. 61, pp. 354–356.

We shall deal first with the skew curve of Type III. ('Phil. Trans.' A, vol. 186, p. 373), because its treatment is less complex and leads at once to some general principles which must be borne in mind, whenever natural selection acts upon an organ exhibiting skew variation.

(15.) *Probable Errors and Error Correlations of the Constants of the Generalized Probability Curve of Type*  $y = y_1 \left(1 + \frac{\gamma x}{p+1}\right)^p e^{-\gamma x}$ .

This is the equation of the curve referred to its mean as origin, where

$$y_1 = n \frac{\gamma e^{-(p+1)} (p+1)^p}{\Gamma(p+1)} \quad \dots \quad (\text{xli}).$$

Further, the moments about the centroid vertical are given by,

$$\mu_2 = \frac{p+1}{\gamma^2}, \quad \mu_3 = \frac{2(p+1)}{\gamma^3}, \quad \mu_4 = \frac{3(p+1)(p+3)}{\gamma^4} \quad \dots \quad (\text{xlii}).$$

or,

$$\gamma = 2\mu_2/\mu_3, \quad p = 4\mu_2^2/\mu_3^2 - 1 \quad \dots \quad (\text{xliii}).$$

The criterion for the application of this curve to any frequency distribution is

$$2\mu_2(3\mu_2^2 - \mu_4) + 3\mu_3^2 = 0,$$

or, if we write  $\beta_2 = \mu_4/\mu_2^2$ ,  $\beta_1 = \mu_3^2/\mu_2^3$ ,

$$6 - 2\beta_2 + 3\beta_1 = 0 \quad \dots \quad (\text{xliv}).$$

Lastly,

$$Sk. = \text{the skewness} = \frac{1}{2}\mu_3/(\mu_2)^{3/2} = \frac{1}{\sqrt{(p+1)}} \quad \dots \quad (\text{xlv}).$$

and the modal frequency,

$$y_0 = \frac{n\gamma p^p}{e^p \Gamma(p+1)} \quad \dots \quad (\text{xlvi}).^*$$

We require to know the probable errors of  $p$ ,  $\gamma$ ,  $y_1$ ,  $y_0$ ,  $\sigma = \sqrt{\mu_2}$ ,  $\mu_2$ ,  $\mu_3$ ,  $\mu_4$ , and the skewness. We must discover the best physical constants to describe such skew frequencies and we shall at the same time succeed in deducing certain—we believe—novel properties of normal frequency distributions as limiting cases of this skew type of distribution.

\* See 'Phil. Trans.,' A, vol. 186, pp. 373-4.

(16.) The first stage in the investigation is to apply the general proposition of our Art. 2, to

$$\log y = \log n + \log \gamma - (p+1) + p \log (p+1) - \log \Gamma(p+1) \\ + p \log \left(1 + \frac{\gamma^x}{p+1}\right) - \gamma^x.$$

We find :

$$\frac{d^2(\log y)}{dx^2} = -\frac{p}{a^2} \frac{1}{(1+x/a)^2}, \text{ if } a = (p+1)/\gamma,$$

$$\frac{d^2(\log y)}{dx dp} = \frac{1}{a} \left( \frac{1}{1+x/a} - \frac{p}{p+1} \frac{1}{(1+x/a)^2} \right),$$

$$\frac{d^2(\log y)}{dx d\gamma} = \frac{p}{p+1} \frac{1}{(1+x/a)^2} - 1,$$

$$\frac{d^2(\log y)}{dp^2} = -\frac{d^2}{dp^2} \log \Gamma(p+1) + \frac{2}{p+1} \frac{1}{1+x/a} - \frac{p}{(p+1)^2} \frac{1}{(1+x/a)^2},$$

$$\frac{d^2(\log y)}{d\gamma^2} = -\frac{1}{\gamma^2} \left( p+1 - 2p \frac{1}{1+x/a} + \frac{p}{(1+x/a)^2} \right),$$

$$\frac{d^2(\log y)}{dp d\gamma} = \frac{1}{\gamma} \left( 1 - \frac{2p+1}{p+1} \frac{1}{1+x/a} + \frac{p}{p+1} \frac{1}{(1+x/a)^2} \right).$$

Let  $I_p = y_1 \int_{-a}^{\infty} \left(1 + \frac{x}{a}\right)^p e^{-\gamma^x} dx$ , then we easily find  $I_{p-1} = \frac{p+1}{p} I_p$ , and  $I_{p-2} = \frac{(p+1)^2}{p(p-1)} I_p$ .

By aid of these we can at once write down the integrals of the above expressions multiplied by  $y$ , since  $n = I_p$ . We find with the notation of p. 243,

$$a_{11} = -\int_{-a}^{\infty} y \frac{d^2(\log y)}{dx^2} dx = \frac{n\gamma^2}{p-1},$$

$$a_{12} = \int_{-a}^{\infty} y \frac{d^2(\log y)}{dx dp} dx = -\frac{n\gamma}{p(p-1)},$$

$$a_{13} = \int_{-a}^{\infty} y \frac{d^2(\log y)}{dx d\gamma} dx = \frac{2n}{p-1},$$

$$a_{22} = -\int_{-a}^{\infty} y \frac{d^2(\log y)}{dp^2} dx = n \left( \frac{d^2}{dp^2} \log \Gamma(p+1) - \frac{p-2}{p(p-1)} \right),$$

$$a_{33} = -\int_{-a}^{\infty} y \frac{d^2(\log y)}{d\gamma^2} dx = \frac{2n(p+1)}{\gamma^2(p-1)},$$

$$a_{23} = \int_{-a}^{\infty} y \frac{d^2(\log y)}{dp d\gamma} dx = \frac{n(p+1)}{\gamma p(p-1)}.$$



Before we consider the determinant and its minors, we may note that

$$\log \Gamma(p+1) = \log \sqrt{(2\pi)} + (p + \frac{1}{2}) \log p - p + \frac{B_1}{1.2.p} - \frac{B_3}{3.4.p^3} + \frac{B_5}{5.6.p^5} - \dots$$

where the B's are the BERNOULLI numbers. Hence

$$\frac{d^2}{dp^2} \{ \log \Gamma(p+1) \} = \frac{2p-1}{2p^2} + \frac{B_1}{p^3} - \frac{B_3}{p^5} + \frac{B_5}{p^7} - ,$$

and we have the convenient form,

$$a_{22} = n \left( \frac{d^2}{dp^2} \log \Gamma(p+1) - \frac{p-2}{p(p-1)} \right) = \frac{n}{p^2} \left\{ \frac{1}{2} \frac{p+1}{p-1} + S \right\},$$

where S is the semi-convergent series  $B_1/p - B_3/p^3 + B_5/p^5 - , \&c.$

Now we have at once,

$$\Delta = \begin{vmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{12} & a_{22} & -a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{vmatrix} = n^3 \begin{vmatrix} \gamma & \frac{\gamma}{p(p-1)} & -\frac{2}{p-1} \\ \frac{\gamma}{p(p-1)} & \frac{1}{p^2} \left( \frac{1}{2} \frac{p+1}{p-1} + S \right) & -\frac{p+1}{\gamma(p-1)p} \\ -\frac{2}{p-1} & -\frac{p+1}{\gamma(p-1)p} & \frac{2(p+1)}{\gamma^2(p-1)} \end{vmatrix}$$

Divide all three columns by  $1/(p-1)$ ; the first row and first column by  $\gamma$ , and then the last row and last column by  $1/\gamma$ ; we find,

$$\Delta = \frac{n^3}{(p-1)^3} \begin{vmatrix} 1 & \frac{1}{p} & -2 \\ \frac{1}{p} & \frac{p-1}{p^2} \left( \frac{1}{2} \frac{p+1}{p-1} + S \right) & -\frac{p+1}{p} \\ -2 & -\frac{p+1}{p} & 2(p+1) \end{vmatrix}$$

Divide the second row and column by  $1/p$ ; add half the last row to the second, and we find,

$$\Delta = \frac{n^3}{p^2(p-1)^3} \begin{vmatrix} 1 & 1 & -2 \\ 0 & (p-1)S & 0 \\ -2 & -(p+1) & 2(p+1) \end{vmatrix}.$$



Lastly,

$$M_{13} = \frac{2n^2S}{p^2(p-1)},$$

whence

$$R_{h\gamma} = M_{13}/(\Delta\Sigma_h\Sigma_\gamma) = \sqrt{\left\{\frac{2}{p+1}S/(\frac{1}{2}+S)\right\}} \quad \dots \quad (\text{liii}).$$

This completes the direct series of probable errors and error correlations. By aid of the above correlations and standard deviations we can now find a further series.

From (xlvi.) we have for the standard deviation  $\sigma$  (about the mean),  $\sigma^2 = \mu_2 = \frac{p+1}{\gamma^2}$ , or  $\sigma = \sqrt{\frac{p+1}{\gamma}}$ . Hence

$$\frac{\Delta\sigma}{\sigma} = \frac{1}{2} \frac{\Delta p}{p+1} - \frac{\Delta\gamma}{\gamma} \quad \dots \quad (\text{liv}).$$

Square both sides of this, divide by  $n$  and sum, we have at once from the definition of a coefficient of correlation

$$\left(\frac{\Sigma_\sigma}{\sigma}\right)^2 = \frac{1}{2} \frac{\Sigma_p^2}{(p+1)^2} + \frac{\Sigma_\gamma^2}{\gamma^2} - \frac{R_{p\gamma}\Sigma_p\Sigma_\gamma}{\gamma(p+1)}.$$

Hence, using (xlix.), (l.), and (li.), we find, after reductions,

$$\Sigma_\sigma = \frac{\sigma}{\sqrt{(2n)}} \left(1 + \frac{1}{2} \frac{1}{(p+1)^2S}\right) \quad \dots \quad (\text{lv}).$$

Multiply (liv.) by  $\Delta h$ , sum and divide by  $n$ , we have

$$\frac{\Sigma_\sigma\Sigma_h R_{h\sigma}}{\sigma} = \frac{1}{2} \frac{\Sigma_p\Sigma_h R_{hp}}{p+1} - \frac{\Sigma_\gamma\Sigma_h R_{h\gamma}}{\gamma}.$$

Whence, by (lii.),

$$R_{h\sigma} = - \frac{\sigma}{\gamma} \frac{\Sigma_\gamma}{\Sigma_\sigma} R_{h\gamma},$$

or, reducing by (l.), (liii.), and (lv.),

$$R_{h\sigma} = - \sqrt{\left(\frac{2}{p+1}\right)} \frac{1}{\sqrt{\left(1 + \frac{1}{2} \frac{1}{(p+1)^2S}\right)}} \quad \dots \quad (\text{lvi}).$$

Next, if  $S_k$  be the skewness, we have from (xlv.)

$$\Delta S_k = - \frac{1}{2} \frac{\Delta p}{(p+1)^{3/2}},$$



or

$$\Sigma_{S_k} = \frac{1}{2} \frac{p}{p+1} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{\{(p+1)S\}}} \quad \dots \quad \text{(lvii).}^*$$

Similarly

$$R_{S_k h} = \frac{S(\Delta S_k \Delta_h)}{\Sigma_{S_k} \Sigma_h} = 0, \text{ since } R_{h p} = 0,$$

or

$$R_{h S_k} = 0 \quad \dots \quad \text{(lviii).}$$

We easily obtain, by multiplying (liv.) by  $\Delta p$ , the result

$$R_{p\sigma} = - \frac{1}{\{1 + 2(p+1)^2 S\}} \quad \dots \quad \text{(lix).}$$

We can now obtain  $R_{\sigma S_k}$ , for every  $\Delta_{S_k}$  is negatively proportional to the corresponding  $\Delta p$ . Hence

$$R_{\sigma S_k} = \frac{1}{\{1 + 2(p+1)^2 S\}} \quad \dots \quad \text{(lx).}$$

We next pass to the mean and modal frequencies as given by (xli.) and (xlvi.). We have, by taking logarithmic differentials,

$$\frac{\Delta y_1}{y_1} = \frac{\Delta \gamma}{\gamma} - J \Delta p,$$

where

$$J = \frac{1}{p+1} + \frac{d}{dp} \{ \log \Gamma(p+1) \} - \log(p+1).$$

\* If  $S' = \frac{B_1}{p+1} - \frac{B_3}{(p+1)^3} + \frac{B_5}{(p+1)^5} - \&c.$ , then it is easy to show that  $\frac{1}{2} + p^2 S' = (p+1)^2 S$ .

Remembering that, if  $S_k = q$ ,  $q = \frac{1}{\sqrt{(p+1)}}$ , we easily deduce

$$\begin{aligned} \Sigma_{S_k} &= \frac{1}{2\sqrt{n}} \frac{1}{\left\{ B_1 - B_3 q^4 + B_5 q^8 + \frac{1}{2} \frac{q^2}{(1-q^2)^2} \right\}^{\frac{1}{2}}} \\ &= \frac{1}{2\sqrt{n}} \frac{1}{\left\{ B_1 + \frac{1}{2} q^2 + (1-B_3) q^4 + \frac{3}{2} q^6 + (B_5+2) q^8 + \dots \right\}^{\frac{1}{2}}} \\ &= \sqrt{\left( \frac{3}{2n} \right)} \frac{1}{\sqrt{\left\{ 1 + 3(Sk.)^2 + \frac{29}{5} (Sk.)^4 + 9(Sk.)^6 \right\}}}, \end{aligned}$$

as far the 7<sup>th</sup> power of the skewness inclusive.

Very generally the probable error of the skewness may be taken as equal to

$$\sqrt{\left( \frac{3}{2n} \right)} \frac{1}{\sqrt{\{1 + 3(Sk.)^2\}}},$$

and it is always less than  $\sqrt{(3/2n)}$ , its value in the case of a normal frequency.

Squaring, introducing the standard deviations, and rearranging, we find

$$\frac{\Sigma_{y_1}^2}{y_1^2} = \frac{\Sigma_{\gamma}^2}{\gamma^2} (1 - R_{\gamma p}^2) + \left( J - \frac{\Sigma_{\gamma} R_{\gamma p}}{\Sigma_p \gamma} \right)^2 \Sigma_p^2 = \frac{1}{2n} \left\{ 1 + \frac{2}{S} (Jp - \frac{1}{2})^2 \right\}.$$

We must now evaluate  $Jp - \frac{1}{2}$ . This is easily shown from the BERNOULLI number expansion for  $\log \Gamma(p + 1)$  to be given by

$$Jp - \frac{1}{2} = \frac{p}{p+1} - p \log \frac{p+1}{p} - T,$$

where

$$T = \frac{B_1}{2p} - \frac{B_3}{4p^3} + \frac{B_5}{6p^5} - \dots$$

Thus we determine

$$\Sigma_{y_1} = \frac{y_1}{\sqrt{(2n)}} \left\{ 1 + \frac{2}{S} \left( p \log \frac{p+1}{p} - \frac{p}{p+1} + T \right)^2 \right\}^{\frac{1}{2}} \dots \dots (lxi.).$$

Expanding the expression in brackets in inverse powers of  $p$  we find

$$\Sigma_{y_1} = \frac{y_1}{\sqrt{(2n)}} \left\{ 1 + \frac{49}{12p} - \frac{28}{3p^2} + \frac{248}{15p^3} - \dots \right\}^{\frac{1}{2}} \dots \dots (lxii.).$$

Result (lxi.), however, with  $S$  and  $T$  calculated to  $1/p^3$ , gives a better value than (lxii.).

To find the modal frequency error we must take the logarithmic differential of (xlvi.) and proceed in the same way. We find almost at once

$$\frac{\Delta y_0}{y_0} = \frac{\Delta \gamma}{\gamma} - \frac{\Delta p}{p} \left( \frac{1}{2} - T \right).$$

Whence on squaring and completing the square of the factor of  $\Sigma p^2$ , we find

$$\frac{\Sigma_{y_0}^2}{y_0^2} = \frac{1}{2n} \left\{ 1 + \frac{2T^2}{S} \right\},$$

and

$$\Sigma_{y_0} = \frac{y_0}{\sqrt{(2n)}} \left\{ 1 + \frac{2T^2}{S} \right\}^{\frac{1}{2}} \dots \dots (lxiii.).$$

Expanding as far as powers of  $1/p^5$  *exclusive* we obtain

$$\Sigma_{y_0} = \frac{y_0}{\sqrt{(2n)}} \left\{ 1 + \frac{1}{12p} \right\}^{\frac{1}{2}} \dots \dots (lxiv.),$$

a very simple expression for the probable error of the modal frequency,  $y_0$ .

In like manner we shall now determine the probable errors of the moments and their error correlations.

Take the logarithmic differentials of (lxii.)

$$\frac{\Delta\mu_2}{\mu_2} = \frac{\Delta p}{p+1} - \frac{2\Delta\gamma}{\gamma},$$

$$\frac{\Delta\mu_3}{\mu_3} = \frac{\Delta p}{p+1} - \frac{3\Delta\gamma}{\gamma},$$

$$\frac{\Delta\mu_4}{\mu_4} = \frac{2(p+2)\Delta p}{(p+1)(p+3)} - \frac{4\Delta\gamma}{\gamma}.$$

Squaring each of these in succession and using the known values of  $\Sigma$ ,  $\Sigma$ ,  $R_{\mu\gamma}$  we find

$$\Sigma_{\mu_2} = \mu_2 \frac{2}{\sqrt{(2n)}} \sqrt{\left\{1 + \frac{1}{2S(p+1)^2}\right\}} \quad \dots \quad (lxv.),$$

$$\Sigma_{\mu_3} = \mu_3 \frac{3}{\sqrt{(2n)}} \sqrt{\left\{1 + \frac{(p+3)^2}{18S(p+1)^2}\right\}} \quad \dots \quad (lxvi.),$$

$$\Sigma_{\mu_4} = \mu_4 \frac{4}{\sqrt{(2n)}} \sqrt{\left\{1 + \frac{(2p+3)^2}{2S(p+1)^2(p+3)^2}\right\}} \quad \dots \quad (lxvii.),$$

Now multiply  $\Delta\mu_2/\mu_2$  by  $\Delta\mu_3/\mu_3$  and we find, after some reductions,

$$R_{\mu_2\mu_3} = \frac{1 + \frac{p+3}{6S(p+1)^2}}{\sqrt{\left\{\left(1 + \frac{1}{2S(p+1)^2}\right)\left(1 + \frac{(p+3)^2}{18S(p+1)^2}\right)\right\}}} \quad \dots \quad (lxviii.).$$

Next multiply  $\Delta\mu_2/\mu_2$  by  $\Delta\mu_4/\mu_4$ , and we ultimately have

$$R_{\mu_2\mu_4} = \frac{1 + \frac{2p+3}{2S(p+1)^2(p+3)}}{\sqrt{\left\{\left(1 + \frac{1}{2S(p+1)^2}\right)\left(1 + \frac{(2p+3)^2}{2S(p+1)^2(p+3)^2}\right)\right\}}} \quad \dots \quad (lxix.).$$

Lastly, multiplying  $\Delta\mu_3/\mu_3$  by  $\Delta\mu_4/\mu_4$ , we deduce, after some reductions,

$$R_{\mu_3\mu_4} = \frac{1 + \frac{2p+3}{6S(p+1)^2}}{\sqrt{\left\{\left(1 + \frac{(p+3)^2}{18S(p+1)^2}\right)\left(1 + \frac{(2p+3)^2}{2S(p+1)^2(p+3)^2}\right)\right\}}} \quad \dots \quad (lxx.).$$



We may add to these results the values of  $\Sigma_{\beta_1}$  and  $\Sigma_{\beta_2}$ , where  $\beta_1$  and  $\beta_2$  are given by (xliv.); we find

$$\Sigma_{\beta_1} = \frac{4}{\sqrt{n}} \frac{p}{(p+1)^2 \sqrt{S}}, \quad \Sigma_{\beta_2} = \frac{6}{\sqrt{n}} \frac{p}{(p+1)^2 \sqrt{S}}. \quad \dots \quad (\text{lxxi.})$$

The distances from the mode to the mean,  $d$ , and from the mean to the end of the range,  $\alpha$ , are given by

$$d = 1/\gamma \quad \text{and} \quad \alpha = \frac{p+1}{\gamma}.$$

Hence

$$\Sigma_{\alpha'} = \frac{d}{\sqrt{(2n)}} \sqrt{\left(1 + \frac{1}{2S}\right)} \dots \dots \dots (\text{lxxii.}),$$

$$\Sigma_{\alpha''} = \frac{\alpha}{\sqrt{(2n)}} \sqrt{\left\{1 + \frac{(p-1)^2}{2S(p+1)^2}\right\}} \dots \dots \dots (\text{lxxiii.}),$$

and further

$$\begin{aligned} R_{\alpha'} &= - \left\{ \sqrt{\left(\frac{2}{\frac{p+1}{2} + S}\right)} \right\} \dots \dots \dots (\text{lxxiv.}), \\ R_{\alpha''} &= - \left\{ \sqrt{\left(\frac{2}{\frac{1}{2} \left(\frac{p-1}{p+1}\right)^2 + S}\right)} \right\} \dots \dots \dots \end{aligned}$$

The results (xlvii.) to (lxxiv.) must be now considered at length.

(17.) ( $\alpha$ .) The frequency curve of the type considered is fully described by the three constants, the mean,  $\gamma$ , and  $p$ . But, since *any* three constants would do equally well—for example, what may be termed the three physical constants: mean, standard deviation (or variation), and skewness—it becomes of some importance to inquire which constants have the least percentage of probable error.

Now (xlviii.) shows us that the probable error in the mean is precisely the same as in the case of the normal curve and

$$= .67449\sigma/\sqrt{n}.$$

Thus, the percentage error in the mean

$$\begin{aligned} &= .67449 \frac{100\sigma}{h} \frac{1}{\sqrt{n}} \\ &= \frac{.67449}{\sqrt{n}} \times \text{coefficient of variation,} \end{aligned}$$

and will certainly be small whenever the coefficient of variation is small. Its value is quite independent of the order of  $p$ .

On the other hand, the percentage probable errors of  $p$  and  $\gamma$  are from (xlix.) and (l.)

$$\frac{67.449}{\sqrt{n}} \frac{1}{\sqrt{S}} \text{ and } \frac{67.449}{\sqrt{(2n)}} \sqrt{\left(1 + \frac{1}{2S}\right)} \text{ respectively.}$$

Here  $S$  is equal to the series  $B_1/p - B_3/p^3 + B_5/p^5 - \dots$  which tends to zero as  $p$  increases.

The errors in  $p$  and  $\gamma$  thus tend to increase indefinitely as  $p$  increases. It may then be asked how the form of the curve can be determined with any degree of accuracy. The answer is simple: Equation (li.) shows us that the correlation between errors in  $p$  and  $\gamma$  tends, as  $p$  increases, to become "perfect," i.e., unity. But as  $p$  increases indefinitely, it has been shown that the frequency curve of this type passes over into the normal form.\* It is the high correlation between errors in  $p$  and  $\gamma$  which renders the curve, when plotted to observations, such an excellent fit. If the errors in  $p$  and  $\gamma$  were independent, this would not be so. At the same time it renders  $p$  and  $\gamma$  unsuitable for tabulation as physical or biological constants of the frequency.

Turning to (lv.) and (lvii.) we see that the standard deviation,  $\sigma$ , and the skewness,  $Sk.$ , are suitable constants for tabulation. Their probable errors do not tend to increase indefinitely with  $p$ , and will always be small, if  $n$  be large.

Hence a frequency distribution of this type is best defined by its mean  $h$ , its standard-deviation  $\sigma$ , and its skewness  $Sk.$  These are constants characteristic of the group, for they are given with small probable errors. If it be desired to draw the form of the frequency-curve, then its algebraic constants,  $p$  and  $\gamma$ , may be found from

$$p = \frac{1}{(Sk.)^2} - 1, \quad \gamma = \frac{1}{\sigma \times Sk.},$$

and the possibly considerable errors in  $p$  and  $\gamma$  will not vary largely its actual shape.

( $\beta$ .) The nature of the probable errors of the other allied constants may now be considered. The mean and modal frequencies per unit variation of organ, or  $y_1$  and  $y_0$  are seen by (lxii.) and (lxiv.) to have small percentage probable errors, and are, therefore, good for use as characteristic physical or biological constants. But it should be noted that the modal frequency is considerably more exact for moderate values of  $p$  than the mean frequency. For example it would be somewhat better to tabulate the modal than the mean frequency of the barometer as a physical characteristic of climate.

The probable errors of the distances from the mean to the mode and from the mean to the terminal of the range are given by (lxxii.) and (lxxiii.). Since  $d = 1/\gamma = \sigma/\sqrt{1+p}$ , we may write the first

$$\Sigma_d = \frac{\sigma}{\sqrt{(2n)}} \sqrt{\left(\frac{1}{1+p} + \frac{1}{2(1+p)S}\right)}.$$

\* 'Phil. Trans.,' A, vol. 186, p. 374.

This remains finite, even if  $p$  be indefinitely great. On the other hand, the probable error of  $a$ , and even its percentage probable error, becomes indefinitely great with  $p$ . It is to be noted that  $a$  in this case becomes infinite.

( $\gamma$ .) Results (lxv.) to (lxvii.) give the probable errors of the second, third, and fourth moments. It will be seen that roughly, for a large  $p$ , the percentage error of the fourth moment is about double that of the second. It might thus appear, at first sight, safer to work with the second than with the fourth, but this is by no means necessarily the case, for to deduce any quantity from one or the other they *must be reduced to the same order*. For example, the square root of  $\mu_2$  must be compared with the fourth root of  $\mu_4$ , and the probable errors of  $\sqrt{\mu_2}$  and  $(\mu_4)^{1/4}$  will be sensibly of the same order.

Remembering that  $\mu_3 = \frac{2(p+1)}{\gamma^3} = \frac{2\sigma^3}{\sqrt{(p+1)}}$ , we may write

$$\Sigma_{\mu_3} = \frac{6\sigma^3}{\sqrt{(2n)}} \sqrt{\left( \frac{1}{p+1} + \frac{(p+3)^2}{18S(p+1)^3} \right)}.$$

This tends to a finite limit as  $p$  increases indefinitely, and we conclude that the probable error of  $\mu_3$  is always finite, and will in general be a small fraction of the cube of the standard deviation. The above remarks are a justification for the use of higher moments in frequency calculations.

Equations (lxviii.) to (lxx.) give the error-correlations between the first three moments. They show that an error in the value of one of these moments will most probably lead to an error in the other two. We see that for  $p$  fairly large  $R_{\mu_2\mu_4}$  is a large correlation, while  $R_{\mu_2\mu_3}$  and  $R_{\mu_3\mu_4}$  are small. In other words a random selection of an even moment makes a far larger correlated change in another even moment than in an odd moment. If  $p$  increase indefinitely we find the ratio  $R_{\mu_3\mu_3}/R_{\mu_3\mu_4}$  approaches the value  $2/3$ ; in other words,  $\mu_3$  is more closely correlated to the higher moment  $\mu_4$  than to the lower moment  $\mu_2$ .

Formulae (lxxi.) give the probable errors of the useful constants  $\beta_1 = \mu_3^2/\mu_2^3$  and  $\beta_2 = \mu_4/\mu_2^2$ . We see that they are small and approach the value zero as  $p$  is indefinitely increased.

( $\delta$ .) Let us restate the formulae for  $p$  indefinitely great, *i.e.*, for the normal curve of frequency

$$y = \frac{n}{2\pi\sqrt{\sigma}} e^{-x^2/(2\sigma^2)}$$

In this case we have  $\mu_2 = \sigma^2$ ,  $\mu_3 = 0$ ,  $\mu_4 = 3\sigma^4$ ,  $\beta_1 = 0$ ,  $\beta_2 = 3$ , skewness = 0, mean and mode coincide. Several of these zero quantities, however, tend to have definite probable errors.

We have



$$\begin{aligned}
 \Sigma_{\rho} &= \sigma/\sqrt{n}, & \Sigma_{\mu_4} &= \frac{12\sigma^4}{\sqrt{(2n)}}, \\
 \Sigma_{\sigma} &= \sigma/\sqrt{(2n)}, & R_{\mu_2\mu_3} &= 0, \\
 \Sigma_{sk} &= \sqrt{\left(\frac{3}{2n}\right)}, & R_{\mu_2\mu_4} &= 1, \\
 \Sigma_{\mu_1} &= \frac{\eta_1}{\sqrt{(2n)}}, & R_{\mu_3\mu_4} &= 0, \\
 \Sigma_{\mu_3} &= \frac{2\sigma^2}{\sqrt{(2n)}}, & \Sigma_{,l} &= \sqrt{\left(\frac{3}{2n}\right)} \sigma = \sigma \Sigma_{,k}, \\
 \Sigma_{\mu_3} &= 2 \sqrt{\sigma^3} = 2\sigma^3 \Sigma_{,l},
 \end{aligned}$$

The first, second, fourth and fifth of these results are old; the rest appear to be novel and of some importance.

In the first place we notice that given a population which is really normal, we should not expect a random selection to exhibit all the signs of normality. Its skewness will differ from zero with a probable error of  $\cdot 67449 \sqrt{\left(\frac{3}{2n}\right)}$ . For example, in a random selection of 600 from a normal population, the skewness will be as likely to exceed as to fall short of  $\cdot 034$ . Hence an exhibition of skewness of less than once to twice  $\cdot 67449 \sqrt{\left(\frac{3}{2n}\right)}$  must not in itself merely be taken to indicate an absence of normality in a general population.

Again, in a random selection from a general population, the mode will differ from the mean, even if the population be normal, with a probable error of  $\cdot 67449 \sqrt{\left(\frac{3}{2n}\right)} \sigma$ . Thus, in a population of 600, a difference between the mean and the mode of  $\cdot 034\sigma$  should not be taken to indicate want of normality. Generally, the divergence between mean and mode in a population must at least exceed once to twice  $\cdot 67449 \sqrt{\left(\frac{3}{2n}\right)} \sigma$ , for us to be able to argue on this ground alone that the population has not a normal distribution.

Again, the third moment not being zero, but having a value of once or twice  $\cdot 67449 \times 2 \sqrt{\left(\frac{3}{2n}\right)} \sigma^3$ , is not in itself an argument for skew frequency.

The above statements are an important addition to the second memoir of this series; they give us the criterion, there wanting, to distinguish between a skewness which is characteristic of a population and one which might arise by the random selection of a population of the given size out of a larger, but really normal, population.

( $\epsilon$ .) We may now note the exceedingly interesting conclusions which these results have for the theory of evolution.

Suppose an organ to have, as so many do, skew variation, then we notice

(i.) Any selection of the organ by size tends to alter its variability, but not its skewness; this follows from (lvi.) and (lviii.). Further, if, as we have supposed, the range be limited on the side of dwarf organs, then any increase of size means a decrease of variability, and *vice versa*.

(ii.) Any selection of variability is a selection of skewness; this follows from (lx.). If a selection be made from a general population, which has less variability, then it will tend to greater normality. In other words, it would appear that stringent selection tends to generate normal distribution. Thus, if out of a skewly distributed population we make a number of random selections, that with the least variability will be most normal. Select at random again out of this latter selection, and the least variable group will again be the most normal, and so on.

Now take a problem of this kind involving *group*, and not *individual*, selection. Let a large general population break itself up at random into groups, and let us suppose these groups, not individuals among them, to carry on a struggle for existence—an inter-group, not an intra-group, struggle. Then, if it be an advantage to a group that its members shall be among themselves close to a type, *i.e.*, less variable, then the more normal groups will survive, for variability is positively correlated with skewness. Now suppose each group to be periodically subdivided at random into new groups—the mathematical description of some process of group reproduction—then we see how normal distribution may be a result of a stringent inter-group selection of groups whose individuals have the closest resemblance to each other—intra-group resemblance.

(iii.) Any selection of the size of an organ produces by (lxxiv.) an alteration in the distances between the mean and the mode, and between the mean and the end of the range.

A random selection which has its mean larger than that of the general population, will, if the mode be on the dwarf side of the mean, tend to have its mode and mean nearer together than are the mode and mean of the general population, while on the other hand, to raise the mean is to raise the dwarf limit to the range.

A considerable number of like results might be stated, but the above will be sufficient to emphasize the general principle that a random and *à fortiori* an artificial selection of the size of an organ, does, whenever its distribution is skew, influence in a definite manner the variability of the organ. It is quite safe to assert that it will also influence the correlation of organs. When we notice how wide-spread is skew variation in nature, we may assert that the general rule is that no modification can be made in any of the features—mean sizes, variabilities and correlations—of a group of organs without at the same time modifying all the others.\*

\* A paper has recently been published by Messrs. DAVENPORT and BULLARD in the 'Proceedings of the American Academy of Science' (see Illustration II. below) on "The Variation and Correlation of the Glands in the Legs of Swine." Unfortunately the authors have overlooked the markedly skew character

As a result of Articles (15) and (16), it is possible to use the frequency curve of type  $y = y_1 (1 + x/a)^p e^{-\gamma x}$  with as much certainty as to the nature and magnitude of the errors made in the constants as has hitherto been possible in the case of the normal distribution  $y = \frac{n}{2\pi\sqrt{\sigma}} e^{-x^2/(2\sigma^2)}$ . The method has been exemplified numerically in twenty-three cases in a memoir on the "Variation of Barometric Frequency" (see 'Phil. Trans.,' A, vol. 190, p. 423). It may not, however, be amiss to illustrate it further in a special case having closer bearings on the theory of evolution.

(18.) *Numerical Illustration—Incidence of Enteric Fever.*

In a memoir in the 'Phil. Trans.,' A, vol. 186, p. 391, it is shown that the curve

$$y = 1894.57 \left(1 + \frac{x}{3.428094}\right)^{3.673,042} e^{-1.071453x}$$

closely represents the distribution with age of 8,689 cases of enteric fever received into the Metropolitan Asylums Board Fever Hospitals. The unit of  $x$  is five years, and the origin is the mode at 14.3025 years. The criterion is not very nearly zero, although small, but the curve is graphically a good fit (see Plate 12, fig. 9).

The following are the numerical values of all the constants :—

Mean = 18.9691 years.  $d$  = mean-mode.

Mode = 14.3025 years.  $= .933,313$  unit.

Sk = skewness = .462,594.

$\gamma = 1.071,453$ .  $\rho = 3.673,042$ .

$a = 3.428,094$ .

$y_0$  = modal frequency = 1894.57.

$y_1$  = mean frequency = 1687.80.

$\sigma$  = standard deviation = 2.01756 units = 10.0878 years.

From these the numerical values of the probable errors and of the correlations between the errors of the constants were found by the processes indicated and the formulæ given above.

We found

$$T = .022,525, \quad S = .044,735,$$

whence

of the distribution. It is, however, clear from their tables and plate that no selection could be made of the absolute number of glands without altering the variability of the gland distribution and the correlation between different systems of glands.



	Probable Error.	Percentage Probable Error.
$p$	·125659	3·4211
$\gamma$	·019130	1·7854
Correlation of errors in $p$ and $\gamma$	=	·9581
$\alpha$	·061202	1·7853
$y_0$	9·8029	·5174
$y_1$	23·5465	1·3951
mean	= ·014600 =	·073 year.
mode	= ·024126 =	·121 year.

These are the constants which determine the position and algebraical equation to the frequency curve, and we see at once that they are all determined with a close degree of accuracy. The largest percentage probable error is in  $p$ , but this is under 3·5 per cent., and, owing to the high correlation between  $p$  and  $\gamma$  a much larger error would produce no sensible change in the shape of the curve.

Two important facts may also be drawn from these results, which indeed follow from the general formulæ, namely :

(i.) The position of the mean is sensibly more exactly determined than the position of the mode. Here about 1·7 times as accurately.

(ii.) The modal frequency, on the other hand, is sensibly more accurate than the mean frequency. Here about 2·8 times as accurate.

Hence the advantage of using the mean as origin of measurement for the curve is accompanied by the counterbalancing, and here relatively greater, disadvantage of the increased inaccuracy of determination of the mean frequency.

Passing to the "physical" constants of the curve, we have

	Probable Error.	Percentage Probable Error.
$\sigma$	·012693	·6291
Sk.	·022845	1·3445
$d$	·016663	1·7854

These fully determine the non-symmetrical nature and spread of the curve, and since the errors in the skewness and in the distance between the mean and mode are less than 1·4 and 1·8 per cent. of the respective values of these quantities, we conclude that skewness and divergence between mode and mean are characteristic features of enteric fever distribution, and not mere anomalies due to a random selection of cases. They are significant constants peculiar to each type of fever distribution and no description of such a distribution is sufficient unless their values are stated.

Before giving a table of the correlations between what we have termed the "physical" constants, it may be well to write down some of the correlations between the errors in the physical and algebraical constants, which arise in the course of their calculation. We find

$$\begin{aligned} R_{py} &= .9581, & R_{pm} &= 0, \\ R_{my} &= .1875, & R_{dy} &= -1. \end{aligned}$$

By aid of these we find the following table of error correlations:—

	Mean.	$y_0$ .	$\sigma$ .	$d$ .	Sk.
Mean.	1	.6469	-.5321	-.1875	0
$y_0$ .	.6469	1	-.8908	-.4260	-.1489
$\sigma$ .	-.5321	-.8908	1	.7905	.1584
$d$ .	-.1875	-.4260	.7905	1	.9592
Sk.	0	-.1489	.1584	.9592	1

Now this table enables us to draw some remarkable conclusions with regard to enteric fever. We see at once that no random selection of a group of individuals, which has any single characteristic differing from that of the general population will, except in the case of mean age of incidence and skewness, leave the other characteristics unmodified. Thus the most probable result of any selection which alters the nature of the distribution of enteric fever can be predicted. The reader will possibly appreciate this better, if we replace the above table by another giving the absolute progressions in years, number of cases per thousand, &c.

PROGRESSION TABLE.

Corresponds to a probable change in the same units of	Unit change of				
	One year in mean age of incidence.	One case per cent. in modal year of frequency.*	One year in standard deviation or "spread."	One year in number of years between modal and mean incidence.	A unit of 1/10 in the skewness.
Mean age of incidence	1	.0909	-.6120	-.1643	0
Modal frequency . . .	4.5852	1	-7.2626	-2.6456	-.3372
"Spread" . . . . .	-.4626	-.1093	1	.6022	.0440
Interval between mode and mean	-.2140	-.0686	1.0377	1	.3498
Skewness . . . . .	0	-.0657	.5702	2.6301	1

\* The frequency of incidence in the modal year  $= y_0 \times \frac{1}{5}$  since the unit is five years  $= 1894.57 \times \frac{1}{5}$ . To make this 1000 we must multiply by  $\frac{1000 \times 5}{1894.57}$ . Similarly  $\Delta y_0 \times \frac{1}{5}$  = error in incidence of modal year. Thus we have to replace  $\Delta y_0$  by  $\frac{1894.57}{1000} \left( \frac{\frac{1}{5} \Delta y_0}{\frac{1}{5} y_0} 1000 \right) = \frac{1894.57}{1000}$  (error per thousand in modal year of incidence).

We see at once from the above table that if the mean age of incidence of enteric fever in any group were raised, the disease would be concentrated in fewer years, the modal and mean incidence would be brought closer together, and the incidence in the modal year of frequency would be heavier. The changes here are very sensible. Thus, if we raised the mean age of attack to that of phthisis, or about nine years, the modal frequency would be increased about 41 per cent., the concentration of the incidence of the fever increased about 40 per cent., while the distance between mode and mean would be reduced to nearly  $\frac{2}{5}$  of its original value. The skewness would not be changed. Much less marked effects would arise from a selection of modal frequency. Any increase of modal frequency tends to slightly raise the mean age of attack, to increase slightly the concentration, to draw the mode towards the mean and reduce the skewness.

The changes produced by closer concentration of the attacks of the disease, *i.e.*, the limitation of its incidence to fewer years, would be of a more marked character, they would raise the mean age of attack and the modal frequency, they would decrease the interval between mode and mean, and reduce the skewness. Concentration of the disease would thus tend to render its distribution more normal.

To increase the interval between mean and mode lowers the mean age of attack, reduces the modal frequency, increases the period of liability to incidence, and much increases the skewness.

Finally, increase of skewness decreases the modal frequency, increases the period of liability and the interval between mean and mode.

These statements with regard to the manner in which enteric fever would affect different groups selected at random from the general population seem of considerable interest, for there is reason to believe that what is thus stated for enteric fever in different groups may be applied to different fevers in one and the same group. For example, the lower the mean age of attack of any fever, the greater its concentration; the less the concentration, the more nearly normal is its distribution, &c., &c.

(19.) *Probable Errors and Error Correlations of the Constants of the Generalised Probability Curve of the Type*  $y = y_1 \left(1 + \frac{x}{a_1}\right)^{m_1} \left(1 - \frac{x}{a_2}\right)^{m_2}$ .

Transfer the origin to one end of the range, and the equation to the curve becomes

$$y = \frac{n}{b} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1) \Gamma(m_2 + 1)} \left(\frac{x}{b}\right)^{m_1} \left(1 - \frac{x}{b}\right)^{m_2} \dots \dots \dots (1xxv.),$$

where  $n$  is the number of observations and  $b$  is the range.

The following values are given in 'Phil. Trans.,' A, vol. 186, pp. 368-9, where  $r = m_1 + m_2 + 2$ :



$$\text{Mean } \bar{x} = \frac{b(m_1 + 1)}{m_1 + m_2 + 2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxvii}),$$

Mode  $x_1 = \frac{m_1 b}{m_1 + m_2} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$  (lxxvii.),

$$d = \bar{x} - x_1 = \frac{b(m_1 - m_2)}{(m_1 + m_2)(m_1 + m_2 + 2)} \cdot \cdot \cdot \cdot \cdot (lxxviii.),$$

$$\sigma = \frac{b}{r} \sqrt{\left\{ \frac{(m_1 + 1)(m_2 + 1)}{r + 1} \right\}} \quad . \quad . \quad . \quad . \quad (\text{lxix}),$$

$$\text{Skewness} = S_k = \frac{m_1 - m_2}{m_1 + m_2} \sqrt{\left\{ \frac{m_1 + m_2 + 3}{(m_1 + 1)(m_2 + 1)} \right\}} \dots \dots \dots (\text{xxx}).$$

For the moments we have

$$\mu_2 = \frac{b^2 (m_1 + 1) (m_2 + 1)}{r^2 (r + 1)} . . . . . \quad (\text{lxvii}).$$

$$\mu_3 = \frac{2b^3(m_1+1)(m_2+1)(m_2-m_1)}{r(r+1)(r+2)} \dots \dots \dots (1xxxii).$$

$$\mu_4 = \frac{3b^4(m_1+1)(m_2+1)\{(m_1+1)(m_2+1)(r-6)+2r^2\}}{r^4(r+1)(r+2)(r+3)}, \quad (\text{xxxxiii}).$$

Lastly, for the mean and modal frequencies of  $y_1 \delta x$  and  $y_2 \delta x$ , we have

$$y_1 = \frac{n}{b} \frac{(m_1+m_2+2)^{3/2}}{\sqrt{\{2\pi(m_1+1)(m_2+1)\}}} \exp. \{ \bar{S}(m_1+m_2+2) - \bar{S}(m_1+1) - \bar{S}(m_2+1) \} \quad (\text{xxiv}),$$

$$y_2 = \frac{n}{b} \frac{(m_1 + m_2 + 1) \sqrt{(m_1 + m_2)}}{\sqrt{(2 \pi m_1 m_2)}} \text{expt.} \{ \bar{S}(m_1 + m_2) - \bar{S}(m_1) - \bar{S}(m_2) \} \quad (1 \text{xxxv}).$$

where

$$\bar{S}(p) = \frac{B_1}{1.2p} - \frac{B_3}{3.4p^3} + \frac{B_5}{5.6.p^5} - \text{etc.} \quad . \quad . \quad . \quad . (lxxxvi).$$

Let

$$I(m_1, m_2) = y_0 \int_0^b \left(\frac{a}{b}\right)^{m_1} \left(1 - \frac{a}{b}\right)^{m_2} dx$$

where

$$y_0 = \frac{a}{b} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1) \Gamma(m_2 + 1)},$$

then  $I(m_1, m_2) = n$ , and we easily find, by the fundamental property of  $\Gamma$  functions, that

$$\left. \begin{aligned} I(m_1 - 1, m_2) &= \frac{m_1 + m_2 + 1}{m_1} n, \\ I(m_1, m_2 - 1) &= \frac{m_1 + m_2 + 1}{m_2} n, \\ I(m_1 - 2, m_2) &= \frac{(m_1 + m_2)(m_1 + m_2 + 1)}{m_1(m_1 - 1)} n, \\ I(m_1, m_2 - 2) &= \frac{(m_1 + m_2)(m_1 + m_2 + 1)}{m_2(m_2 - 1)} n, \end{aligned} \right\} \dots \dots (lxxxvii.).$$

From (i.) we have

$$\log y = \log n - \log b + \log \chi + m_1 \log \frac{a}{b} + m_2 \log \left(1 - \frac{a}{b}\right) (lxxxviii.),$$

where

$$\chi = \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)}.$$

It will be needful to find  $\frac{d^2(\log \chi)}{dm_1^2}$ ,  $\frac{d^2(\log \chi)}{dm_2^2}$ , and  $\frac{d^2(\log \chi)}{dm_1 dm_2}$ .

$$\begin{aligned} \frac{d^2(\log \chi)}{dm_1^2} &= \frac{d^2 \Gamma(m_1 + m_2 + 2)}{dm_1^2} - \frac{d^2 \Gamma(m_1 + 1)}{dm_1^2} \\ &= \frac{d^2 \Gamma(m_1 + m_2 + 2)}{d(m_1 + m_2 + 1)^2} - \frac{d^2 \Gamma(m_1 + 1)}{dm_1^2} = \epsilon_3 - \epsilon_1 \dots \dots (lxxxix.). \end{aligned}$$

Similarly

$$d^2(\log \chi)/dm_2^2 = \epsilon_3 - \epsilon_2 \dots \dots (xc.),$$

$$d^2(\log \chi)/dm_1 dm_2 = \epsilon_3 \dots \dots (xci.),$$

where

$$\left. \begin{aligned} \epsilon_1 &= d^2 \Gamma(m_1 + 1)/dm_1^2, & \epsilon_2 &= d^2 \Gamma(m_2 + 1)/dm_2^2, \\ \epsilon_3 &= d^2 \Gamma(m_1 + m_2 + 2)/d(m_1 + m_2 + 1)^2 \end{aligned} \right\} \dots \dots (xcii.).$$

$\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_3$  can now be readily expressed in semi-convergent series admitting of calculation.

$$\left. \begin{aligned} \epsilon_1 &= \frac{2m_1 - 1}{2m_1^2} + \frac{S(m_1)}{m_1^2} \\ \epsilon_2 &= \frac{2m_2 - 1}{2m_2^2} + \frac{S(m_2)}{m_2^2} \\ \epsilon_3 &= \frac{2(m_1 + m_2 + 1) - 1}{2(m_1 + m_2 + 1)^2} + \frac{S(m_1 + m_2 + 1)}{(m_1 + m_2 + 1)^2} \end{aligned} \right\} \dots \dots (xciii.),$$

where

$$\left. \begin{aligned} S(p) &= B_1/p - B_3/p^3 + B_5/p^5 - \dots \\ \frac{S(p)}{p^2} &= \frac{d^2}{dp^2} \{ \bar{S}(p) \} \end{aligned} \right\} \dots \dots (xciv.).$$

and

It is clear that if  $m_1$  and  $m_2$  be at all large, which they frequently will be, we may omit the series S, or even reduce  $\epsilon_1, \epsilon_2, \epsilon_3$  to  $1/m_1, 1/m_2$ , and  $1/(m_1 + m_2 + 1)$ , respectively.

Making use of (lxxxvii.) and (lxxxviii.) we easily find

$$a_{11} = nb_{11} = -\int_0^b y \frac{d^2(\log y)}{dx^2} dx = \frac{n}{b^2} (m_1 + m_2)(m_1 + m_2 + 1) \left( \frac{1}{m_1 - 1} + \frac{1}{m_2 - 1} \right) \quad (\text{xcv}).$$

$$a_{12} = nb_{12} = -\int_0^b y \frac{d^2(\log y)}{dx db} dx = -\frac{n}{b^2} \frac{(m_1 + m_2)(m_1 + m_2 + 1)}{m_2 - 1} \quad \dots \quad (\text{xcvi}).$$

$$a_{13} = nb_{13} = -\int_0^b y \frac{d^2(\log y)}{dx dm_1} dx = -\frac{n}{b} \frac{m_1 + m_2 + 1}{m_1} \quad \dots \quad (\text{xcvii}).$$

$$a_{14} = nb_{14} = -\int_0^b y \frac{d^2(\log y)}{dx dm_2} dx = \frac{n}{b} \frac{(m_1 + m_2 + 1)}{m_2} \quad \dots \quad (\text{xcviii}).$$

$$a_{22} = nb_{22} = -\int_0^b y \frac{d^2(\log y)}{db^2} dx = \frac{n}{b^2} (m_1 + m_2 + 1) \frac{m_1 + 1}{m_2 - 1} \quad \dots \quad (\text{xcix}).$$

$$a_{23} = nb_{23} = -\int_0^b y \frac{d^2(\log y)}{db dm_1} dx = \frac{n}{b} \quad \dots \quad (\text{c}).$$

$$a_{24} = nb_{24} = -\int_0^b y \frac{d^2(\log y)}{db dm_2} dx = -\frac{n}{b} \frac{m_1 + 1}{m_2} \quad \dots \quad (\text{ci}).$$

$$a_{33} = nb_{33} = -\int_0^b y \frac{d^2(\log y)}{dm_1^2} dx = n(\epsilon_1 - \epsilon_3) \quad \dots \quad (\text{cii}).$$

$$a_{34} = nb_{34} = -\int_0^b y \frac{d^2(\log y)}{dm_1 dm_2} dx = -n\epsilon_3 \quad \dots \quad (\text{ciii}).$$

$$a_{44} = nb_{44} = -\int_0^b y \frac{d^2(\log y)}{dm_2^2} dx = n(\epsilon_2 - \epsilon_3) \quad \dots \quad (\text{civ}).$$

The next stage is to calculate the determinant

$$\Delta' = \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{vmatrix}$$

and the minors  $B_{ss'}$ , &c., of  $b_{ss'}$ , &c. We shall then determine

$$\Sigma_b^2 = \frac{1}{n} \frac{B_{11}}{\Delta'}, \quad \Sigma_b^2 = \frac{1}{n} \frac{B_{22}}{\Delta'}, \quad \Sigma_{m_1}^2 = \frac{1}{n} \frac{B_{33}}{\Delta'}, \quad \Sigma_{m_2}^2 = \frac{1}{n} \frac{B_{44}}{\Delta'} \quad \dots \quad (\text{cv}).$$



and the correlations

$$\left. \begin{aligned} R_{hb} &= \frac{B_{12}}{\sqrt{(B_{11}B_{22})}}, & R_{hm_1} &= \frac{B_{13}}{\sqrt{(B_{11}B_{33})}}, & R_{hm_2} &= \frac{B_{14}}{\sqrt{(B_{11}B_{44})}} \\ R_{bm_1} &= \frac{B_{23}}{\sqrt{(B_{22}B_{33})}}, & R_{bm_2} &= \frac{B_{24}}{\sqrt{(B_{22}B_{44})}}, & R_{m_1m_2} &= \frac{B_{34}}{\sqrt{(B_{33}B_{44})}} \end{aligned} \right\} \dots \dots \text{(cvi.).}$$

The algebraic expressions for the expanded determinant and its minors are very lengthy, and it will be found easiest in any numerical case to calculate the numerical values of the  $b_{11}$ ,  $b_{12}$ ,  $b_{13}$ ,  $b_{14}$ ,  $b_{22}$  . . . , and then find the values of the determinant and its minors numerically. So soon as the above four standard deviations and six error correlations have been calculated, the determination of the probable errors in the fundamental constants of the frequency distribution becomes easy.

We have for the mean organ  $M$ , if  $h$  now defines the origin of coordinates,

$$\begin{aligned} \Delta M &= \Delta h + \Delta x \\ &= \Delta h + \frac{m_1 + 1}{m_1 + m_2 + 2} \Delta b + \frac{b(m_2 + 1) \Delta m_1}{(m_1 + m_2 + 2)^2} - \frac{b(m_1 + 1) \Delta m_2}{(m_1 + m_2 + 2)^2} \dots \dots \text{(cvii.).} \end{aligned}$$

Whence

$$\begin{aligned} \Sigma M^2 &= \Sigma h^2 + \left( \frac{m_1 + 1}{r} \right)^2 \Sigma b^2 + \frac{b^2(m_2 + 1)^2}{r^4} \Sigma m_1^2 + \frac{b^2(m_1 + 1)^2}{r^4} \Sigma m_2^2 \\ &\quad + 2 \frac{m_1 + 1}{r} \Sigma h \Sigma b R_{hb} + \frac{2b(m_2 + 1)}{r^2} \Sigma h \Sigma m_1 R_{hm_1} - \frac{2b(m_1 + 1)}{r^2} \Sigma h \Sigma m_2 R_{hm_2} \\ &\quad + \frac{2b(m_1 + 1)(m_2 + 1)}{r^3} \Sigma b \Sigma m_1 R_{bm_1} - \frac{2b(m_1 + 1)}{r^3} \Sigma b \Sigma m_2 R_{bm_2} \\ &\quad - \frac{2b^2(m_1 + 1)(m_2 + 1)}{r^4} \Sigma m_1 \Sigma m_2 R_{m_1m_2} \dots \dots \dots \text{(cviii.).} \end{aligned}$$

Similarly, the modal value of the organ can be found from\*

$$\Delta M_0 = \Delta h + \frac{m_1}{m_1 + m_2} \Delta b + \frac{bm_2}{(m_1 + m_2)^2} \Delta m_1 - \frac{bm_1}{(m_1 + m_2)^2} \Delta m_2 \dots \text{(cix.).}$$

\* The easiest numerical method in this, as in the previous case, is to proceed as follows :--Write

$$\Sigma M_0 X_{M_0} = \Sigma h X_h + \frac{m_1}{m_1 + m_2} \Sigma b X_b + \frac{bm_2}{(m_1 + m_2)^2} \Sigma m_1 X_{m_1} - \frac{bm_1}{(m_1 + m_2)^2} \Sigma m_2 X_{m_2},$$

where the  $X$ 's are umbral symbols, and let  $N_1$ ,  $N_2$ ,  $N_3$ ,  $N_4$  be the numerical values of the coefficients on the right. Then put

$$\Sigma M_0 X_{M_0} = N_1 X_h + N_2 X_b + N_3 X_{m_1} - N_4 X_{m_2}.$$

Now square this equation, and, whenever a product,  $\chi_q \chi_{q'}$ , occurs, multiply it by the corresponding error correlation,  $R_{qq'}$ , already calculated, putting  $R_{qq'} = 1$  if  $q = q'$ . Then, actually,

$$\begin{aligned} \Sigma M_0^2 &= N_1^2 + N_2^2 + N_3^2 + N_4^2 + 2N_1N_2R_{hb} + 2N_1N_3R_{hm_1} - 2N_1N_4R_{hm_2} + 2N_2N_3R_{bm_1} \\ &\quad - 2N_2N_4R_{bm_2} - 2N_3N_4R_{m_1m_2}. \end{aligned}$$

BARLOW'S Tables rapidly give the squares and CRELLE'S Tables, or a Brunsviga, the products.

In like manner from (lxxviii.)

$$\begin{aligned}\frac{\Delta d}{d} &= \frac{\Delta b}{b} + \Delta m_1 \left\{ \frac{1}{m_1 - m_2} - \frac{1}{m_1 + m_2} - \frac{1}{m_1 + m_2 + 2} \right\} \\ &\quad + \Delta m_2 \left\{ -\frac{1}{m_1 - m_2} - \frac{1}{m_1 + m_2} - \frac{1}{m_1 + m_2 + 2} \right\} \\ &= \Delta b/b + c_1 \Delta m_1 + c_2 \Delta m_2,\end{aligned}$$

say, where the numerical values of  $c_1$  and  $c_2$  can easily be found in any actual case. Hence

$$\Sigma_d^2/d^2 = \Sigma_b^2/b^2 + c_1^2 \Sigma_{m_1}^2 + c_2^2 \Sigma_{m_2}^2 + 2c_1 c_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} + \frac{2c_1}{b} \Sigma_b \Sigma_{m_1} R_{bm_1} + \frac{2c_2}{b} \Sigma_b \Sigma_{m_2} R_{bm_2} \quad (\text{cx.}).$$

Again, from (lxxix.)

$$\frac{\Delta \sigma}{\sigma} = \frac{\Delta b}{b} + \Delta m_1 \left( -\frac{1}{r} - \frac{1}{2(r+1)} + \frac{1}{2(m_1+1)} \right) + \Delta m_2 \left( -\frac{1}{r} - \frac{1}{2(r+1)} + \frac{1}{2(m_2+1)} \right).$$

Or  $\Delta \sigma/\sigma = \Delta b/b + c_1 \Delta m_1 + c_2 \Delta m_2$ ,

$$\frac{\Sigma_\sigma^2}{\sigma^2} = \frac{\Sigma_b^2}{b^2} + c_1^2 \Sigma_{m_1}^2 + c_2^2 \Sigma_{m_2}^2 + 2c_1 c_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} + \frac{2c_1}{b} \Sigma_b \Sigma_{m_1} R_{bm_1} + \frac{2c_2}{b} \Sigma_b \Sigma_{m_2} R_{bm_2} \quad (\text{cxi.}).$$

Further from (lxxx.)

$$\begin{aligned}\frac{\Delta S_k}{S_k} &= \Delta m_1 \left\{ -\frac{1}{m_1 - m_2} - \frac{1}{m_1 + m_2} + \frac{1}{2(m_1 + m_2 + 3)} - \frac{1}{2(m_1 + 1)} \right\} \\ &\quad + \Delta m_2 \left\{ -\frac{1}{m_1 - m_2} - \frac{1}{m_1 + m_2} + \frac{1}{2(m_1 + m_2 + 3)} - \frac{1}{2(m_2 + 1)} \right\} \\ &= f_1 \Delta m_1 + f_2 \Delta m_2, \text{ say.}\end{aligned}$$

Hence

$$\Sigma_{S_k}^2 = S_k^2 \{ f_1^2 \Sigma_{m_1}^2 + f_2^2 \Sigma_{m_2}^2 + 2f_1 f_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} \} \quad \dots \quad (\text{cxii.}).$$

From the results given above we can deduce the effects on size, range, variability, or skewness of a selection at random of any one of these four.

Writing (cvii.)

$$\Delta M_c = \Delta h + g_1 \Delta b + g_2 \Delta m_1 - g_3 \Delta m_2$$

we find

$$\begin{aligned}R_{M, \sigma} &= \frac{\sigma}{\Sigma_{M, \sigma}} \left\{ \frac{\Sigma_b \Sigma_h R_{bh}}{b} + \frac{g_1}{b} \Sigma_b^2 + \frac{g_2}{b} \Sigma_b \Sigma_{m_1} R_{bm_1} - \frac{g_3}{b} \Sigma_b \Sigma_{m_2} R_{bm_2} + c_1 \Sigma_h \Sigma_{m_1} R_{hm_1} \right. \\ &\quad + c_1 g_1 \Sigma_{m_1} \Sigma_b R_{bm_1} + c_1 g_2 \Sigma_{m_1}^2 - c_1 g_3 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} + c_2 \Sigma_h \Sigma_{m_2} R_{hm_2} + c_2 g_1 \Sigma_b \Sigma_{m_2} R_{bm_2} \\ &\quad \left. + c_2 g_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} - c_2 g_3 \Sigma_{m_2}^2 \right\} \quad \dots \quad (\text{cxiii.}).\end{aligned}$$

$$R_{M,b} = \frac{1}{\Sigma_{M_r}} \left\{ \Sigma_h R_{hb} + g_1 \Sigma_b + g_2 \Sigma_{m_1} R_{m_1 b} - g_3 \Sigma_{m_2} R_{m_2 b} \right\} \quad \dots \quad (\text{cxiv}).$$

$$R_{M,s_k} = \frac{S_k}{\Sigma_{S_k} \Sigma_{M_r}} \left\{ f_1 \Sigma_h \Sigma_{m_1} R_{hm_1} + g_1 f_1 \Sigma_{m_1} \Sigma_b R_{bm_1} + f_1 g_2 \Sigma_{m_1}^2 - f_1 g_3 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} + f_2 \Sigma_h \Sigma_{m_2} R_{hm_2} \right. \\ \left. + f_2 g_1 \Sigma_b \Sigma_{m_2} R_{bm_2} + f_2 g_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} - f_2 g_3 \Sigma_{m_2}^2 \right\} \quad \dots \quad (\text{cxv}).$$

These results show us that it is, in the general case of skew variation, impossible to select any one of the quantities—mean size, range, variability, or skewness of an organ, without at the same time in all probability modifying all the others.

For example, the frequency of the incidence of certain types of diseases at different ages follows a distribution of this character. Hence, if any special class of the community had a mean age of incidence differing from that of the general population, we should expect correlated changes in such other characteristics of the disease as (i.) its first appearance; (ii.) its last appearance; (iii.) its tendency to heavier incidence above or below the mean age of incidence; (iv.) the concentration of its incidence about the mean age of incidence for this selected class.

Precisely similar series of changes would arise in the case of a random selection of individuals having the variability of a certain organ greater or less than that of the general population, there would be correlated changes in the size, range, and skewness of the distribution of this organ.

Turning to the mean and modal frequencies, we have

$$\frac{\Delta y_1}{y_1} = -\frac{\Delta b}{b} + \left\{ \frac{3}{2} \frac{1}{m_1 + m_2 + 2} - \frac{1}{2} \frac{1}{m_1 + 1} + \frac{d\bar{S}(m_1 + m_2 + 2)}{d(m_1 + m_2 + 2)} - \frac{d\bar{S}(m_1 + 1)}{d(m_1 + 1)} \right\} \Delta m_1 \\ + \left\{ \frac{3}{2} \frac{1}{m_1 + m_2 + 2} - \frac{1}{2} \frac{1}{m_2 + 1} + \frac{d\bar{S}(m_1 + m_2 + 2)}{d(m_1 + m_2 + 2)} - \frac{d\bar{S}(m_2 + 1)}{d(m_2 + 1)} \right\} \Delta m_2 \\ = -\Delta b/b + h_1 \Delta m_1 + h_2 \Delta m_2, \text{ say} \quad \dots \quad (\text{cxvi}).$$

Similarly,

$$\frac{\Delta y_2}{y_2} = -\frac{\Delta b}{b} + \left\{ \frac{1}{m_1 + m_2 + 1} + \frac{1}{2(m_1 + m_2)} - \frac{1}{2m_1} + \frac{d\bar{S}(m_1 + m_2)}{d(m_1 + m_2)} - \frac{d\bar{S}(m_1)}{dm_1} \right\} \Delta m_1 \\ + \left\{ \frac{1}{m_1 + m_2 + 1} + \frac{1}{2(m_1 + m_2)} - \frac{1}{2m_2} + \frac{d\bar{S}(m_1 + m_2)}{d(m_1 + m_2)} - \frac{d\bar{S}(m_2)}{dm_2} \right\} \Delta m_2 \\ = -\Delta b/b + k_1 \Delta m_1 + k_2 \Delta m_2, \text{ say} \quad \dots \quad (\text{cxvii}).$$

Here  $h_1$ ,  $h_2$ ,  $k_1$ ,  $k_2$  can be easily calculated, if we note that

$$d\bar{S}(p)/dp = -B_1/2p^2 + B_3/4p^4 - B_5/6p^6 + \dots \quad (\text{cxviii}). \\ = -T/p, \text{ where } T \text{ is the same as on p. 272.}$$



$\Sigma_{y_1}$  and  $\Sigma_{y_2}$  can be found in the usual manner by squaring after the insertion of the numerical values.

From (lxxxi.)–(lxxiii.) the probable errors and correlations of the moments can be found if required, the calculation being numerically somewhat laborious, but presenting nothing of novelty.

The probable error of the criterion

$$\kappa = 3\beta_1 - 2\beta_2 + 6,$$

where  $\beta_1 = \mu_3^2/\mu_2^3$  and  $\beta_2 = \mu_4/\mu_2^2$ , may be found as follows: Put  $\epsilon = (m_1 + 1)(m_2 + 1)$ ,  $r = m_1 + m_2 + 2$ ; then we find

$$\kappa = \frac{12r^2(r+1+\epsilon)}{(r+2)^2(r+3)\epsilon} \dots \dots \dots \text{(cxix.)}$$

and accordingly

$$\begin{aligned} \frac{\Delta\kappa}{\kappa} &= \left( \frac{2}{r} - \frac{2}{r+2} - \frac{1}{r+3} + \frac{1}{r+1+\epsilon} \right) \Delta r + \left( \frac{1}{r+1+\epsilon} - \frac{1}{\epsilon} \right) \Delta\epsilon \\ &= \left( \frac{2}{r} - \frac{2}{r+2} - \frac{1}{r+3} + \frac{1}{r+1+\epsilon} - \frac{r+1}{(r+1+\epsilon)(m_1+1)} \right) \Delta m_1 \\ &\quad + \left( \frac{2}{r} - \frac{2}{r+2} - \frac{1}{r+3} + \frac{1}{r+1+\epsilon} - \frac{r+1}{(r+1+\epsilon)(m_2+1)} \right) \Delta m_2 \\ &= i_1 \Delta m_1 + i_2 \Delta m_2 \dots \dots \dots \text{(cxx.)} \end{aligned}$$

where  $i_1$  and  $i_2$  admit of easy calculation. Hence

$$\Sigma_{\kappa}/\kappa^2 = i_1^2 \Sigma_{m_1}^2 + i_2^2 \Sigma_{m_2}^2 + 2i_1 i_2 \Sigma_{m_1} \Sigma_{m_2} R_{m_1 m_2} \dots \dots \dots \text{(cxxi.)}$$

The value of  $\Sigma_{\kappa}$  can thus be found, and the steadiness of the curve to its type ascertained.

### *Illustration.—Glands of the Fore-legs of Swine.*

In the ‘Proceedings of the American Academy of Arts and Sciences,’ vol. 32, p. 87, 1896, is a memoir by C. B. DAVENPORT and C. BULLARD, on the variation in number of the Müllerian glands in the fore-legs of 4,000 swine. The paper especially attracted our attention, because the authors are content to describe the frequency distribution of these glands by means of a normal curve. They write, after discussing the plotting of the normal curve on their diagram (pp. 90–91):—

“These and other characters of the ‘probability’ curve are indicated in that shown in dotted line\* in the accompanying diagram. The diagram also shows the curve of

\* The authors actually represent the normal curve by an 18-sided polygon.



it gives 30 per mille instead of 10 per mille with no glands. These difficulties are entirely met by the skew curve, which gives no frequency whatever of negative glands (see figure).

Taking the number of glands in the right fore-leg of female swine, we have the frequency series :

No. of glands .	0	1	2	3	4	5	6	7	8	9	10
Frequency . .	15	209	365	482	414	277	134	72	22	8	2
Per mille . . .	7.5	104.5	182.5	241	207	138.5	67	36	11	4	1

We have worked with the frequency per mille for convenience of reduction, although the actual number of observed cases, 2,000, is used, of course, in the determination of the probable errors.

Using the method of the paper in the 'Phil. Trans.,' A, vol. 186, p. 367, we found

$$\begin{aligned}
 \text{mean} &= 3.501 \text{ glands.} & \sigma &= 1.680,774 \\
 \mu_2 &= 2.824,999 & \beta_1 &= 0.259,1825 \\
 \mu_3 &= 2.417,278 & \beta_2 &= 3.110,8211 \\
 \mu_4 &= 24.826,297 & 6 + 3\beta_1 - 2\beta_2 &= 0.555,905
 \end{aligned}$$

Thus the criterion is greater than zero, or the frequency distribution is of Type I., or has a limited range.

Proceeding we found

$$\begin{aligned}
 r &= 19.985119 & \epsilon &= 72.71918 \\
 m_1 &= 3.783718 & m_2 &= 14.201402 \\
 a_1 &= 3.79623 & a_2 &= 14.24837 \\
 b &= 18.0446 & y_0 &= 237.263 \\
 d &= 0.522996 & \text{Sk.} &= 0.311164 \\
 \text{Mode} &= 2.978 & \text{Start of curve} &= -0.818 \text{ gland.}
 \end{aligned}$$

Thus it would appear that both the distance ( $d$ ) from mean to mode and the skewness are very sensible, and that, unless their probable errors be very large, it is quite impossible to represent the results by a normal curve.

We may note that the range starts from  $-0.818$  gland and runs to  $17.227$  glands, so since it gives zero at  $-1$  gland, we see that it sensibly confines the possible number of glands between 0 and 17, but we should have to examine considerably more than 2,000 swine to have a probability of more than 10 glands occurring. The



total range given is thus both in magnitude and position extremely satisfactory, and supposing only the frequency, not the actually measured quantity, *i.e.*, number of glands, to be known, the theory would have given a very accurate determination of the limits of possibility, especially the start of possibility with the whole number of glands.

In order to work out easily the determinant,  $\Delta$ , and its minors, we found it desirable to bring out certain factors and reduce the formulæ given above to slightly different forms, which, as they are likely to be of general service, are here repeated.

Let  $\alpha = m_1\epsilon_1 - (m_1 + m_2)\epsilon_3$ ,  $\beta = m_2\epsilon_2 - (m_1 + m_2)\epsilon_3$ , where  $\epsilon_1, \epsilon_2, \epsilon_3$  have the values given on p. 284, then we found

$$\Delta = \frac{n^4(m_1 + m_2 + 1)}{b^4 m_1^2 m_2^2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2 & 0 & 1 & 0 \\ -(m_1 - 1) & m_1\alpha + m_2\beta & -m_1\alpha & -(m_1 + m_2 - 2) \\ m_2(m_1 - 1) & m_2\beta & m_1m_2\epsilon_3 & m_1 - 2m_2 + 1 \\ m_2 + 1 & -1 & 0 & 2(m_1 + m_2 + 1) \end{vmatrix},$$

$$A_{11} = \frac{n^3}{b^2 m_1^2 m_2^2 (m_2 - 1)} \begin{vmatrix} m_1\alpha + m_2\beta & -m_1\alpha & -1 \\ m_2\beta & m_1m_2\epsilon_3 & -(m_1 + 1) \\ -(m_2 - 1) & -m_1(m_2 - 1) & (m_1 + 1)(m_1 + m_2 + 1) \end{vmatrix},$$

$$A_{22} = \frac{n^3(m_1 + m_2 + 1)}{b^4 m_2^2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2 & 1 & 0 \\ m_2(m_1 - 1) & m_2^2(\epsilon_2 - \epsilon_3) & m_1 - 2m_2 + 1 \\ m_2 + 1 & -1 & 2(m_1 + m_2 + 1) \end{vmatrix},$$

$$A_{33} = \frac{n^3(m_1 + m_2 + 1)}{b^4 m_1^2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2 & -1 & 0 \\ -(m_2 + 1)(m_1 - 1) & m_1^2(\epsilon_1 - \epsilon_3) & m_2 - 2m_1 + 1 \\ m_2 + 1 & 0 & 2(m_1 + m_2 + 1) \end{vmatrix},$$

$$A_{44} = \frac{n^3(m_1 + m_2 + 1)}{b^2 m_1^2 m_2^2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} (m_1 + m_2)(m_1 + m_2 - 2) & 0 & (m_1 - 1)(m_2 - 1) \\ 0 & m_1\alpha + m_2\beta & -m_1\alpha \\ m_1 + m_2 + 1 & m_2\beta & m_1m_2\epsilon_3 \end{vmatrix},$$

$$A_{12} = -\frac{n^2(m_1 + m_2 + 1)}{b^3 m_1 m_2^2 (m_2 - 1)} \begin{vmatrix} -1 & -m_1m_2\epsilon_3 & 0 \\ 0 & m_2\beta & -1 \\ -(m_1 + m_2) & -(m_1 + 1)(m_2 - 1) & m_2 + 2m_1 + 1 \end{vmatrix},$$

$$A_{13} = \frac{n^3(m_1 + m_2 + 1)}{b^3 m_1 m_2 (m_2 - 1)} \begin{vmatrix} 0 & \alpha & -1 \\ 1 & -m_2\epsilon_3 & 0 \\ -(m_1 + m_2) & (m_2 - 1) & (m_1 + 1) \end{vmatrix},$$

$$A_{14} = - \frac{n^3(m_1 + m_2 + 1)}{b^2 m_1^2 m_2^2 (m_2 - 1)} \begin{vmatrix} 0, & m_1 \alpha + m_2 \beta, & -m_1 \alpha \\ m_2 - 1, & m_2 \beta, & m_1 m_2 \epsilon_3 \\ -(m_1 + m_2), & -1, & -m_1 \end{vmatrix},$$

$$A_{23} = - \frac{n^3(m_1 + m_2 + 1)}{b^4 m_1 m_2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2, & -1, & 0, \\ m_2(m_1 - 1), & -m_1 m_2 \epsilon_3, & m_1 - 2m_2 + 1 \\ m_2 + 1, & 0, & 2(m_1 + m_2 + 1) \end{vmatrix},$$

$$A_{24} = \frac{n^3(m_1 + m_2 + 1)}{b^3 m_1 m_2^2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2, & -(m_2 + 1)(m_1 - 1), & -(m_1 - 1) \\ 1, & -m_1 m_2 \epsilon_3, & m_2 \beta \\ 0, & m_2 - 2m_1 + 1, & -(m_1 + m_2 - 2) \end{vmatrix},$$

$$A_{34} = - \frac{n^3(m_1 + m_2 + 1)}{b^3 m_1^2 m_2 (m_1 - 1)(m_2 - 1)} \begin{vmatrix} m_1 + m_2, & -(m_1 - 1), & m_2(m_1 - 1) \\ -1, & m_1 \alpha, & -m_1 m_2 \epsilon_3 \\ 0, & -(m_1 + m_2 - 2), & m_1 - 2m_2 + 1 \end{vmatrix}.$$

In our particular case we found

$$\begin{aligned} \epsilon_1 &= .232,4012, & \epsilon_2 &= .067,9945, & \epsilon_3 &= .051,3099, \\ m_1 \alpha &= -.164,4934, & m_2 \beta &= .607,8513. \end{aligned}$$

With the aid of the values for  $m_1$  and  $m_2$  given above, the determinantal parts of the A's were then calculated. If these be  $\delta$ ,  $\alpha_{11}$ ,  $\alpha_{22}$ ,  $\alpha_{33}$ ,  $\alpha_{44}$ ,  $\alpha_{12}$ ,  $\alpha_{13}$ ,  $\alpha_{14}$ ,  $\alpha_{23}$ ,  $\alpha_{24}$ ,  $\alpha_{34}$ , we have

$$\begin{aligned} \delta &= .153,7969, & \alpha_{12} &= .274,9969, \\ \alpha_{11} &= .348,3713, & \alpha_{13} &= .111,8280, \\ \alpha_{22} &= 13.018,7332, & \alpha_{14} &= .211,9650, \\ \alpha_{33} &= 47.671,9443, & \alpha_{23} &= 22.706,5156, \\ \alpha_{44} &= 13.357,1309, & \alpha_{24} &= 11.507,0153, \\ & & \alpha_{34} &= 25.088,6121. \end{aligned}$$

From these the standard deviations and correlations of errors in the algebraical constants are at once found. We have

$$\begin{aligned} \Sigma_h &= .2325, & R_{h m_1} &= .9387, \\ \Sigma_{m_1} &= .7784, & R_{h m_2} &= .7548, \\ \Sigma_{m_2} &= 5.5908, & R_{h b} &= .7143, \\ \Sigma_b &= 3.7602, & R_{m_1 m_2} &= .91145, \\ & & R_{m_1 b} &= .8726, \\ & & R_{m_2 b} &= .9942, \end{aligned}$$

Then as a step towards the determination of other probable errors, the standard deviation and umbral equation\* for  $\gamma = m_1 + m_2$  were found

$$\Sigma_\gamma = 6.3085,$$

$$\chi_\gamma = \text{Antl. } 1.091,2932\chi_{m_1} + \text{Antl. } 1.947,5528\chi_{m_2}.$$

This led to

$$R_{\gamma m_1} = .9312, \quad R_{\gamma b} = .9888, \quad R_{\gamma k} = .7848.$$

By aid of these auxiliary results the probable errors of all the algebraical and "physical" constants were determined.

PROBABLE Error Table.

Constant.		Probable error.	Percentage probable error.
Algebraic constants.	$m_1$ . . . . .	0.5250	13.8762
	$m_2$ . . . . .	3.7709	26.5533
	$a_1$ . . . . .	0.1748	4.6056
	$a_2$ . . . . .	2.4351	17.0903
"Physical" constants.	Range . . . . .	2.5362	14.0554
	Start of range . . . . .	0.1568	—
	Mode . . . . .	0.0398	—
	Mean . . . . .	0.0253	—
	Standard deviation . . . . .	0.0183	1.0911
	Mean to mode . . . . .	0.0294	5.6308
	Skewness . . . . .	0.0158	5.0655
	Modal frequency . . . . .	3.2455	1.3679

Now it will be clear from an examination of these results that all the "physical" constants are determined with great accuracy.† The mean is subject to less probable error than the mode, the modal frequency has a slightly less probable error than the mean, and as it is less than 1.4 per cent. in the former case, either are closely known. The skewness and distance from mean to mode are known respectively with less than 5.1 and with 5.6 probable errors. Thus they are both significant constants. In other words, the curve differs significantly from a normal curve, and it is erroneous to represent the frequency by such a normal curve. The range which ought to be such that there is no frequency at  $-1$  gland, gives no frequency at

\* See footnote, p. 286, and later, p. 305. It may be as well to remind the reader that here, as in the other illustrations, logarithms of the full, not the cited values, were used in the calculations.

† The probable percentage errors  $m_1$ ,  $m_2$ ,  $a_1$ ,  $a_2$  are high, but this, as we have several times pointed out, is of small importance, as, owing to their high correlation, the actual shape of the curve is not changed sensibly by large changes in  $m_1$  and  $m_2$ .



— ·818 gland with a probable error of  $\pm \cdot 157$ . It is, therefore, clear that our method gives the start of the range with very considerable accuracy. The whole length of the range runs to 17·227 glands, with a probable error of 2·536. We may, accordingly, conclude that the maximum possible number of glands is hardly likely to be less than 16 or more than 20. We consider that this example is a good illustration of the accuracy with which the principal “physical” characteristics of a distribution may be obtained by aid of skew curves, and how they provide much information which is not given by the use of the normal curve.

The next point is the determination of the umbral equations giving the error correlations of the “physical” constants. They are, if Antl. stands for antilogarithm :

$$\begin{aligned} \chi_{\text{mean}} = & \text{Antl. } \cdot 792,4156 \chi_h - \text{Antl. } 1\cdot 380,2040 \chi_b - \text{Antl. } 1\cdot 153,9620 \\ & + \text{Antl. } 1\cdot 508,1033 \chi_{m_2}, \end{aligned}$$

$$\chi_{\text{range}} = \chi_b,$$

$$\chi_{v_0} = - \text{Antl. } 1\cdot 011,7885 \chi_b - \text{Antl. } \cdot 248,2856 \chi_{m_1} + \text{Antl. } 1\cdot 097,6534 \chi_{m_2},$$

$$\chi_{\sigma} = \text{Antl. } 1\cdot 109,9660 \chi_b + \text{Antl. } \cdot 168,8507 \chi_{m_1} - \text{Antl. } 1\cdot 151,0582 \chi_{m_2},$$

$$\chi_d = \text{Antl. } \cdot 397,2701 \chi_b - \text{Antl. } \cdot 274,1702 \chi_{m_1} - \text{Antl. } 1\cdot 810,3180 \chi_{m_2},$$

$$\chi_{sk.} = - \text{Antl. } \cdot 381,5919 \chi_{m_1} + \text{Antl. } \cdot 367,7012 \chi_{m_2}.$$

Multiplying these out pair and pair, we found

ERROR Correlation Table.

	Mean.	Range.	Modal frequency.	Standard deviation.	Mean to mode.	Skewness.
Mean . . . . .	1	·0232	·3400	—·3493	·0500	·1309
Range . . . . .	·0232	1	·6284	·0906	·2132	·2175
Modal frequency . . . .	·3400	·6284	1	—·6944	—·1473	—·0141
Standard deviation . . .	—·3493	·0906	—·6944	1	·5891	·4394
Mean to mode . . . . .	·0500	·2132	—·1473	·5891	1	·9847
Skewness . . . . .	·1309	·2175	—·0141	·4394	·9847	1

Hence, proceeding to multiply rows and divide columns by the corresponding standard deviations, we have, after altering the units, the following

PROGRESSION Table.

Corresponds to probable change in same units of	Unit change of					
	One gland in the mean.	One gland in the range.	One per cent. in the modal frequency.	One gland in the standard deviation.	One gland in the interval from mean to mode.	$\frac{1}{10}$ in the skewness.
Mean . . . . .	1	·0002	·0063	—·4818	·0430	·0210
Range . . . . .	2·3231	1	1·1651	12·5279	18·3604	3·4992
Modal frequency . . . . .	18·3876	·3389	1	—51·7973	—6·8411	—·1225
Standard deviation . . . . .	—·2533	·0007	—·0093	1	·3668	·0511
Interval from mean to mode	·0583	·0025	—·0032	·9459	1	·1840
Skewness . . . . .	·8156	·0135	—·0016	3·7761	5·2706	1

An examination of this table brings out several interesting features of the frequency distribution of Müllerian glands in the fore-legs of swine. If a group of swine were isolated, and found to have a higher mean number of glands, then this group would most probably have an increased possible range, but at the same time a decreased variability and a marked increase of skewness. This increase of the possible range with a decreased variability is especially notable, since the rough-and-ready class of statistician is very apt to treat the range observed as a measure of variability; we have here a case in which the same cause, raising of the mean, produces opposite effects on range and variability. Increase of range, it will next be observed, produces very little effect on any of the physical constants, but such effect as there is, is an increase of them all. To increase the modal frequency is to increase the range and to reduce both the variability and the skewness. Thus the more mediocre swine there exist in any group, the more nearly their distribution will be normal. Change in the variability is the cause which on the whole produces most effect. Increased variability means lowered mean and less mediocrity, but much increased skewness. Finally increased skewness denotes probable increase of range, variability, and mean.

As we have suggested in a previous illustration the principles of multiple correlation easily enable us to predict the probable change in a random selection in which two or more of the characters differ from those of the general population.

(20.) *Probable Errors and Error-Correlations of the Constants of the Generalised Probability Curve of Type*

$$y = y_0 \frac{1}{\{1 + (x/a)^2\}^m} e^{-v \tan^{-1}(x/a)} \quad \text{. . . . . (cxxii.)}$$

This curve is discussed at length, 'Phil. Trans.,' A, vol. 186, pp. 376-80. The chief constants are given as follows, if  $m = \frac{1}{2}(r + 2)$ ,  $z = v^2 + r^2$ , and  $h$  denote the origin :—

Moments—

$$\mu_2 = \frac{a^2}{r^2(r-1)}(r^2 + v^2) \quad \text{. . . . . (cxxiii.)}$$

$$\mu_3 = -\frac{4a^3v(r^2 + v^2)}{r^3(r-1)(r-2)} \quad \text{. . . . . (cxxiv.)}$$

$$\mu_4 = \frac{3a^4(r^2 + v^2)\{(r+6)(r^2 + v^2) - 8r^2\}}{r^4(r-1)(r-2)(r-3)} \quad \text{. . . . . (cxxv.)}$$

$$\text{Distance of centroid from origin} = -av/r \quad \text{. . . . . (cxxvi.)}$$

$$\text{Size of mean organ} = h - \frac{av}{r} \quad \text{. . . . . (cxxvii.)}$$

$$\text{Size of modal organ} = h - \frac{av}{r+2} \quad \text{. . . . . (cxxviii.)}$$

$$\text{Distance from mean to mode} = d = \frac{2va}{r(r+2)} \quad \text{. . . . . (cxxix.)}$$

$$\text{Skewness} = \frac{2v}{r+2} \sqrt{\frac{r-1}{r^2 + v^2}} \quad \text{. . . . . (cxxx.)}$$

$$\text{Standard deviation} = \sigma = \frac{a}{r\sqrt{r-1}} \sqrt{r^2 + v^2} \quad \text{. . . (cxxxii.)}$$

$$y_0 = \frac{n}{a} e^{1/2\pi} / G(r, v) \quad \text{. . . . . (cxxxii.)}$$

where

$$G(r, v) = \int_0^\pi \sin^r \theta e^{v\theta} d\theta$$

and

$$G(r, v) = \frac{r(r-1)}{r^2 + v^2} G(r-2, v) \quad \text{. . . . . (cxxxiii.)}$$

is the formula of reduction.



Further, we have the following BERNOULLI number series for  $G(r, \nu)$ , where  $\tan \phi = \nu/r$ :—

$$\log \{e^{-i\pi\nu} G(r, \nu)\} = \log \sqrt{(2\pi/r)} + (r+1) \log \cos \phi + \nu\phi \\ + \sum_0^{\infty} \left\{ \frac{B_{2s+1}(-1)^s}{(2s+1)(2s+2)r^{2s+1}} (1 - 2^{2s+2} \cos^{2s+1} \phi \cos \overline{2s+1} \phi) \right\} \quad (\text{cxxxiv}).$$

To find  $y_1 \delta x$ , the mean frequency, we have only to put  $x = -a\nu/r$  in (i.), and we have

$$\log y_1 = \log y_0 + (r+2) \log \cos \phi + \nu\phi \\ = \log n - \log a - \log \{e^{-i\pi\nu} G(r, \nu)\} + (r+2) \log \cos \phi + \nu\phi \\ = \log n - \log a - \log \sqrt{(2\pi/r)} + \log \cos \phi - \chi,$$

where  $\chi$  stands for the summation in (cxxxiv.). Hence

$$y_1 = \frac{n}{a} \sqrt{\left(\frac{r}{2\pi}\right)} e^{-\chi} \cos \phi \quad \dots \quad (\text{cxxxv}).$$

or,

$$= \frac{n}{\sqrt{(2\pi)\sigma}} \sqrt{\left(\frac{r}{r-1}\right)} e^{-\chi} \quad \dots \quad (\text{cxxxvi}).$$

As typical constants we require the probable errors of the mean, the standard deviation, the skewness, and the mean frequency. It is clear that these will require us first to find  $\Sigma_h$ ,  $\Sigma_r$ ,  $\Sigma_\nu$ ,  $\Sigma_a$ , and  $R_{ha}$ ,  $R_{h\nu}$ ,  $R_{h\nu}$ ,  $R_{ha}$ ,  $R_{r\nu}$ ,  $R_{ra}$ ,  $R_{\nu a}$ .

We shall only indicate briefly the steps towards finding the integrals of the second differentials of  $\log y$ .

$$\log y = \log y_0 - m \log \{1 + (x/a)^2\} - \nu \tan^{-1}(x/a),$$

$$\frac{d(\log y)}{dx} = -\frac{\nu}{a} \frac{1}{\{1 + (x/a)^2\}} - \frac{(2mx)/a^2}{1 + (x/a)^2},$$

$$\frac{d^2(\log y)}{dx^2} = -\frac{2}{a^2} \left\{ -\frac{\nu x/a}{\{1 + (x/a)^2\}^2} - \frac{m}{1 + (x/a)^2} + \frac{2m}{(1 + x^2/a^2)^2} \right\}$$

$$= -(2/a^2) \{ -\nu \sin \theta \cos^3 \theta - m \cos^2 \theta + 2m \cos^4 \theta \}$$

$$\int_{-\infty}^{+\infty} y \frac{d^2 \log y}{dx^2} dx \\ = -\frac{2n}{a^2 G(r, \nu)} \left\{ -e^{i\pi\nu} \int_{-\pi/2}^{\pi/2} \sin \theta \cos^{r+3} \theta e^{-i\theta} d\theta - mG(r+2, \nu) + 2mG(r+4, \nu) \right\},$$

whence, remembering that  $2m = r+2$ , and integrating the first integral by parts, we find

$$\int_{-\infty}^{+\infty} y \frac{d^2 \log y}{dx^2} dx = - \frac{2n}{a^2} \left\{ \left( \frac{v^2}{r+4} + (r+2) \right) \frac{G(r+4, v)}{G(r, v)} - \frac{1}{2} (r+2) \frac{G(r+2, v)}{G(r, v)} \right\}$$

$$= - \frac{n}{a^2} \frac{(r+4) \{v^2 + (r+2)^2\}}{v^2 + (r+4)^2} \frac{G(r+2, v)}{G(r, v)}$$

or,

$$- a_{11} = - \frac{n}{a^2} \frac{(r+1)(r+2)(r+4)}{v^2 + (r+4)^2} \dots \dots \dots \text{(cxxxvii.)}$$

Precisely similar reductions lead us to

$$- a_{12} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dx da} dx = - \frac{nv(r+1)(r+2)}{a^2 \{v^2 + (r+4)^2\}} \dots \dots \text{(cxxxviii.)}$$

$$- a_{13} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dx dr} dx = \frac{nv(r+1)}{a \{v^2 + (r+2)^2\}} \dots \dots \text{(cxxxix.)}$$

$$- a_{14} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dx dv} dx = - \frac{n}{a} \frac{(r+1)(r+2)}{v^2 + (r+2)^2} \dots \dots \text{(cxl.)}$$

$$- a_{22} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{da^2} dx = - \frac{n}{a^2} (r+1) \frac{v^2 + 2(r+4)}{v^2 + (r+4)^2} \dots \dots \text{(cxli.)}$$

$$- a_{23} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{da dr} dx = \frac{n}{a} \frac{v^2 + (r+2)}{v^2 + (r+2)^2} \dots \dots \text{(cxlii.)}$$

$$- a_{24} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{da dv} dx = - \frac{n}{a} \frac{v(r+1)}{v^2 + (r+2)^2} \dots \dots \text{(cxliii.)}$$

$$- a_{33} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dr^2} dx = - n \frac{d^2}{dr^2} \{ \log G(r, v) \} \dots \dots \text{(cxliv.)}$$

$$- a_{34} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dr dv} dx = - n \frac{d^2}{dr dv} \{ \log G(r, v) \} \dots \dots \text{(cxlv.)}$$

$$- a_{44} = \int_{-\infty}^{+\infty} y \frac{d^2 (\log y)}{dv^2} dx = - n \frac{d^2}{dv^2} \{ \log G(r, v) \} \dots \dots \text{(cxlvi.)}$$

It will now be needful to find easily calculable series for the second differentials of  $\log G(r, v)$ . These can be obtained from (cxxxiv.). We find

$$\frac{d}{dr} \{ \log G(r, v) \} = - \frac{1}{2r} + \log \cos \phi + \frac{\sin^2 \phi}{r}$$

$$- \sum_0^{\infty} \frac{B_{2s+1}(-1)^s}{(2s+2)r^{2s+2}} \{ 1 - 2^{2s+2} \cos^{2s+2} \phi \cos(2s+2)\phi \} \dots \text{(cxlvii.)}$$

$$\frac{d}{dv} \{ \log G(r, v) \} = \frac{1}{2} \pi - \frac{\sin \phi \cos \phi}{r} + \phi$$

$$+ \sum_0^{\infty} \frac{(-1)^s B_{2s+1}}{(2s+2)r^{2s+2}} 2^{2s+2} \cos^{2s+2} \phi \sin(2s+2)\phi \dots \dots \text{(cxlviii.)}$$

Hence

$$\frac{d^2}{dr^2} \{\log G(r, \nu)\} = \frac{1}{r^2} \left\{ \cdot 5 + \sin^2 \phi (r - 1 - 2 \cos^2 \phi) \right. \\ \left. + \sum_0^\infty \frac{(-1)^s B_{2s+1}}{r^{2s+1}} \{1 - 2^{2s+2} \cos^{2s+3} \phi \cos(2s+3)\phi\} \right\} \text{(cxlix.)},$$

$$\frac{d^2}{d\nu^2} \{\log G(r, \nu)\} = \frac{1}{r^2} \left\{ \cos^2 \phi (2 - 1 + 2 \sin^2 \phi) \right. \\ \left. + \sum_0^\infty \frac{(-1)^s B_{2s+1}}{r^{2s+1}} 2^{2s+2} \cos^{2s+3} \phi \cos(2s+3)\phi \right\} \dots \text{(cl.)},$$

$$\frac{d^2}{dr d\nu} \{\log G(r, \nu)\} = \frac{1}{r^2} \left\{ \sin \phi \cos \phi (2 \cos^2 \phi - r) \right. \\ \left. - \sum_0^\infty \frac{(-1)^s B_{2s+1}}{r^{2s+1}} 2^{2s+2} \cos^{2s+3} \phi \sin(2s+3)\phi \right\} \dots \text{(cli.)}.$$

These allow of the fairly rapid calculation of  $\alpha_{33}$ ,  $\alpha_{34}$ ,  $\alpha_{44}$ . The values of the standard deviations of the errors, and of the error correlations, can then all be calculated from the determinant

$$\Delta = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{34} \\ \alpha_{14} & \alpha_{24} & \alpha_{34} & \alpha_{44} \end{vmatrix}$$

and its minors.\*

Let  $b_{pp'} = \frac{1}{n} \alpha_{pp'}$ , and let  $\Delta' = \frac{1}{n^4} \Delta$ , then if  $B_{pp'}$  be the minor corresponding to  $b_{pp'}$  in  $\Delta'$ , we must work out for any special numerical case

$$\begin{aligned} \Sigma_r^2 &= \frac{1}{n} \frac{B_{11}}{\Delta'}, & \Sigma_\alpha^2 &= \frac{1}{n} \frac{B_{22}}{\Delta'}, \\ \Sigma_{r'}^2 &= \frac{1}{n} \frac{B_{33}}{\Delta'}, & \Sigma_\nu^2 &= \frac{1}{n} \frac{B_{44}}{\Delta'}, \\ R_{ah} &= \frac{B_{12}}{\sqrt{(B_{11} B_{22})}}, & R_{rh} &= \frac{B_{13}}{\sqrt{(B_{11} B_{33})}}, \\ R_{vh} &= \frac{B_{14}}{\sqrt{(B_{11} B_{44})}}, & R_{ar} &= \frac{B_{23}}{\sqrt{(B_{22} B_{33})}}, \\ R_{av} &= \frac{B_{24}}{\sqrt{(B_{22} B_{44})}}, & R_{rv} &= \frac{B_{34}}{\sqrt{(B_{33} B_{44})}}. \end{aligned}$$

\* As in the former case, these were all developed, but the extreme length of the resulting formulæ gives them no advantage over working in any special case with the numerical determinants.



In general none of these correlations vanish, and their values must all be found before the errors and correlations of the chief characteristics of the frequency can be found.

The following results, easily obtained by aid of the relation  $\tan \phi = \nu/r$ , will be of service

$$\Sigma_{\phi}^2 = \frac{\cos^4 \phi}{r^2} \{ \Sigma_{\nu}^2 + \tan^2 \phi \Sigma_r^2 - 2 \tan \phi \Sigma_{\nu} \Sigma_r R_{\nu r} \} \quad \text{. . . . . (clii.),}$$

$$R_{\phi \nu} = \frac{\cos^3 \phi}{r \Sigma_{\phi}} \{ \Sigma_{\nu} - \tan \phi \Sigma_r R_{\nu r} \} \quad \text{. . . . . (cliii.),}$$

$$R_{\phi r} = \frac{\cos^3 \phi}{r \Sigma_{\phi}} \{ \Sigma_{\nu} R_{\nu r} - \tan \phi \Sigma_r \} \quad \text{. . . . . (cliv.),}$$

$$R_{\phi a} = \frac{\cos^3 \phi}{r \Sigma_{\phi}} \{ \Sigma_{\nu} R_{\nu a} - \tan \phi \Sigma_r R_{r a} \} \quad \text{. . . . . (clv.),}$$

$$R_{\phi h} = \frac{\cos^3 \phi}{r \Sigma_{\phi}} \{ \Sigma_{\nu} R_{\nu h} - \tan \phi \Sigma_r R_{r h} \} \quad \text{. . . . . (clvi.).}$$

By (cxxvii.) and (cxxx.) if  $\bar{x}$  be the mean size of organ,

$$\Delta \bar{x} = \Delta \bar{h} - \tan \phi \Delta a - \frac{a''}{\cos^2 \phi} \Delta \phi,$$

$$\frac{\Delta \sigma}{\sigma} = \frac{\Delta a'}{a'} + \tan \phi \Delta \phi - \frac{1}{2} \frac{\Delta r'}{r' - 1}.$$

Hence

$$\begin{aligned} \Sigma_{\bar{x}}^2 &= \Sigma_h^2 + \tan^2 \phi \Sigma_a^2 + \frac{a^2}{\cos^4 \phi} \Sigma_{\phi}^2 - 2 \tan \phi \Sigma_h \Sigma_a R_{ha} - \frac{2a}{\cos^2 \phi} \Sigma_h \Sigma_{\phi} R_{\phi h} \\ &\quad + 2 \tan \phi \frac{a}{\cos^2 \phi} \Sigma_a \Sigma_{\phi} R_{a\phi} \quad \text{. . . . . (clvii.),} \end{aligned}$$

$$\begin{aligned} \frac{\Sigma_{\bar{x}}^2}{\sigma^2} &= \frac{\Sigma_a^2}{a^2} + \tan^2 \phi \Sigma_{\phi}^2 + \frac{1}{4(r-1)^2} \Sigma_r^2 + \frac{2}{a} \tan \phi \Sigma_a \Sigma_{\phi} R_{a\phi} - \frac{1}{a(r-1)} \Sigma_a \Sigma_r R_{ar} \\ &\quad - \tan \phi \frac{1}{r-1} \Sigma_{\phi} \Sigma_r R_{\phi r} \quad \text{. . . . . (clviii.),} \end{aligned}$$

and

$$\begin{aligned} R_{\bar{x}\sigma} &= \frac{\sigma}{\Sigma_{\sigma} \Sigma_{\bar{x}}} \left\{ \frac{\Sigma_a \Sigma_h R_{ah}}{a} - \frac{\tan \phi}{a} \Sigma_a^2 - \frac{\Sigma_a \Sigma_{\phi} R_{a\phi}}{\cos^2 \phi} + \tan \phi \Sigma_h \Sigma_{\phi} R_{h\phi} - \tan^2 \phi \Sigma_a \Sigma_{\phi} R_{a\phi} \right. \\ &\quad \left. - \frac{a \sin \phi}{\cos^3 \phi} \Sigma_{\phi}^2 - \frac{1}{2(r-1)} \Sigma_h \Sigma_r R_{hr} + \frac{\tan \phi}{2(r-1)} \Sigma_a \Sigma_r R_{ar} + \frac{a}{2 \cos^2 \phi (r-1)} \Sigma_r \Sigma_{\phi} R_{r\phi} \right\} \quad \text{(clix.).} \end{aligned}$$

From (cxxx.) we have, if  $S_k$  = skewness,

$$S_k = \frac{2 \sin \phi}{r+2} \sqrt{r-1}.$$

Or, taking logarithmic differentials,

$$\Delta S_k/S_k = \cot \phi \Delta \phi - \left( \frac{1}{r+2} - \frac{1}{2} \frac{1}{r-1} \right) \Delta r,$$

whence,

$$\Sigma_{S_k}^2/S_k^2 = \cot^2 \phi \Sigma_{\phi}^2 + \frac{(r-4)^2}{4(r-1)^2(r+2)^2} \Sigma_r^2 - \cot \phi \frac{r-4}{(r-1)(r+2)} \Sigma_{\phi} \Sigma_r R_{r\phi}. \quad (\text{clx}).$$

In a similar manner  $R_{\bar{x}S_k}$  and  $R_{\sigma S_k}$  can be found if desired. None of these quantities will, as a rule, vanish, and as very many measurements on animals give curves of the tangent type, we conclude that in general *all selection of the size of an organ alters its variability and the skewness of its distribution, and again all selection of variability connotes alteration of the size and skewness of the selected organ.*

The probable errors of  $\mu_2$ ,  $\mu_3$ , and  $\mu_4$ , as well as the error-correlations of these quantities, can all be found from the differentials of (cxxiii.), (cxxiv.), and (cxxv.); the calculation is laborious, but presents no novelty.

Lastly, the probable errors of the mean and modal frequencies may be deduced.

For the mean frequency we start from (cxxxv.) and use (cxxxiv.).

This requires us to know  $\Delta \chi$ , where

$$\chi = \sum_0^{\infty} \frac{(-1)^s B_{2s+1}}{(2s+1)(2s+2)r^{2s+1}} (1 - 2^{2s+2} \cos^{2s+1} \phi \cos(2s+1)\phi) \dots \quad (\text{clxi}).$$

We have, as in (cxlvii.) and (cxlviii.),

$$\begin{aligned} \Delta \chi &= -\frac{\Delta r}{r} \sum_0^{\infty} \frac{(-1)^s B_{2s+1}}{(2s+2)r^{2s+1}} (1 - 2^{2s+2} \cos^{2s+2} \phi \cos(2s+2)\phi) \\ &\quad + \frac{\Delta \nu}{r} \sum_0^{\infty} \frac{(-1)^s B_{2s+1}}{(2s+2)r^{2s+1}} 2^{2s+2} \cos^{2s+2} \phi \sin(2s+2)\phi \\ &= -c_1 \Delta r/r + c_2 \Delta \nu/r, \end{aligned}$$

where  $c_1$  and  $c_2$  admit of fairly easy calculation.

Hence, by (cxxxv.), we find,

$$\begin{aligned} \Delta y_1/y_1 &= -\Delta a/a + \left( \frac{1}{2} + \sin^2 \phi + c_1 \right) \Delta r/r - (c_2 + \cos \phi \sin \phi) \Delta \nu/r. \\ \Sigma_{y_1}^2/y_1^2 &= \Sigma_a^2/a^2 + \left( \frac{1}{2} + \sin^2 \phi + c_1 \right)^2 \Sigma_r^2/r^2 + (c_2 + \cos \phi \sin \phi)^2 \Sigma_{\nu}^2/\nu^2 \\ &\quad - \frac{2}{ra} \left( \frac{1}{2} + \sin^2 \phi + c_1 \right) \Sigma_a \Sigma_r R_{ar} + \frac{2(c_2 + \cos \phi \sin \phi)}{a\nu} \Sigma_a \Sigma_{\nu} R_{a\nu} \\ &\quad - \frac{2(\frac{1}{2} + \sin^2 \phi + c_1)(c_2 + \cos \phi \sin \phi)}{r^2} \Sigma_r \Sigma_{\nu} R_{r\nu} \dots \dots \dots (\text{clxii}). \end{aligned}$$

If the problem be to find the modal frequency  $y_2 \delta x$ , we easily deduce  $y_2$  by putting

$x = -\frac{\nu}{a(r+2)}$  in the equation to the curve. Writing  $\tan \phi' = \nu/(r+2)$  and  $\chi'$  the same function of  $r+2$  that  $\chi$  is of  $r$ , we have

$$y_2 = \frac{n}{a} \sqrt{\left(\frac{r+2}{2\pi}\right)} \cos \phi' \frac{r+1}{r+2} e^{-\chi'} \quad \dots \quad (\text{clxiii}).$$

Since  $\frac{r+1}{\sqrt{r+2}}$  is greater than  $\sqrt{r}$ , and  $\phi'$  and  $\chi'$  are less than  $\phi$  and  $\chi$  respectively, it follows that  $y_2$  is greater than  $y_1$ , as it should be. Further we find

$$\frac{\Delta y_2}{y_2} = -\frac{\Delta a}{a} + \left(\frac{1}{2} \frac{r+3}{r+1} + \sin^2 \phi' + c_1'\right) \frac{\Delta r}{r+2} - (c_2' + \cos \phi' \sin \phi') \frac{\Delta \nu}{r+2} \quad \dots \quad (\text{clxiv}).$$

Here  $c_1'$  and  $c_2'$  are the same functions of  $r+2$  and  $\phi'$  that  $c_1$  and  $c_2$  are of  $r$  and  $\phi$ . The usual process of squaring and introducing the standard deviations into the square terms and the product of standard deviations and correlations into the product terms will give us  $\Sigma_{y_2}^2$ .

## (21.) *Illustration.—Stature of Children.*

In order to illustrate the difficulties which may arise in determining the probable errors of the constants and the error correlations, we have selected for this illustration not a curve markedly skew, but one which is extremely nearly normal. The problem in this case is accordingly the following one: Are the values of the constants obtained for the distribution and distinguishing it from a normal distribution really significant? The difficulties which arise in the course of the arithmetical work depend upon the fact that, as the distribution is nearly normal, its constants approach the values at which the type of the skew curve passes over into the normal curve, and consequently not only will their probable errors be large, but, as in all cases of approach to limits, they will depend upon expressions tending to become indeterminate. Thus in the evaluation of the determinant  $\Delta$  and its minors, we at once found our results depended on the ratio of the differences of very small quantities. We were accordingly in this case obliged to calculate our constituents to a degree of accuracy which will, in general, be quite unnecessary, and which was only possible and straightforward owing to the ready help of a large sized Brunsviga. That the method, even in a critical case of this kind, gives correct results is evidenced by the agreement of our values of the constants with those (probable errors of mean and standard deviation) which can be readily calculated by other processes.

The example we have selected is that given for the stature of 2192 St. Louis school girls of 8 years of age in 'Phil. Trans.,' A, vol. 186, p. 386.

The equation to the frequency curve is



$$x = 14.9917 \tan \theta,$$

$$y = 235.323 \cos^{32.8023} \theta e^{-4.56967\theta},$$

the axis of  $x$  being *positive towards dwarfs* and the origin 2.2241 on the positive side of the mean. The unit of  $x$  is 2 cms. of height, and all the constants except the mean height are given in two-centimetre units.

We have the following values of the constants:—

$\mu_2 = 7.70739,$	Mean height = 118.271 centims.,
$\mu_3 = -2.38064,$	$\sigma = 2.77622,$
$\mu_4 = 192.17419,$	$y_1 = \text{modal frequency} = 324.18,$
	$y_2 = \text{mean frequency} = 323.76,$
$d = .135,606,$	Sk. = skewness = .04885,
$r = 30.8023,$	$m = 16.4011,$
$\nu = 4.56967,$	$\alpha = 14.9917.$

It will be seen at once that the skewness is small, that the mean and mode are close together, and the mean and modal frequencies are almost identical. Our problem is: Are these differences significant or not?

Let  $n = 2192$ , the total population; then the values of  $\alpha, r, \nu$  given above were assumed to be absolutely exact, and  $\Delta$  calculated with its constituents to 9 places of figures, as it depends on the differences of very small quantities. We shall indicate one or two stages in the arithmetical work

$$\frac{\Delta}{n^4} = \begin{vmatrix} .131,108,064 & .017,214,971 & -.008,837,638 & .063,438,906 \\ .017,214,971 & .010,392,042 & -.003,264,669 & .008,837,638 \\ -.008,837,638 & -.003,264,669 & .001,144,775 & -.004,422,657 \\ .063,438,906 & .008,837,638 & -.004,422,657 & .030,799,043 \end{vmatrix}.$$

The evaluation of this determinant and its minors was then carried out by means of the Brunsviga, and we found

$\frac{\Delta}{n^1} = \frac{.104,824,472}{10^{12}},$	$\frac{A_{12}}{n^3} = -\frac{1200.842,528}{10^{12}},$
$\frac{A_{11}}{n^3} = \frac{670.195,695,496}{10^{12}},$	$\frac{A_{13}}{n^3} = -\frac{5059.387,378}{(10)^{12}},$
$\frac{A_{22}}{n^3} = \frac{4606.123,523}{10^{12}},$	$\frac{A_{14}}{n^3} = -\frac{1762.570,609}{10^{12}},$
$\frac{A_{33}}{n^3} = \frac{76025.131,845}{10^{12}},$	$\frac{A_{23}}{n^3} = \frac{18675.261,289}{10^{12}},$
$\frac{A_{44}}{n^3} = \frac{4828.382,384}{10^{12}},$	$\frac{A_{24}}{n^3} = \frac{3833.460,555}{10^1},$
	$\frac{A_{34}}{n^3} = \frac{15979.332,581}{10^1}$

We can now, remembering that

$$\Sigma_p = \sqrt{(A_{pp}/\Delta)}, \quad \text{and} \quad R_{pq} = A_{pq}/\sqrt{(A_{pp}A_{qq})},$$

write down the standard deviations and error correlations of the algebraical constants

$$\begin{aligned} \Sigma_h &= 1.7078, & R_{ha} &= -.6835, \\ \Sigma_a &= 4.4773, & R_{hr} &= -.7088, \\ \Sigma_r &= 18.1898, & R_{hv} &= -.9798, \\ \Sigma_v &= 4.5840, & R_{ar} &= .9980, \\ & & R_{av} &= .8129, \\ & & R_{rv} &= .8340. \end{aligned}$$

Here  $h$  marks the position of the origin of the curve, and the numerical values are only retained to four places of figures, although, of course, in the further calculations the logarithms of the full values of the  $\Sigma$ 's and  $R$ 's have been used.

It will be noticed at once that though  $a$ ,  $r$ , and  $v$  have very considerable probable errors, the correlation between them is very high. In other words, as the curve approaches its limiting shape,  $a$ ,  $r$ , and  $v$  may vary very considerably, but owing to their close correlation this will not sensibly affect the geometrical shape of the curve.

The next stage was to determine the standard deviations and error correlations of certain subsidiary constants. Here, as in the determination later of the like quantities for the "physical" characters, we found the umbral notation of great service. It consists, as we have seen, in writing down a difference equation between any constants, and then replacing the differences  $\delta u$  by  $\Sigma_u \chi_u$ ,  $\delta v$  by  $\Sigma_v \chi_v$ , &c., where  $\chi_u$ ,  $\chi_v$ , &c., are quantities which obey the relations  $\chi_u^2 = 1$ ,  $\chi_v^2 = 1$ ,  $\chi_u \chi_v = R_{uv}$ . Thus, if  $\tan \phi = v/r$ , we find for the umbral equation

$$\Sigma_\phi \chi_\phi = \frac{\cos^2 \phi \Sigma_v}{r} \chi_v - \frac{\sin \phi \cos \phi}{r} \Sigma_r \chi_r.$$

Whence, putting in the numerical values, we have

$$\Sigma_\phi \chi_\phi = \text{Antl. } 1.163,2115 \chi_v - \text{Antl. } 2.933,0706 \chi_r$$

where Antl. stands for antilogarithm, in which form we found it easiest to keep the umbral coefficients. The square of this result gave at once

$$\Sigma_\phi = .087,926$$

and dividing out by its logarithm, we have the pure umbral equation

$$\chi_\phi = \text{Antl. } .219,0937 \chi_v - \text{Antl. } 1.988,9728 \chi_r.$$

Our object was then to find such pure umbral equations connecting all the "physical" constants with the algebraic constants. Their products will then give the error correlations of all the "physical" constants in terms of the correlations already known between the algebraical constants.

For example, multiplying the above equation for  $\chi_\phi$  by  $\chi_h, \chi_a, \chi_r, \chi_v$  we have, since  $\chi_h\chi_a = R_{ha}, \chi_a\chi_r = R_{ar},$  &c., are already known,

$$\begin{aligned} R_{h\phi} &= -\cdot9317, \text{ actually } \log(-R_{h\phi}) = \bar{1}\cdot969,2668, \\ R_{a\phi} &= \cdot3733, \quad \text{,,} \quad \log R_{a\phi} = \bar{1}\cdot572,0115, \\ R_{r\phi} &= \cdot4063, \quad \text{,,} \quad \log R_{r\phi} = \bar{1}\cdot608,8746, \\ R_{v\phi} &= \cdot8430, \quad \text{,,} \quad \log R_{v\phi} = \bar{1}\cdot925,8379. \end{aligned}$$

It was these logarithms, of course, which were used in the further calculations.

Since  $h$  is measured negatively (*i.e.*, towards dwarfs,  $x$  is positive), we must write for transferring origin to the mean

$$x' = x + a \tan \phi,$$

where  $a \tan \phi$  is the distance between the old origin and the mean, or if  $m$  be used to represent the mean we have

$$m = h + a \tan \phi.$$

Hence we find the umbral equation

$$\Sigma_m \chi_m = \text{Antl. } \cdot232,4493\chi_h + \text{Antl. } \bar{1}\cdot822,3179\chi_a + \text{Antl. } \cdot129,4233\chi_\phi.$$

Hence we determine

$$\Sigma_m = \cdot0549,$$

and the pure umbral equation

$$\chi_m = \text{Antl. } 1\cdot492,5897\chi_h + \text{Antl. } 1\cdot082,4583\chi_a + \text{Antl. } 1\cdot389,5637\chi_\phi.$$

In precisely the same way all the other "physical" constants, *i.e.*, the standard deviation,  $\sigma$ , the mean frequency,  $y_2$ , the distance between mean and mode,  $d$ , and the skewness,  $Sk$ , were found, and the umbral equations investigated. It is only necessary here to give the results.

Quantity.		Probable error.	Percentage probable error.
Algebraic constants.	$h$ , position of origin . . . . .	1.1519	—
	$a$ . . . . .	3.0199	20.1438
	$r$ . . . . .	12.2688	39.8309
	$v$ . . . . .	3.0919	67.6612
"Physical" constants.	Position of mean . . . . .	0.03705	—
	Position of mode . . . . .	0.05950	—
	Mean frequency, $y_2$ . . . . .	4.4362	1.3703
	Standard deviation, $\sigma$ . . . . .	0.02984	1.0750
	Mean to mode, $d$ . . . . .	0.0497	36.6420
	Skewness, $Sk$ . . . . .	0.02661	54.4690



Now it will be seen at once that the probable errors in the algebraic constants are large, but that the probable errors in the position of the mean, of the mode, and in the magnitudes of the mean frequency and standard deviation are small. The position of the mean is sensibly more correct than that of the mode. On the other hand, the distance of the mean from the mode and the skewness have large probable errors, not, however, so large but what these quantities are probably significant. The frequency distribution probably differs significantly from the normal distribution, but the difference is small and would require a very large number of observations to determine it with extreme accuracy. That there is a significant divergence from normality is also indicated by the sensible difference between the percentage errors in  $y_2$  and  $\sigma$ , which would be equal for a normal distribution. Had we taken a normal distribution, the probable error of the mean would have been  $\cdot 0400$ , and of the standard deviation,  $\cdot 02831$ . In fact, the standard deviation of the standard deviation, if calculated for the normal curve =  $\cdot 04197$ , if calculated by our present method =  $\cdot 044246$ , and if calculated by a modified form of the fourth moment formula given by CZUBER\* =  $\cdot 044240$ . This shows that the arithmetic of our process has been substantially correct.

We now place together the umbral equations for the correlations of the errors in the "physical" constants. They are

$$\begin{aligned} \chi_m &= \text{Antl. } 1\cdot492,5897\chi_h + \text{Antl. } 1\cdot082,4583\chi_a + \text{Antl. } 1\cdot389,5637\chi_\phi \\ \chi_\sigma &= \text{Antl. } 1\cdot272,7484\chi_a - \text{Antl. } 1\cdot282,1306\chi_r + \text{Antl. } 1\cdot913,0028\chi_\phi \\ \chi_{y_2} &= \text{Antl. } 1\cdot816,6966\chi_\phi - \text{Antl. } 1\cdot167,3480\chi_a + \text{Antl. } 1\cdot155,7688\chi_r \\ \chi_d &= \text{Antl. } \cdot266,3616\chi_r + \text{Antl. } 1\cdot740,1625\chi_a - \text{Antl. } \cdot323,8256\chi_\phi \\ \chi_{sk.} &= \text{Antl. } 1\cdot865,6420\chi_\phi - \text{Antl. } \cdot017,0466\chi_r \end{aligned}$$

From these results any correlation between pairs of errors, "physical" or algebraic, can be found at once. The following table gives the chief results:—

CORRELATION Coefficients between Errors in Constants.

	$m.$	$\sigma.$	$y_2.$	$d.$	$sk.$
$m$	1	$\cdot 0772$	$-\cdot 0584$	$\cdot 0826$	$\cdot 0426$
$\sigma$	$\cdot 0772$	1	$-\cdot 7062$	$\cdot 1177$	$\cdot 1431$
$y_2$	$-\cdot 0584$	$-\cdot 7062$	1	$\cdot 1779$	$\cdot 4086$
$d$	$\cdot 0826$	$\cdot 1177$	$\cdot 1779$	1	$\cdot 6843$
$sk.$	$\cdot 0426$	$\cdot 1431$	$\cdot 4086$	$\cdot 6843$	1

\* 'Theorie der Beobachtungsfehler,' p. 133.

Now it is clear that although the curve is nearly normal, there is still sensible correlation between quantities—*e.g.*, mean and  $\sigma$  or  $d$ —which would have no correlation between them if the curve were absolutely normal. This will be clearer if, as in the previous illustration, we replace this table by a table of regression coefficients.

	$m.$	$y_2.$	$\sigma.$	$d.$	$sk.$
$m$	1	-.0005	.0959	.0616	.05935
$y_2$	-6.9883	1	-104.9745	15.8875	68.1271
$\sigma$	.0622	-.00475	1	.0707	.16045
$d$	.1108	.0020	.1960	1	1.2780
$sk.$	.0306	.00245	.12755	.3665	1

This table has now finally to be thrown into more suitable units and attention paid to the fact that  $m$  increases towards dwarfs. We have, after the proper changes, the following results:—

PROGRESSION Table.

Corresponds to probable changes in the same units of	Unit change of				
	One centim. in mean stature.	One child per hundred in frequency of mean stature.	One centim. in standard deviation.	One centim. in interval from mean to mode.	1/10 in the skewness.
Mean stature . . . . .	1	.0032	-.0959	-.0616	-.0119
Mean frequency . . . . .	1.0798	1	-16.1745	2.4536	2.1042
Standard deviation . . . . .	-.0622	-.0308	1	.0707	.0321
Interval from mean to mode	-.1108	.0129	.1960	1	.2556
Skewness . . . . .	-.1530	.0794	.6378	1.8333	1

This table is extremely suggestive. It shows us that a random selection of girls of eight which had an increase of stature would have a less standard deviation, less distance between the mode and mean and less skewness. In other words, a selection giving taller children would be less variable and more nearly normal. Now as children grow older their stature increases, is less variable, and is more normal in its distribution. Thus, a selection of taller children from among children of eight would broadly tend to reproduce the characters of the stature distribution of older

children. In the same manner a selection of shorter children is more variable and less normal than the distribution of the general population of eight years of age, *i.e.*, tends to reproduce the characteristics of a younger population. Generally, a random selection, which increases variability, very sensibly increases skewness and decreases stature. What, perhaps, would hardly be expected, is that increase of skewness as well as increase of interval from mean to mode, *i.e.*, greater divergence from normality, increases the frequency of the mean stature.

It will be clear that by aid of this table we are able to predict the probable changes in all the other physical characters of the distribution when any sub-class has been selected at random from the general population with a difference of one character. If two or more characters differ in the sub-class, the probable changes in the other characters can be found by the principles of multiple correlation from the correlation table on page 307.

### (22.) *Conclusion.*

This study of the probable errors and error correlations shows us that these quantities can be determined for the most complex system of organs in the case of normal correlation, and in the case of either normal or skew variation with considerable ease. It is only in the case of skew variation that the arithmetic becomes at all laborious. But numerical examples suffice to show that the errors here made are of the same order as in the case of normal variation, if we confine our attention to the characteristic features of the frequency, *e.g.*, the mean or modal frequency, the standard deviation, the skewness, &c. Certain constants of the algebraic form of the frequency curves have large probable errors, but these errors are so highly correlated, that their existence does not suffice to substantially modify either the form of the curve, or the "physical" characteristics of the distribution calculated from such values.

For the theory of evolution certain very important principles flow, beyond the mere advantage of knowing the probable errors made in the measurement of racial or organic characters. Above all we note the importance of a random selection in altering in a systematic manner all racial constants. In most cases even size cannot be altered without alteration of the size, variation and correlation of all correlated organs. This principle is developed more at length in a memoir, nearly completed, on the influence of *directed* selection, which covers as a special case that of random selection.

Later, we hope to apply the general theorem from which our memoir starts to determine the probable errors in the constants of the components into which a heterogeneous frequency distribution may be resolved by the method of the first memoir of this series.\* It applies equally to such an investigation.

\* The importance of such a determination was emphasized by Professor GEORGE DARWIN in the discussion which took place at the reading of that memoir.



[NOTE.—Added May 25, 1898. One point ought to have been more fully dealt with in the above memoir, namely, the probable error of the criterion  $\kappa = 6 + 3\beta_1 - 2\beta_2$ , upon which the selection of the type of the frequency depends. Clearly, if the probable error of this criterion is as large as the criterion itself, there can be no stability of type, or the frequency may change over from one type to another.

On page 289 we have found the standard deviation of the criterion in terms of known quantities for the curve

$$y = y_1 (1 + x/\alpha_1)^{m_1} (1 - x/\alpha_2)^{m_2}.$$

It is in fact given by the umbral equation

$$\chi_\kappa \Sigma_\kappa / \kappa = i_1 \Sigma_{m_1} \chi_{m_1} + i_2 \Sigma_{m_2} \chi_{m_2} \dots \dots \dots (\text{clxv.})$$

where  $i_1$  and  $i_2$  are functions of  $m_1$  and  $m_2$  given in (cxx.) and  $\chi_{m_1} \chi_{m_2} = R_{m_1 m_2}$  is known from (cvi.).

The standard deviation of the criterion for the curve of type

$$y = y_0 e^{-\nu \tan^{-1} (x/a)} \{1 + (x/a)^2\}^n$$

may be found by taking differentials of

$$\kappa = 6 + 3\beta_1 - 2\beta_2 = -\frac{12}{r-3} \left\{ 4 \sin^2 \phi \frac{r-1}{(r-2)^2} + 1 \right\},$$

a value readily obtainable from ‘Phil. Trans.’ A, vol. 186, p. 377. We thus find the umbral equation

$$\begin{aligned} \chi_\kappa \Sigma_\kappa &= \left\{ 96 \sin^2 \phi \frac{r^2 - 3r + 1}{(r-3)^2 (r-2)^2} + \frac{12}{(r-3)^2} \right\} \Sigma_r \chi_r - \frac{96 \sin \phi \cos \phi (r-1)}{(r-3)(r-2)^2} \Sigma_\phi \chi_\phi \\ &= i_1' \Sigma_r \chi_r - i_2' \Sigma_\phi \chi_\phi, \text{ say,} \end{aligned} \quad (\text{clxvi.})$$

where  $\Sigma_r$ ,  $\Sigma_\phi$  and  $\chi_r \chi_\phi = R_{r\phi}$  are given by (clii.) and (cliv.).

Applying these results to the numerical examples, we find:—

(a.) For the glands of swine

$$\frac{\Sigma_\kappa}{\kappa} \chi_\kappa = -\cdot 070,5459 \Sigma_{m_1} \chi_{m_1} - \cdot 038,4629 \Sigma_{m_2} \chi_{m_2},$$

whence the probable error of  $\kappa = \cdot 67449 \Sigma_\kappa = \cdot 1012$ ; or,

$$\kappa = \cdot 5559 \pm \cdot 1012.$$

(b.) For the stature of children

$$\Sigma_\kappa \chi_\kappa = \cdot 015,6206 \Sigma_r \chi_r - \cdot 018,0067 \Sigma_\phi \chi_\phi,$$

whence the probable error of  $\kappa = \cdot 67449 \Sigma_\kappa = \cdot 1919$ ; or,

$$\kappa = -\cdot 4330 \pm \cdot 1919.$$

In both cases, therefore, we may consider that the sign of  $\kappa$  is beyond question, or that the type selected is really a significant character of the frequency.

With regard to the probable error made in estimating a criterion to be zero, and using a curve of type

$$y = y_1 (1 + x/a)^{\nu} e^{-\gamma},$$

we must remark that, the criterion being assumed zero is equivalent to assuming that its probable error is zero. Accordingly the only satisfactory method of testing whether a curve really falls under this type is to work out the probable error of its criterion on the hypothesis that it belongs to one or other of the two types, with positive or negative criterion as the case may be. If the probable error of the criterion thus calculated is sensibly as large as the criterion itself, then we may assume that the frequency distribution is of the type

$$y = y_1 (1 + x/a)^{\nu} e^{-\gamma}.]$$





VIII. *A Compensated Interference Dilatometer.**By* A. E. TUTTON, *Assoc. R.C.S.**Communicated by* Capt. ABNEY, *C.B., F.R.S.*

Received March 8,—Read April 28, 1898.

THE instrument now described is a form of Fizeau interference dilatometer, in which the sensitiveness of the interference method is so largely increased as to render it unnecessary to employ a plate or block of the solid, whose expansion is to be measured, of greater thickness than 5 millimetres. The author has been led to adopt it for the accurate determination of the thermal expansion of the crystals of artificial chemical preparations, which it is frequently impossible to obtain sufficiently large, and at the same time homogeneous, to furnish plates a centimetre thick, as demanded by the forms of apparatus hitherto described.

The exquisite method devised by FIZEAU ('Compt. Rend.,' vol. 58, p. 923, and vol. 62, p. 1133; 'Ann. Chim. Phys.,' [4], vol. 2, p. 143, and [4], vol. 8, p. 335) depends essentially upon the determination of the difference of expansion, which accompanies rise of temperature, between the screws of a small metallic tripod and the object under investigation which is supported by it. In the form of apparatus now described the expansion of these screws is compensated and eliminated, thus rendering the total expansion of the object available for measurement. Hence, it is possible to obtain a result with a small crystal of as satisfactorily accurate a character as was formerly only to be obtained with a much larger crystal. Besides the introduction of this compensating principle, the new instrument combines, in the author's opinion, the best features of the several forms of Fizeau dilatometer previously described; at the same time it is essentially different from any one of these previous forms, and includes many details of a novel character. The author is largely indebted to the Memoirs of BENOIT ('Trav. et Mémoires du Bureau Int. des Poids et Mesures,' vol. 1, 1881, p. 1, and vol. 6, 1888, p. 106), concerning the Fizeau apparatus belonging to the Bureau International des Poids et Mesures, Paris, and the classical work which he has accomplished by the use of it; and to the later one of PULFRICH ('Zeitschrift für Instrumentenkunde,' 1893, p. 365) concerning an improved form of apparatus embodying the important modifications introduced into the method by ABBE in the year 1884 and described by WEIDMANN in 1889 ('WIEDEMANN'S Annalen,' vol. 38, p. 453).

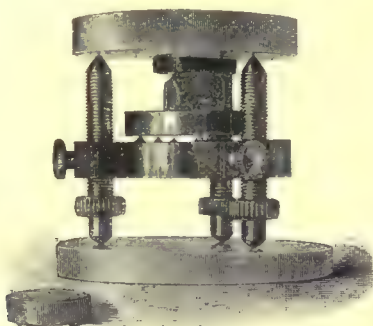
## DESCRIPTION OF THE APPARATUS.

*The Interference Tripod.*

The first essential of an interference dilatometer is the tripod support for the object and for the glass lens or plate to be suspended, without touching, above the latter. The upper surface of the object and the lower surface of the glass are the two surfaces, by reflection of pure monochromatic light from which interference curves or bands are to be produced; and it is the movement of these bands upon alteration of the temperature which affords the measurement of the change in thickness of the air space which separates the two surfaces. For the passage of two successive curves or bands past any given spot determines precisely that a change of thickness equal to half a wave-length, of the particular light employed, has occurred during the interval.

The tripod employed by BENOIT was constructed of platinum-iridium; that of ABBE and PULFRICH was of steel. The tripod of the apparatus now described is constructed of platinum-iridium, which, in addition to its obvious advantage on the score of unalterability, and its exceptionally small coefficient of expansion, is specially suitable for a reason which will be apparent when the method of compensation is described. It is represented in fig. 1.

Fig. 1.



It is entirely constructed out of a single homogeneous casting of the alloy of platinum with 10 per cent. of iridium, which was supplied by Messrs. JOHNSON and MATTHEY for the purpose, in order that the thermal expansion should be uniform throughout its whole extent. It consists of a solid horizontal table 7 millims. thick, circular in shape, but with three short equi-distant projecting arms, through which are threaded the three screws, of very fine pitch, each 35 millims. long, upon the bluntly-pointed upper ends of which the glass cover-wedge is to be supported. The distance between the centres of any two adjacent screws is 30 millims. The screws are provided, near their lower rounded ends, with milled

heads for convenience of rotation; and each of the arms is slit from one side, the slit passing through the centre of the screw hole and slightly beyond the latter, and provided with a small clamping screw, which draws the two sides of the slit together, binding the main vertical screw and thus enabling it to be firmly fixed in any position. Both sides of the table are polished, and on one side three concentric series of raised points are provided, three equi-distant ones in each series, and each series increasing in height as the centre is receded from. These serve as rests for objects of different sizes.

### *The Compensation.*

It is an interesting fact that pure aluminium expands 2.6 times as much as platinum-iridium for the same increment of temperature, and it is this fact which the author has utilised for the purpose of effecting compensation for the expansion of the platinum-iridium screws. A circular disc of aluminium, whose thickness is about ten twenty-sixths of the length of the parts of the platinum-iridium screws projecting through and above the table, is laid upon the latter. Provided adequate care has been taken to adjust the screws to the length calculated for complete compensation, from a knowledge of the two coefficients of expansion, it will be evident that the expansion of the apparatus is entirely eliminated. The length of screw provided affords, after compensation, an available space of some millims. above the aluminium block in which to place the crystal or other substance to be investigated, and as this space remains unaltered by change of temperature, any alteration of the thickness of the air layer between the upper surface of the object and the lower surface of the glass disc laid upon the screws must be entirely due to the expansion of the crystal. Moreover, the relative positions of the aluminium block and the crystal may be interchanged, it being immaterial which is uppermost. For the case in which the crystal rests on the block, a series of circular aluminium blocks of 25 millims. diameter and of respectively 12, 10, 8, 6 and 4 millims. thickness are provided, suitable for use with all the various sizes of crystals which are likely to be met with. The thinnest one is shown in position in fig. 1, resting upon the raised points. They were constructed out of a homogeneous casting of the purest aluminium obtainable. It is only necessary to select the block which is most suitable for use in connection with the particular size of crystal to be employed, and to set the screws to the proper corresponding length, which has been calculated, verified experimentally, and recorded in a table once for all.

The polish which many crystalline solids take upon their ground plane surfaces is rarely of a character equal to that of glass, and it is particularly desirable that the two reflecting surfaces relevant to the interference should reflect as equally as possible, in order that the interspaces between the bands shall be as dark as possible. Moreover, it is likewise desirable that the lower reflecting surface should be fairly extended, in order to afford an ample field of interference bands; and that a crystal



should not have to be rejected as unsuitable because either one or both of the parallel horizontal faces, perpendicular to which the expansion is to be measured, is, relatively to the other dimensions of the crystal, small. For these reasons the author usually lays upon the crystal a circular plate of glass, sufficiently thick to be adequately rigid, the lower surface of which is ground dull and the upper polished and worked perfectly plane. The glass employed is black astronomical glass, in order that no disturbance may arise from an illuminated lower surface, as would be more or less the case with colourless glass. The expansion of this glass, which has been accurately determined, is very nearly the same as that of platinum-iridium, so that the very slightly corrected measure of the thickness of the plate has simply to be added to the length of the screws calculated for compensation by the aluminium, and the screws set to this total length. Three series of such discs, of respectively 25, 15, and 10 millims. diameter, have been provided for use with crystals of greater or less lateral extension; for it is unwise to employ the 25-millim. disc with crystals of small diameter, a columnar type placed on end for instance, and the smaller ones still provide an adequate field of bands. One of medium size is shown resting on a crystal in fig. 1. Each series consists of three plates, varying from 2 to 4 millims. in thickness, so that there is ample choice afforded. Such a choice is preferably made as will leave, when the crystal and compensator are in position, an air space, between the upper surface of the glass plate and the lower surface of the large disc resting on the screws, of less than half a millimetre, in order that brilliant interference bands may be obtained. The size mostly employed by the author with crystals of artificial salts is 10 millims. diameter and 2 millims. in thickness.

[The second method of using the compensator, namely, above the crystal, is the one which the author prefers to employ whenever possible. Three series of aluminium discs are provided for use with this method, of respectively 15, 10, and 6 millims. diameter. Each disc is polished as truly plane as possible on one surface, intended to be the upper surface when in position on the crystal, and carries on the other side three equi-distant points, the ends of miniature screws of the same aluminium firmly screwed into the disc. Each series consists of two discs, the thicknesses of which, including the points, are respectively 4 and 5 millims. One of these compensators is shown resting on the ground in fig. 1. By this method the lower surface of the crystal rests directly on the three points of the platinum-iridium table, the particular three varying with the size of the crystal but being usually the innermost three, and the aluminium compensator rests by its three points upon the upper surface of the crystal. Hence any movement due to very slight convexity of the surfaces, or possible rolling upon a speck of dust included between the surfaces, is entirely avoided by this arrangement of three-point contact only. Moreover, the calculations are simplified and sources of error reduced by avoiding the use of a glass crystal-covering disc. In this connection it is particularly fortunate that aluminium does not take a very high polish, but one which

happens to reflect light about equally with the lower surface of the glass cover-wedge laid on the screws, and thus affords most excellent bands. The smallest compensators, of 6 millims. diameter, still exhibit a field of about eight bands of normal width, ample for the purpose of the observations; the points of these compensators are only 5 millims. apart, so that the method is applicable down to crystals of only slightly greater extent of surface than is adequate for these points to rest upon.—May, 1898.]

### *The Cover-Wedge.*

The thick glass disc of 40 millims. diameter, which is placed over the screws of the tripod, and whose lower plane surface is to form the upper of the two surfaces relevant to the production of interference, is not lenticular, as employed by FIZEAU and by BENOIT, but is possessed also of an upper plane surface, as used by ABBE and by PULFRICH, in order that parallel rectilinear bands and not curved ones may be produced. The surfaces of this plate are not precisely parallel, but are inclined at an angle of thirty-five minutes, forming in reality a wedge of extremely small angle. By this device the reflection of the illuminating light from the upper surface, which would otherwise illuminate the interspaces between the bands, is deflected out of the field of vision.

### *General Arrangement of the Dilatometer.*

The interference apparatus which has now been described is supported during the observations in a manner similar in principle to that employed by ABBE, but differing in the constructive details; and the heating apparatus is an air bath of special construction instead of an oil bath. Moreover, the interference tripod rests on non-conducting material in direct contact with the heated air, whose temperature is measured by the bath thermometers, and its actual temperature is determined by a third thermometer, so bent that the bulb is in direct contact with it. Further, this portion of the dilatometer, which may be termed the expansion apparatus, is, unlike the ABBE arrangement, separated from the illuminating and observing apparatus, which latter is removed to a considerable distance in order to be well out of the range of the heated atmosphere in the neighbourhood of the air bath.

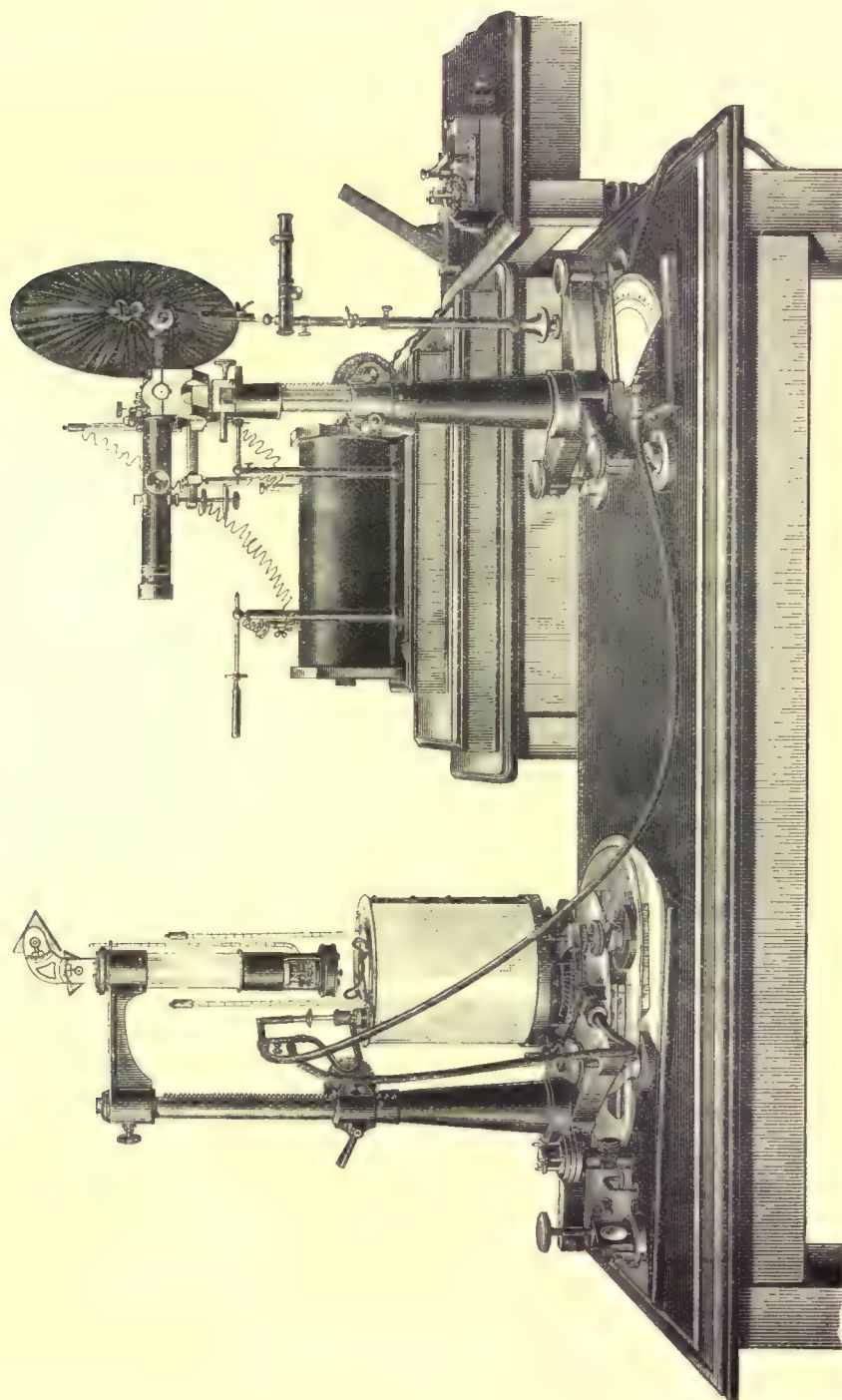
A general view of the dilatometer and its accessories is given in fig. 2. The two parts, the expansion apparatus on the left, and the illuminating and observing apparatus on the right, are mounted at the two ends of a rigid slate table six feet long.

### *The Expansion Apparatus.*

The details of the expansion apparatus will be more readily understood with the aid of the section, fig. 3. The tripod, *a*, stands upon a thick circular glass plate, *b*, of 50 millims. diameter, which forms the floor of the interference chamber. This



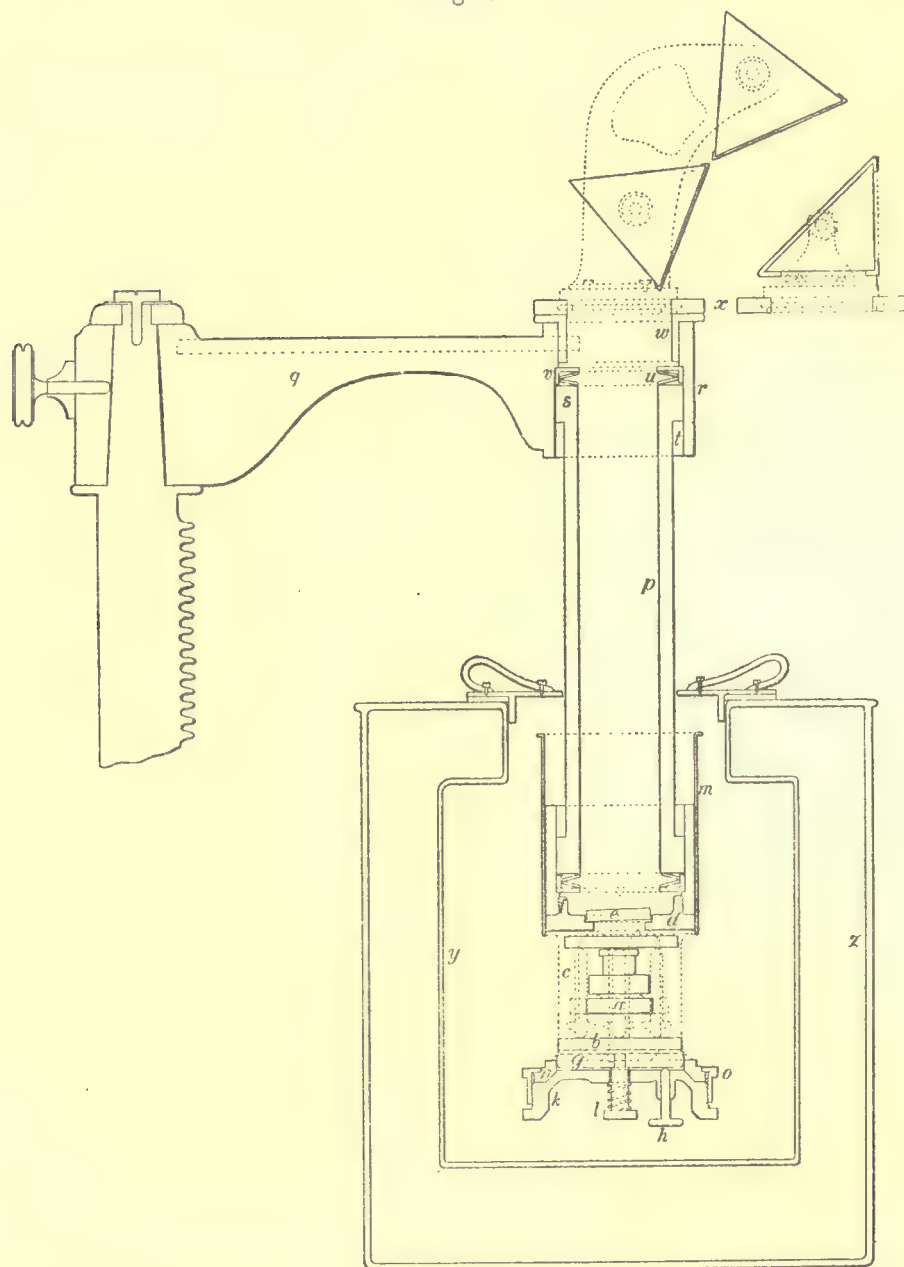
Fig. 2.





chamber consists of a wide gun-metal tube, *c*, very little longer than its diameter, and the cylindrical wall of which is so considerably cut away in three equi-distant parts as to form three relatively large windows separated by three equi-distant

Fig. 3.



pillars adequate to maintain rigidity. It is closed at the top by a stout diaphragm, *d*, whose aperture is filled by a circular glass plate, *e*, 27 millims. in diameter, which serves the double object of acting as a non-conducting roof to the interference chamber, to prevent upward radiation and consequent loss of heat into the supporting

tube, and of correcting the slight prismatic deviation of the light rays introduced by making the cover-wedge slightly wedge-shaped. The correction is achieved by making the disc, *e*, equally wedge-shaped. In fact, the two discs are cut from the same piece of plate glass, which has been ground down so that the two perfectly plane surfaces are 35 minutes out of parallelism. The direction of the wedge is marked by a straight line engraved parallel to the long edge of the slab before cutting the discs from it. The latter are so cut that this engraved line comes near the margin of each, forming a chord of the circle. It is only necessary to place the two discs so that the engraved chords are parallel and on the same side of the centre, but so that the marked surface of the larger disc of the interference apparatus lies uppermost while that of the smaller diaphragm disc lies underneath, when the refractive effect is fully counteracted. In order that no reflections shall be visible due to the two surfaces of the counteracting diaphragm wedge, the latter is slightly tilted by means of two very small screws driven through the rabbet of the aperture from the under side in such positions that the line joining them is parallel to the engraved chord of the wedge; when the disc is laid in its rabbeted aperture it rests upon the two screw points and a third equi-distant point on the surface of the rabbet, the lines joining the three points of contact thus forming an equilateral triangle. The slight tilt given to the disc when the screws are properly adjusted is sufficient to throw the reflected rays so far out of the optical axis of the system as to be no longer visible through the micrometer eyepiece.

The chamber terminates above the diaphragm roof in a screw thread, *f*, which engages with one on the lower end of the supporting apparatus. Below, the chamber ends in an adjustable table, *g*, upon which rests a disc of asbestos millboard, which, in turn, supports the glass floor disc. The adjustment is effected by three milled-headed screws, *h*, worked from underneath, which raise or lower the table above the rigid gun-metal base, *k*, of the chamber through which the screws pass. This rigid base is also perforated by a central hole, through which passes a short cylindrical rod fixed to the under side of the adjustable table; the rod terminates in a boss, *l*, between which and the under side of the fixed base a spring is confined, which causes the adjustable table to be always pulled firmly down upon the ends of the three screws. The chamber can be closed when desirable, in a light-tight manner, by a concentric outer tube, *m*, capable of sliding over it from above, with sufficient friction to maintain it up when the chamber is required to be open during adjustment of the interference apparatus. The author removes it altogether when the interference chamber is immersed in the air bath for the purpose of observations at the higher temperatures. The rigid base of the chamber and its adjustable table may be readily detached from the chamber if required, but the windows are sufficiently large to admit the tripod without this necessity arising merely for that purpose. To provide for its occasional desirability, however, the lower end of the chamber wall terminates in an annulus and flange, *n*, which rest upon a corresponding rabbet cut



in the rigid base; rotation is prevented by a slot and pin. The rigid base and the flange are bound together by means of a large milled nut, *o*, which has above an inner flange projecting over the chamber flange just referred to, and whose thread engages with one turned in the periphery of the base.

The interference chamber is supported at the lower end of a non-conducting tube of Berlin porcelain, *p*, supported from an arm, *q*, carried by a stout gun-metal columnar pedestal with three-legged base, provided with levelling screws resting on toe-plates. The column is provided with a strongly-made vertical adjustment by rack and pinion, worked by a milled ebonite disc, and provided with an arresting clamping screw, manipulated by a lever to securely fix it at the required height. Sufficient rack is given to enable the interference chamber to be raised above the height of the air bath, and conversely to be readily lowered into the latter. Particular care has been taken to render the working of the rack smooth and rigid, in order that no movement of the adjusted interference apparatus shall occur during the operation. The arm is rotatable about the upper part of the column, and can be fixed when desirable by a clamping screw. The outward end of the arm carries, as part of the same casting, a short thick-walled tube, *r*, of the same diameter as the interference chamber; at the lower end of this tube the porcelain tube is supported. In order to be prepared for the possible occasional fracture of a porcelain tube during heating, half-a-dozen such tubes were specially prepared, from the author's pattern, at the Berlin porcelain works. They are glazed outside and biscuit within, and have each a flange at either end; by subsequent trimming in the lathe the flanges have all been reduced exactly to the same size, so that a fractured tube can be readily replaced by another. This is also facilitated by the method of suspension, which is likewise one that renders fracture very improbable, as it admits of freedom of expansion. The flange, *s*, passes easily up the metal supporting tube carried by the arm, and after its insertion the two halves of a collar, *t*, are passed up the metal tube under the flange until they are flush with the lower end of the tube, when they are secured by screws passing through the tube and collar from the outside. In order that the porcelain tube, thus easily suspended, may be sufficiently rigid for the purpose in view, slight pressure is brought to bear on the flange from above by a pair of circular bent springs, *u*, confined between the flange and another internal collar, *v*, in the upper part of the metal tube. The lower flange of the porcelain tube is similarly attached to a short metal tube, which carries inside, at its lower end, an adequately long screw thread, which engages with the one at the upper end of the interference chamber, by which means the latter is attached.

The incident light rays are directed into the expansion apparatus from the illuminating apparatus, and the reflected rays back into the latter (which, being arranged for autocollimation, serves also as observing apparatus) by means of one of two interchangeable pieces of deflecting apparatus. The first is a single total reflecting prism; this is employed during the preliminary adjustment of the interference apparatus, with the aid of ordinary white light, and also when it is only desired to produce the



interference bands by monochromatic flames. The second is a train of two flint glass refracting prisms, which is employed for the production and observation of the bands produced, with the aid of a Geissler tube containing rarefied hydrogen and mercury, by the monochromatic light corresponding to the red and greenish-blue hydrogen lines, C and F of the spectrum, and to the green mercury line found so suitable by PULFRICH. The interchangeability of the two is very simply attained. In the top of the tubular termination of the arm, above the internal collar, there fits a very much shorter tube, *w*, terminating above in a stout flange; it is capable of rotating, for the purpose of azimuth adjustment, in the main tube, without the possibility of vertical motion, by means of an annular groove, into which project a couple of screws driven through the outer tube from opposite sides. The flange, whose central aperture is of slightly greater diameter than that of the porcelain tube, carries at two opposite sides raised and grooved guides, in which are capable of sliding the correspondingly rabbeted basal annular supports, *x*, of the single or double prism arrangements.

The single reflecting prism is supported in suitable bearings by central axles rigidly attached to the metallic case, and provided with milled bosses at the ends for convenience of rotating the prism; this enables the latter to be adjusted for altitude, and as the axles fit fairly tightly in the bearings, the setting remains unaltered after adjustment.

The pair of refracting prisms are mounted in a light framework, whose shape is calculated to ensure rigidity. Each prism is cased in metal, and the case carries centrally at its two sides short but thick axles, narrowing somewhat after a distance equal to the thickness of the mount, and terminating in a longer screw thread. The axles gear at their thickest part in corresponding bearings in the supporting framework. Over the two projecting axial screws on one side of the mount are passed in each case first an arm carrying at its end a silver arc as if for a vernier, and which slips over the screw and narrower portion of the axle only and is arrested at the thickening, and then a milled nut engaging with the thread, by which the arm may be clamped firmly to the axle and hence to the prism. As the thick part of the axle projects a little beyond its bearing, however, the arm is not clamped to the mount. The silver arc carries a central indicating mark, and by rotation of the milled head and the prism travels over a divided silver quadrant carried by the mount, and thus the prism may be set to any angular reading which may at any subsequent time be reproduced. The axial screws at the other side of the mount receive in each case first a washer and then a milled nut similar to that on the other side; the thick part of the axle being in this case slightly shorter than the thickness of the frame, the prism is clamped firmly to the mount on screwing home the nut, and so the prism may be rigidly fixed to the indicated circle reading. The refracting angle of each of these prisms is approximately  $57^\circ$ , which in the case of the particular flint glass employed gives a total minimum deviation due to the two of  $90^\circ$  for the

middle part of the spectrum, and of about  $88^{\circ}$  for the red hydrogen line C, and  $92^{\circ}$  for the greenish blue hydrogen line F.

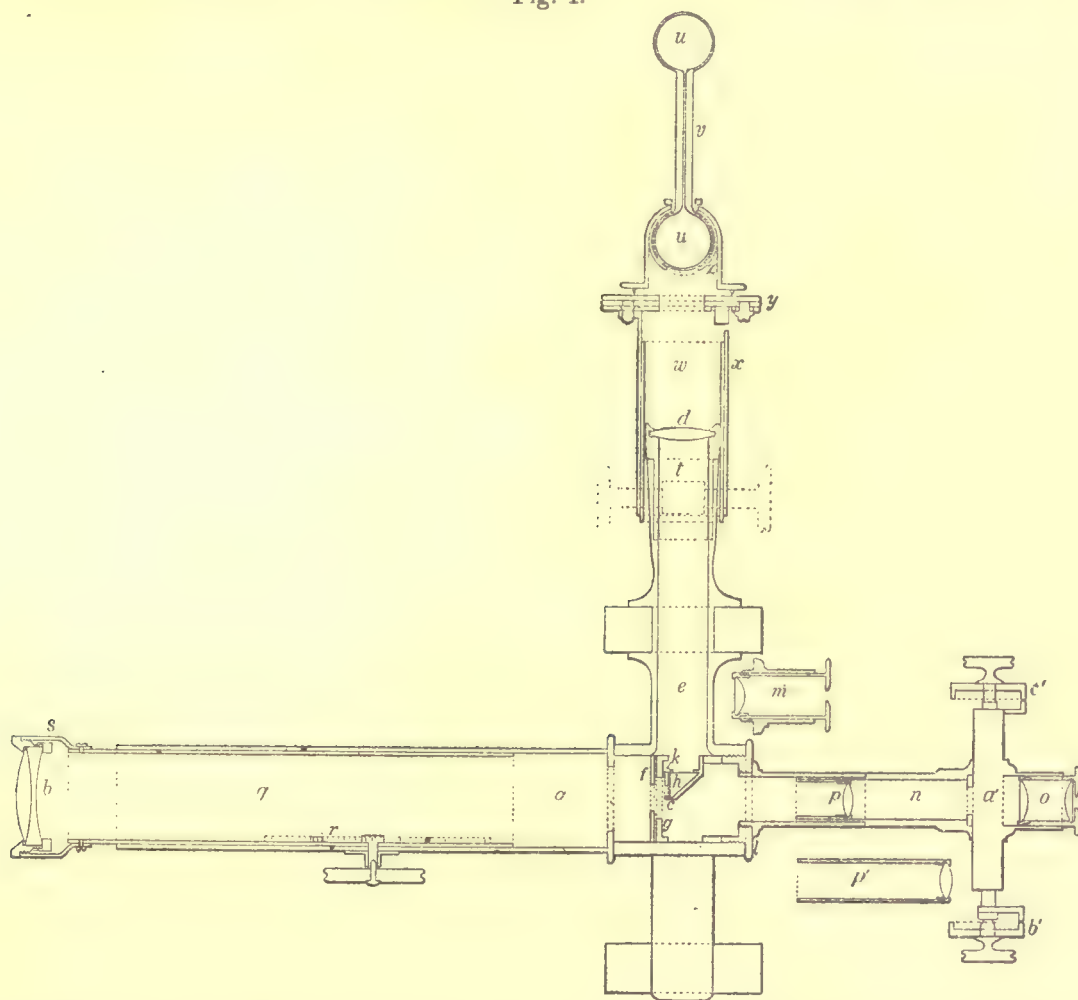
The heating apparatus consists of a double-chambered cylindrical air bath, mounted on a stout annular support resting by means of three levelling screws upon toe-plates also carried by the movable slab. During adjusting operations, and observations at the temperature of the room, the bath is removed from its stand. It is constructed throughout of brazed copper, and is covered outside with a thick coat of asbestos cloth. The inner chamber, *y*, is similar in shape to the outer, *z*, from which it is separated all round by the air space of about 4 centims. Within this inner bath the interference chamber of the dilatometer is to be lowered, by means of the rack and pinion, for observations at higher temperatures. The opening in the top of the bath is necessarily a little larger than the widest part of the interference chamber, the flange of the rigid base, *k* (fig. 3). The inner bath is supported within the outer one by the tubular wall of the opening,  $3\frac{1}{2}$  centims. deep, which connects the top of the outer with the top of the inner bath. The top of this tubular opening is closed, when the interference chamber has been lowered into position, with the two overlapping halves of an asbestos-lined and closely-fitting lid, which encircles the porcelain tube of the dilatometer, and is provided with suitable handles above and a deep rim beneath, which latter passes down into the tubular neck sufficiently far to obviate any appreciable diffusion of cold air from outside. Through a wide tubular opening at the top and near the side of the bath there is inserted into the outer bath, for the purpose of enabling the temperature within the bath to be regulated with constancy, a Muenke thermostat, provided with a regulator to control the size of the flame of the gas-burner when the mercury in the thermostat closes the orifice in the steel gas-supply tube. There are also two short tubes, on opposite sides of the centre, which pass air-tight through the tops of both baths into the inner one, for the passage of two thermometers. Both the thermostat and the thermometers are suspended free of the metal by non-conducting stoppers, and the thermometers are arranged so that the bulbs are on either side of and in close proximity to the interference tripod. A third thermometer, of such special construction that its bulb lies in actual contact with the tripod, passes through a hole in the front half of the lid of the bath. Details concerning the thermometers will be given at a later stage. A twenty-jet Fletcher ring gas-burner is employed as source of heat, the size of the jets being controlled by the thermostat, which has been adjusted so as to obtain a constant upper temperature, as near the particular desired limit as possible. Of very great utility in attaining constant temperatures has been found a gas-tap provided with a long pointed lever arm, travelling over a graduated quadrant, divided to read directly to degrees. By its aid the temperature can be very nicely regulated, leaving only such little further regulation to the thermostat as is rendered necessary by the very slight variation in the pressure of the gas supplied from a meter furnished with a governor.



*Illuminating and Observing Apparatus.*

The illuminating and observing apparatus consists of an autocollimating telescope, mounted separately upon a stout pedestal similar to that which supports the expansion apparatus, provided likewise with a very strong rack and pinion vertical adjustment, in order to enable the telescope to be adjusted to the height required by the particular deflecting arrangement mounted on the expansion apparatus. The

Fig. 4.



construction of the optical system will be more clearly evident from the section given in fig. 4. Another view of the apparatus is given in fig. 5, so as to exhibit more clearly the illuminating arrangement.

The main optical tube,  $\alpha$ , carries at the end furthest from the observer an achromatic objective,  $b$ , whose focus is at  $c$ , which is likewise the focus of the rays from the illuminating lens,  $d$ , carried at the end of the side tube,  $e$ . At  $c$  is an iris diaphragm,  $f$ , immediately behind which are a fixed diaphragm,  $g$ , pierced by a relatively large



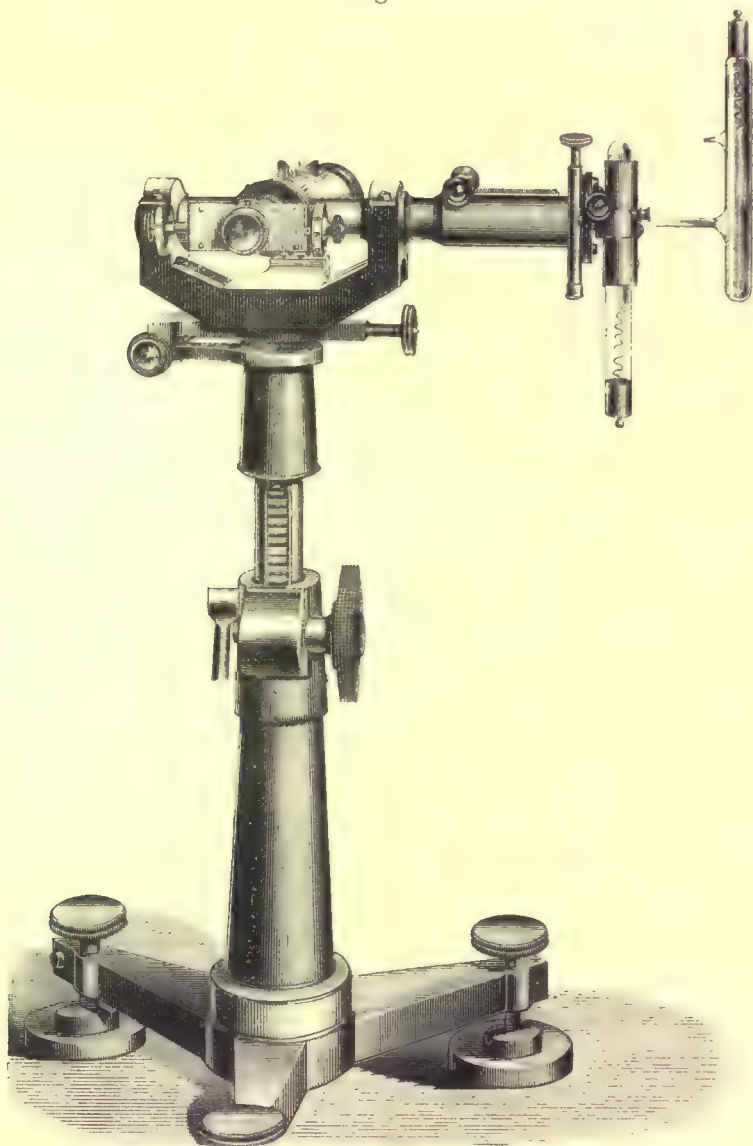
aperture, and a totally-reflecting prism,  $h$ , covering one-half of the aperture and so arranged that the rays from the illuminating lens,  $d$ , are reflected through this half of the aperture towards the objective,  $b$ , which latter they are capable of filling with light. The surface of the prism nearest the objective is intended to be masked by one or other of a series of interchangeable stops,  $k$ , having larger or smaller rectangular apertures. A short distance after the prism comes a stout milled flange,  $l$ , to the thread within the aperture of which either of two interchangeable observing optical arrangements is capable of being attached. One,  $m$ , is an ordinary eyepiece, for use in adjusting the interference apparatus by observation of the images of the rectangular signal aperture,  $k$ , in front of the reflecting prism, reflected from the various reflecting surfaces of that apparatus. The other,  $n$ , is an optical combination of a micrometer two-lens eyepiece,  $o$ , with a third movable lens,  $p$ . The system is such that when the lens  $p$  is at that end of its path nearest the prism, the reflecting surfaces of the interference apparatus and the interference bands which they exhibit are clearly focussed to the observer looking through the eyepiece  $o$ , when the expansion apparatus is approached very near to the observing apparatus; and when  $p$  is at the end of its range nearest to the eyepiece  $o$ , the same result is achieved for the separation of the two parts of the apparatus by about five feet. There is no necessity for a separate adjusting apparatus, as used by PULFRICH, for when the first of the two conditions just referred to obtains, the two parts of the apparatus are so close together that it is quite easy to manipulate the screws of the interference tripod while actually observing. The relatively large windows of the interference chamber, as distinguished from PULFRICH's closed chamber, greatly facilitate this.

For convenience in moving the expansion apparatus between the two positions, the toe-plates of its pedestal do not, like those of the telescope, rest directly upon the long slate table upon which the instrument is mounted, but on a smaller rigid slab of hard mahogany, lined underneath with thick baize, and furnished with a couple of stout brass handles.

In addition to the vertical rack and pinion motion provided in the pedestal of the telescope, the latter can be adjusted for altitude by a tilting arrangement. It is mounted about trunnions and the axis of rotation is made coincident with that of the illuminating side tube, which latter forms a prolongation of the hollow axle on one side. This is convenient when employing a source of light which does not move with the apparatus as does the Geissler tube, the illuminating centre remaining in the prolongation of the axis of the side tube whatever the tilt of the telescope. The latter may be laid in the bearings with the illuminating tube on either side according to convenience; it is only necessary to remove the two screws which in each case secure the caps of the bearings in order to effect a change of side. The telescope is suspended sufficiently high above the transverse part of the bearing casting to admit of considerable inclination from the horizontal position, and the depression or elevation is effected by a milled-headed screw which passes through a rigid arm radiating

from and forming part of the bearing casting, and which presses upwards on rotation against a V-piece attached to the optical tube. Two such V-pieces are provided, on the under and upper sides of the tube. It is convenient to have a means of recording the proper tilt of the telescope corresponding to the adjustment of each of the spectral lines (and therefore of the corresponding signal images and interference

Fig. 5.



bands), whose light is to be employed in the observations. This is delicately achieved by placing a sharp edged flange above the milled head of the tilting screw and employing it as the indicator of a vertical scale, divided on silver to half millimetres, and suspended from the horizontal arm which carries the screw.

The method of suspension is by a slider whose rectangular aperture is slightly larger



than the dimensions of the arm at this point, affording means of adjustment by four small adjusting screws driven through the top of the slider, whose points are pulled tightly down upon the arm by two larger milled-headed binding screws passing through the under side of the slider, one on each side of the scale. By means of these screws the scale is readily adjusted to its proper position with respect to the indicating flange edge. The scale is 60 millims. long, this distance corresponding to the traverse by the telescope of rather more than the whole length of the spectrum. Hence a delicate means is afforded of setting the telescope to the proper tilt for the use of either C or F hydrogen light, or that corresponding to the green mercury line, without actually observing the signal image for that colour at every observation. It is only necessary to ascertain once for all, with occasional verifications, the scale readings corresponding to these colours.

The bearing casting of the telescope carries below a vertical conical axis, which rotates within a corresponding hollow cone or vertical bearing, thus admitting of rotation of the telescope in the horizontal plane; the greater portion of this vertical conical bearing passes down into the wide internal boring of the pedestal, and it is rigidly attached to the latter by a screw thread turned on the thickened upper part of its cylindrical exterior and engaging with one in the upper part of the boring of the pedestal. The telescope is prevented from rising in the vertical bearing by a screw driven into the lower end of the solid cone, and a washer broad enough to cover also the end of the bearing. This vertical bearing carries at the top as part of the same casting, immediately above the thread by which it is attached to the pedestal, a broad collar, passing on one side into an arm, intended for use in connection with a fine adjustment fitting placed immediately above it. This latter takes the form of another collar passing round the thickened upper end of the conical axis and continued on two opposite sides into arms. That on one side is short, and through it passes a milled-headed clamping screw to attach the fitting rigidly to the cone, and hence to the telescope; the opposite arm is of the same length as the fixed one carried by the vertical bearing, and its end carries below two elbow pieces, forming two vertical claws, between which the narrower end of the fixed arm passes, with the necessary amount of free play within which the fine adjustment can be effected. The end of the fixed arm is pressed between the fine adjustment screw passing through one claw, and a spring piston carried by the other. Hence, in order to effect fine adjustment for azimuth, it is only needful to tighten the clamping screw, and then to manipulate the fine adjusting screw.

The telescope is provided with a rack and pinion adjustment for the objective, which is of 4 centims. aperture, and is placed at the end of an inner tube,  $q$ , which carries the rack,  $r$ . This enables the objective to be brought exactly to the focus of the signal aperture of the reflecting prism. There are also provided the means of slightly tilting the objective, in order to throw the troublesome reflection from the first concave lens surface just behind the diaphragm of the micrometer, and hence



out of the field of view. This is achieved partly in the mount of the achromatic combination itself, and partly as a final adjustment, by screwing the mount, not to the draw tube itself, but to a cap or short tube, *s*, slipping so easily over the latter that when a pair of screws are passed through both at opposite ends of a horizontal diameter, the cap is free to tilt slightly about them as an axis. By means of another pair of screws at the ends of a vertical diameter, passing through the cap tube only, the latter may be fixed at the slight tilt required. The objective cap is further provided, for use when desirable, with an outer lengthening tube not shown in the figures, projecting outwards 4 centims. beyond the objective, in order to shield it from any extraneous side light.

The reflecting prism, *h*, is cased in metal on the hypotenuse side towards the eyepiece. Between the glass surface facing the objective and the fixed diaphragm there is just room to insert, through a niche cut in the outer tube, any one of the series of interchangeable stops, *k*, suitable guides being provided to prevent actual contact with the glass during the insertion or withdrawal. The stop is in its proper position when it has been pushed home, as far as a handle at the top will permit. The rectangular aperture is placed as near one edge as possible, and when the stop is in position this edge is almost identical with the vertical diameter of the large circular diaphragm aperture, and consequently with the common edge of the semicircular free aperture and the prism; also the centre of the stop lies on the horizontal diameter of the circular aperture. Hence the axis of the bundle of rays incident from the rectangular signal aperture is as nearly as possible identical with the optical axis, and if the rays reflected from the interference apparatus are brought to a focus in the free aperture, so that the image of the signal is almost in contact with this same vertical diameter, the two paths are practically identical, and the best conditions for interference attained. The apertures of the stops vary from 2 millims. by 1 millim. to 3 by 5 millims., and two series are provided, one with the longest sides of the apertures horizontal, and the others with them vertical; the former are naturally preferable for use with the refracting deflection apparatus of the dilatometer. The smallest size is, of course, preferable, provided the source of light is sufficiently powerful. With the author's Geissler tube the illumination of the bands is many times more powerful for C or F light than it is when using a sodium flame. An additional stop is also provided, which is furnished with a means of varying the vertical height of the aperture, and which is intended for use as a fine slit. At that position in the metallic strip, which will be in the horizontal diameter of the field of the telescope when the stop is in position, a rectangular aperture of 4 millims. width is commenced, and continued upwards as far as and right through the handle. This long aperture runs, of course, parallel to the edges of the strip, and very close to the edge on the side which is to be nearest to the optical axis when in position. Through the handle a movable strip, the same width as the aperture, but longer, so as to project through the handle, passes, adequately guided by miniature guides in the

edges of the aperture and corresponding bevels on the slider, the lower horizontal end edge of which forms the upper jaw of the slit. Both this edge and that of the main strip which forms the lower end of the aperture and acts as the lower jaw of the slit, are bevelled. The top of the movable slider projecting above the handle is furnished with a smaller handle, for convenience in setting it so as to adjust the fineness of the slit. This slit enables the spectrum of the illuminating source of light to be studied. As the light traverses the two refracting prisms twice, the apparatus affords a dispersion equal to a spectroscope of four such prisms, and hence the study of the spectrum is highly interesting.

The iris diaphragm,  $f$ , is placed almost in contact with the fixed diaphragm,  $g$ ; it is manipulated from outside by means of a lever handle traversing a slot, upon the edge of which is a graduated silver arc, the indications of which, corresponding to definite apertures, are recorded by an indicator attached to the handle.

The lens of the illuminating tube is carried in a short sliding tube,  $t$ , so that the most favourable position for the illumination of the rectangular signal aperture can be attained. When white light is being employed for adjusting purposes, or when sodium light is being used for observations of the bands, this is all that is necessary. When a capillary Geissler tube for longitudinal vision is being employed as source of light, a further arrangement is needful for adjusting the capillary so as to attain the maximum illumination of the signal. The Geissler tube employed is of the H type recommended by PULFRICH, which consists of two wide terminal tubes,  $u$ , arranged vertically and fitted with aluminium spiral terminals, and a horizontal capillary connecting tube,  $v$ , the brilliant glow in which, regarded end on, is the source of light. The tubes, after Dr. RIEDEL'S pattern, were supplied by ZEISS. They contain a hydrogen vacuum, and a globule of mercury in one of the wide limbs, which is placed furthest from the lens, and on gently warming which the green light due to mercury vapour, corresponding to the green line of the mercury spectrum, makes its appearance. This particular radiation, when separated from the hydrogen radiations by the train of prisms of the expansion apparatus, forms the most effective monochromatic source of illumination for the production of interference bands when the two reflecting surfaces are at a considerable distance.

The supporting and adjusting arrangement for the Geissler tube affords the means of both centering and adjusting, so as to bring the capillary exactly into the prolongation of the optical axis of the illumination tube, and also for moving it longitudinally in this axis to the most suitable distance from the lens. It consists of a tube,  $w$ , which fits fairly tightly over the broad flange which forms the lens mount of the sliding tube; this flange is of slightly greater diameter than the illuminating tube itself, hence the latter is not injured by the placing in position and removal of this attachment. On the outside of this attached tube slides another,  $x$ , movable over it by means of a pinion gear carried at the end of the tube nearest the observer and travelling with it, which gears with a rack carried by the inner



attached tube. At the outer end of the movable tube is carried a three-disc centering apparatus,  $y$ ; the second and third discs are complementarily pivoted about points near their periphery, the second about the first, and the third about the second; they are maintained rigidly in position by similar pins on the other side of the centre, working in slots, and their movement is effected by a pair of screws and spring pistons, one carried by the first and another by the third disc, working against projections carried on each side of the middle disc and which are movable in larger slots, cut likewise in the first and third discs. The three discs are pierced by central apertures sufficiently large to admit the rays from the capillary to the lens. Attached to the third disc is a vertical tubular holder,  $z$ , for the Geissler tube, one limb of which slides easily in it, a slit being provided in its outer side for the passage of the capillary as far as the centre. Owing to the presence on this limb of some of the author's Geissler tubes of a little sealed side tube, which had been used for the purpose of the exhaustion, the sliding into position of the Geissler tube is brought about from below. When the capillary is raised as far as the slit permits, it is approximately central; a loop support is then attached below it to maintain it in position, by means of a pair of milled-headed screws, which secure the loop to the stout bracket which attaches the tube holder to the third centering disc. The inner side of the tubular holder has a central aperture corresponding to those of the three discs. The bracket is attached in such a manner to the third disc as to provide a means of adjustment for altitude, the claws, one on either side of the holder, not being secured by the pair of screws directly in contact with the disc, but being able to swivel more or less about a horizontal axis formed by a pair of short pins, one carried by each claw between the two screw holes and on the side which would otherwise be in contact with the disc. The slit in which the capillary slides being adequately large, adjustment for azimuth is readily attained. Hence every required adjustment of the capillary is provided for. In order that the Geissler tube may be firmly held, and also that the glass may not be in direct contact with the metal tube, a pair of caoutchouc rings of suitable thickness are introduced as packing, one near the top and one near the lower end of the holder. As the Geissler tube fitting requires to be frequently removed, and it cannot be laid down without the risk of mercury getting into the capillary, an inclined cloth-covered supporting pillar, on a suitable stand, is provided for its reception when not in use. The pillar is of slightly less diameter than the inner tube of the fitting, so as to pass inside it until arrested by the centering discs. The Geissler tube can thus be always left in position in the fitting. It is shown on the side table in fig. 2.

The milled flange,  $l$  (fig. 4), which completes the main optical tube, and to the aperture of which the observing eyepiece or micrometer combination is attached, has been constructed in duplicate, one of metal, for use whenever the Geissler tube is not necessary, and another of ebonite, for use in connection with the Geissler tube. For it is found that when the latter is employed, the relatively powerful electrical



induction discharge, corresponding to a 5-centim. spark in air, leaks slightly along the metal of the observing apparatus, and if no insulator is interposed, small sparks are discharged from the eyepiece to the observer's eyebrows. The thick ebonite disc acts as a perfect insulator, and prevents this disconcerting occurrence.

The ordinary eyepiece,  $m$ , slides in a short tube which is directly screwed to the flange,  $l$ , the amount of slide being adequate to enable the observer to sharply focus the vertical edge of the reflecting prism,  $h$ , which divides the iris diaphragm into two halves, a semicircle of light (clear aperture) and a dark half covered by the prism.

The micrometer optical combination,  $n$ , slides for about 3 centims. in a similar but slightly longer tube attachment to the flange, it is then prevented from sliding further by the thickening of the tube. This thickened part eventually passes into a still thicker one, which is directly attached to the micrometer box,  $a'$ ; in front of the latter is attached a short tube within which slides the Ramsden double eyepiece,  $o$ , which focusses the micrometer spider-lines. Three such eyepieces, of graduated magnifying power, are provided, suitable for all the widths of interference bands likely to be employed. As the movable lens  $p$  is only required in two positions, one corresponding to the close approximation of the expansion apparatus and the telescope, and the other to their removal to the two ends of the slate table, two inner sliding tubes are provided, of respectively 15 and 57 millims. length, each carrying at one end an inner thread corresponding with one on the lens mount. When the lens is attached to the shorter tube, and the latter is pushed in the long tube of the combination until the ends of the two tubes are flush with each other, the combination, when in position, focusses the bands for close quarters; when the longer tube,  $p'$ , is employed, and similarly pushed in until flush, the lens being then in the beginning of the thickest part of the long tube of the combination, the bands are focussed when the expansion apparatus and the telescope are separated at opposite ends of the table. If angle marks are placed on the table as indicators for the position of the corners of the movable slab upon which the expansion apparatus is mounted, corresponding to the sharpest focus under the two conditions, it is easy to bring the slab exactly to one or other of these positions.

The micrometer employed by the author is one in which the spider-lines are moved, not the eyepiece as in the Abbe apparatus. Moreover, it is a double one, carrying a divided drum on each side of the box. Each drum is divided into 100 parts, every 10 being figured. It may be set so that the zero corresponds exactly to any desired position of the spider-line or lines by means of a milled clamping boss carried at the end of the axle, which is here of square section; the aluminium drum itself, being on a round part of the axle, is free to move independently of the spider-lines until the boss is pushed firmly against it and fixed there by a screw driven into the axle, which does not extend so far as to be quite flush with the boss; the drum is then held firmly between the boss on the outside and

the thickening of the axle on the inner side. There are two movable vertical spider-lines, and a fixed horizontal one; below and parallel with the latter is the finely-serrated edge of a diaphragm, forming the lower boundary of the field. Every fifth niche of this edge is more deeply indented, and every tenth more still. One of the drums,  $b'$ , moves one only of the vertical spider-lines, and thus enables it to be set at any recorded distance from the other, or to measure in drum units the distance of any object in the field from the fixed line. The distance between any two teeth or niches of the diaphragm edge is equal to the distance traversed by the spider-line for one complete revolution of the drum. The other,  $c'$ , moves both vertical spider-lines simultaneously, the movement from one tooth or niche to the next corresponding to one complete revolution of the drum. Hence, each niche corresponds to 100 drum units, and each of the deepest niches to 1000. The micrometer thus affords every convenience for the measurement of the thickness of the bands and the precise location of any one with respect to any point on the horizontal spider-line, and will be equally suitable for any proposed method of procedure. It is particularly convenient for determining the position of a band by bringing it between the two parallel vertical spider-lines, which can be arranged at such a distance apart as is suitable for the purpose.

The fixed point with reference to which the position of the bands at the two limiting temperatures is to be measured, and the number of bands passing which during the change of temperature is to be ascertained, is of the kind adopted by PULFRICH, namely, the centre of a silvered circular spot in the middle of the lower surface of the glass wedge of the interference apparatus. The spot is about 1 millim. in diameter, and the author has removed the central portion so as to leave a silver ring, the inner circular edge of which is employed as the reference circle. Such a ring is readily made by silvering the whole surface by means of a milk-sugar ammoniacal silver solution, and removing the greater portion by ordinary means, and the portion about and within the ring with the aid of a rigidly supported needle and a microscopist's turn-table. The spider-lines are to be placed at such a distance apart as is just slightly smaller than the diameter of the inner silver circle. This enables the centre of the spot to be accurately located by setting the pair of spider-lines so that an equally small arc is cut off from the silver circle on each side. The inner portion is found by the author to be very advantageously left clear, as it enables a band passing the centre to be actually observed at the centre.

To illuminate the Geissler tube, the author employs a 15-centim. spark induction coil, actuated to the extent of a 5-centim. spark. A 100-volt direct supply current is used, filtered through five 32-candle-power lamps arranged in parallel circuit. As each lamp allows 1.2 amperes to pass, the current supplied to the primary coil is 6 amperes, and this current affords a very satisfactory and constant illumination of the tube.



*The Thermometers.*

The two thermometers, whose lower halves were immersed in the inner bath on each side of the interference chamber, were the two excellent instruments supplied by FUESS, with the most recent form of heating apparatus in connection with his largest gonio-spectrometer No. 1A. They had previously been tested several times, and twice during the course of this expansion work their zero and  $100^{\circ}$  points were again determined with great care. The zero of both remained exact throughout, and the correction for boiling-point never exceeded  $0^{\circ}\cdot 18$ . Experience has shown, however, that the temperature of the interior of the interference chamber never quite attains the temperature of the inner bath at the higher limits. The difference for the neighbourhood of  $70^{\circ}$  is about  $2^{\circ}$ , and for the higher limit of  $120^{\circ}$  it is rarely less than  $4^{\circ}$ . These bath thermometers were merely used, therefore, for the purpose of attaining and maintaining a constant temperature in the inner bath. For it is found that if the temperature of the inner bath can be maintained constant to  $0^{\circ}\cdot 2$ , which can readily be attained by the combined use of the graduated gas-tap and thermostat, the actual temperature of the tripod remains constant to well within  $0^{\circ}\cdot 1$ , and as it is capable of exact measurement, and the bands move precisely with the temperature of the tripod, all error of temperature disappears.

The measurement of the actual temperature of the tripod is attained by a third thermometer, specially constructed for the author by Messrs. NEGRETTI and ZAMBRA. This thermometer is bent at right angles just above the bulb, and it is so suspended alongside the interference tube that the small elbow carrying the bulb passes right into the interior of the interference chamber itself, without contact with any part of the walls of the chamber, and the cylindrical bulb rests in actual contact with the upper surface of the platinum-iridium table and one of the screws. Special care has been taken to determine its fixed points from time to time; its capillary was a specially selected one, and its zero and boiling points have required only the minutest corrections. It was so constructed that the  $70^{\circ}$  mark came just outside the bath, so that no correction for exposed stem is required for the lower limit. For the higher limit of  $120^{\circ}$  this correction is necessary, and to enable it to be made another much smaller thermometer was attached to the stem, with the bulb opposite to the  $90^{\circ}$  mark. That the indications of this thermometer actually express the temperature of the tripod and whatever it supports, is proved by the fact that the bands follow it exactly, their motion being arrested simultaneously with that of the mercury column. The importance of actual contact of the bulb of the thermometer with the interference tripod has been very exhaustively proved. Mere hanging of a straight thermometer alongside the chamber, as in the apparatus employed by PULFRICH, affords no guarantee, especially with a closed interference chamber, that the tripod and its contents actually attain the temperature indicated thereby. In the author's experience, even with an open chamber, it never does. The use of oil in the bath cannot



do more than slightly reduce the source of error, besides introducing an error due to the impossibility of stirring the oil, on account of the disturbance which a stirrer would cause to such highly delicate observations as are here in question.

### *Recording Apparatus.*

ABBE has shown that it is not necessary to actually follow and count the number of bands which pass the silver spot during the interval between the two limiting temperatures, it being possible to deduce the number by calculation from the data afforded by observations of the initial and final positions of the bands adjacent to the silver spot for light of two different wave-lengths. It is necessary, however, that these initial and final positions should be determined with the utmost precision. In addition to employing this method, which has been elaborately worked out by PULFRICH (*loc. cit.*), the author prefers to remove the slightest possibility of doubt as to the number of bands which pass the reference spot by actually following and counting them, as a mistake of a single band is highly important when the observations relate to the very slight differences of expansion between the members of a series of isomorphous salts. Moreover, it is far more satisfactory to observe the transit of the bands throughout, in order to be assured that it has been unbrokenly uniform, and that there has consequently been no derangement of the adjustment, or cracking, or irregular expansion of the crystal. In order that no error of counting may occur, a method has been adopted by means of which a permanent record of the passage of each band is obtained. A small recording apparatus has been constructed, of such a nature that, by pressing down a key with the finger the moment each band passes the silver spot, a puncture is made in a paper tape, and on releasing the key the tape is moved on a short space ready for the next puncture. When the bands become stationary again, on the attainment of the higher limit of temperature, the tape is cut off behind the last puncture, and thus a permanent record of the number of bands which have made their transit is obtained, which can be counted at leisure, and verified as frequently as may be desired.

The recorder is shown to the right on the accessory table in fig. 2. A drum, carrying a roll of Morse tape, is suspended on an axle inside a rectangular box, the front side of which is hinged below in order that the box may be thrown open, for the purpose of replenishing the drum. For this latter purpose the axle is fixed, and the drum provided with a corresponding central bore; the drum can consequently be drawn forwards right off the axle, and replenished, and is again exactly in position when pushed on the axle as far as it will go, a stout boss attached to a disc screwed to the back of the box arresting its further progress. In order to prevent the drum coming forwards more or less off the axle during working, which would pull the tape out of position, a spring is fixed to the inside of the front lid, and when the latter is closed and is fastened by the two hooks and pins provided

for the purpose, this spring presses lightly against the drum and maintains it in position. The tape comes up through a slot in the top of the box, and passes over a small roller raised on supporting bearings about half an inch above the surface. From this roller the tape passes horizontally through the punch and thence between a pair of caoutchouc rollers, which compress it sufficiently to be able to determine its movement when the lower driving roller is rotated. The upper roller is free to rotate on its axle simultaneously with the lower one, but in the opposite direction, by simple contact with the latter; its axle is adjustable for height in order to obtain the necessary amount of pressure between the two rollers, by means of two little milled-headed screws carried on the top of these supports; the screws press down upon rectangular sliders carrying the axle and sliding in slots cut in the upright supports. The axle of the lower roller is continued backwards just beyond the back of the box, where it carries, rigidly attached to it, a ratchet wheel. The movement of the latter is determined by a click, maintained in position against the teeth by a spring; the click is pivoted to and in front of a short lever, the fulcrum of which is the end of the roller axle, upon which it freely turns. The teeth of the ratchet are so arranged that upward movement of the lever and click causes the latter merely to pass round the tooth, while downward movement causes the click to force the ratchet round by one tooth. The lever is connected with a longer one by a short vertical link; the longer one is pivoted at the end of a prolongation of the drum axle, and its movement is directed by another vertical link, which is hinged to the back end of the punch lever. This latter is supported by two fulcrum upright supports, which form the bearings of the horizontal axle about which it is capable of movement. The longer front end terminates in a knob above and a cylindrical rod beneath; the latter passes slightly through a hole in the top of the box when the lever end is up at its normal height. The lever is kept up by a strong spiral spring wound round the rod and slightly compressed between the lever end and the top of the box. Between the knob and the fulcrum the lever carries below a thickening, directly over the punch head, not in contact with the latter, but separated from it by about 4 millims. The small cylindrical punch is provided at the top with a circular head carrying an annular groove, in which gear the two prongs of the forked end of a strong strip of steel spring. The spring is fastened at its other end to a suitable raised support, and the punch passes down, but not quite so far as the tape, through a guiding block provided with a corresponding vertical bore and fixed to the upper plate of the punch bed. Underneath this, leaving a space sufficiently thick for the passage of the tape, is the steel plate of the punch bed, whose sharp-edged circular aperture corresponds to the punch, which fits it sufficiently tightly. It will be at once evident that on pressing down the knob of the lever and thus compressing the spiral spring round the rod, after traversing the 4 millims. space the punch will be pressed through the tape, and the click of the ratchet will be slipped upwards round one tooth; on releasing the knob, the spring pushes it upwards again, but the click now drives the



ratchet round for the space of the tooth, thus causing the caoutchouc rollers to draw the tape onwards for a corresponding space. The size of the ratchet teeth is so arranged, relatively to the amount of movement given to the click, that the movement of the ratchet and the tape does not occur until the punch has been extracted from the tape.

#### *Reading Telescope.*

A further accessory is a small reading telescope, by means of which the thermometers can be watched by the observer without leaving his seat. This is mounted upon a standard provided with the usual means of adjustment for height, altitude, and azimuth. An inner sliding rod, capable of being arrested at any height by means of a clamping screw passing through the upper boss of the fixed standard, affords the means of adjusting the height of the telescope. Azimuth is arranged for by attaching the telescope to a short tube sliding over the lengthening rod, and fixable at any position by means of a slit and tightening collar carried at one end of the tube. Altitude is provided for by the means of attachment to the sliding tube just referred to, which carries a short projecting horizontal rod over which slips tightly a corresponding short, narrow, horizontal tube, carried by the wider collared tube through which the telescope is inserted; rotation of the narrow tube over the rod enables any desired tilt to be given to the telescope, or the latter to be arranged precisely horizontally with the aid of a spirit level laid along the optical tube. A further adjustment is attainable by means of a joint in the lengthening rod, which enables the telescope to be brought to the most convenient position for the eye when it is impossible to do so by moving the standard bodily. A strong clamping screw is provided with the joint. At the summit of the lengthening rod is carried a clamp which supports a large circular screen of dark-green silk, which serves the double purpose of protecting the observer's eyes from the glare of the vacuum tube, and interposing a broad non-conducting surface between him and the electrical connections of the tube. The whole arrangement is seen in position in fig. 2 (p. 318).

#### *Accessory Interference Apparatus for Observing the Fizeau Phenomenon.*

The "Fizeau phenomenon" of periodical secondary interference, produced when light not strictly homogeneous is employed for the generation of the bands, may be very conveniently observed with the aid of the little accessory to the interference apparatus which is shown resting on the movable slab of the expansion apparatus in fig. 2, and which enables the separation of the two reflecting surfaces to be varied at pleasure.

It consists of a special base similar to the ordinary one (*k* of fig. 3), and adjustable table (*g* of fig. 3); the latter carries three adjustable screws to act as an interference tripod for the support of the large glass cover-wedge, and a platform, adjustable for



height, for the support of the object whose upper horizontal surface is to act as the lower reflecting surface. The base has a much larger central aperture than the ordinary one (*k* of fig. 3) in order to permit the passage of a stout, hollow cylinder, rigidly carried below by the table; otherwise it is similar to the ordinary one, having three screws passing through it from beneath, pressing upwards against the table for the purpose of adjusting the latter, which is pulled firmly down upon them by a stout spiral spring, confined between the under side of the base, and a collar carried by the hollow cylinder. Within this hollow cylinder slides another, also very stout walled, and closed at the upper end, which is truly plane perpendicular to the axis and forms the vertically movable platform. The fit of the two cylinders is a very close one, and the movement is effected by a pinion, whose manipulating milled head gears in a rack sunk in the movable internal cylinder. It is imperative that there should be absolutely no alteration of the parallelism of the vertical axis during movement, as the minutest amount of such alteration is sufficient to entirely alter the position and width of the bands; hence the closeness of the fit and the relatively considerable length of the cylinders. In order that the amount of separation of the two surfaces may be accurately determined, the milled head of the pinion carries at its inner side a divided drum, reading directly to quarter millimetres with the aid of an indicator carried by the outer cylinder. The three tripod screws are adjustable for height by manipulation of the milled flanges carried some little distance below the pointed upper ends, which enable the screws to be more or less screwed into the fixed nuts carried on the surface of the table. The base carries a screw thread on its periphery, similar to that on the ordinary one, for attachment by means of the tapped flange to the lower end of the interference chamber.

#### *Thickness Measurer.*

The measurements of the thickness of the objects investigated, and of the various accessory aluminium blocks and glass discs employed in the work, are carried out by the author with the aid of an admirably accurate thickness measurer of the same type as that furnished by ZEISS, for use in connection with the Abbe dilatometer, and described by PULFRICH in the 'Zeitschrift für Instrumentenkunde,' 1892, p. 307. An improvement is introduced, however, into the method of suspending the counterpoised vertical silver scale, which does not form the continuation of the agate-pointed contact rod, but is suspended in front of the latter, in such a manner as to allow of a delicate means of adjusting the zero of the scale exactly to the centre of the pair of parallel horizontal spider-lines of the micrometer eyepiece of the microscope, when the agate point is resting upon the glass disc fixed in the base.

The instrument is shown in fig. 6, from which the nature of the improved method of suspension will be apparent. Round the rod, at the suitable height, a collar is fixed by a clamping screw; the collar is continued forwards and sideways into a bracket,

between the two arms of which the scale is indirectly suspended. The upper end of the scale slides tightly in a short jacket which is directly suspended from the arms of the bracket by two screw-pins. The jacket is pierced in front by a square aperture, through which passes a projecting piece, slightly less in size, screwed firmly to the scale; above and below the aperture the jacket carries two adjusting screws between which the projecting piece is firmly gripped, and which afford the desired means of adjustment.

Fig. 6.



A sliding piece, fixed to the scale half way down, but sliding round the rod, maintains the scale strictly parallel to the rod. The mode of counterpoising the latter so that the agate point may not exert undue pressure upon the object, is the same as in the Zeiss instrument. The scale is divided directly to fifths of millimetres, each division being consequently 0.2 millim. The drum of the micrometer is divided into 100 parts, the tens being figured as units. For one complete revolution the pair of horizontal spider-lines travel vertically through half a scale division, that is, 0.1 millim



The tenths of millimetres are also indicated by the teeth of a serrated diaphragm, which forms the left edge of the field of the micrometer eyepiece, and whose centre, corresponding to the zero of the scale, is specially indented. The figures on the drum thus represent the second place of decimals of millimetres; and the small divisions of the drum represent the third place. Hence the instrument reads directly to 0.001 millim.

#### ADJUSTMENT AND USE OF THE APPARATUS, AND DETERMINATION OF ITS CONSTANTS.

Before the apparatus can be employed for the determination of the coefficient of the absolute expansion of any substance, it is necessary to determine the expansion of the platinum-iridium alloy of which the interference tripod is constructed, in order that the expansion of the three screws of the tripod may be known. Having ascertained this, the expansion of any substance may be determined by FIZEAU'S method, provided a homogeneous specimen of the substance can be obtained in the form of a stout block, furnished with two parallel surfaces at least a centimetre apart. For the author's purpose of being able to determine the expansion of a substance with a block only half a centimetre thick, it is essential to proceed immediately with the determination of the expansion of the specimen of aluminium from which the series of compensators are cut. A block between 12 and 13 millims. thick was employed for this purpose. If the method is adopted of employing the compensator above the crystal, these are all the data required before being able to employ the apparatus directly for the determination of the absolute expansion of a crystal or other small object. If the compensator is employed below the crystal, a determination of the expansion of the glass of the small plates, to be used as covering plates with crystals whose surfaces take a duller polish than glass, also requires to be carried out. A block of the glass, 13 millims. thick, was similarly employed for this purpose. The operations of making these preliminary determinations of the constants of the interference apparatus will so fully include all the necessary manipulative and adjusting operations in connection with the use of the apparatus, that the description of the mode of conducting them, and the discussion of the mode of calculating the coefficients of expansion from the experimental results obtained, will at once be proceeded with.

#### *Determination of the Expansion of the Platinum-Iridium Interference Tripod.*

This is the most difficult of all the operations in connection with the work, on account of the fact that it is necessarily performed with the two reflecting surfaces, the under surface of the large glass cover-wedge laid upon the three screws and the upper surface of the platinum-iridium table, separated at the relatively long distance



of somewhat over a centimetre. At this distance it is most difficult to generate satisfactory bands with C or F hydrogen light, and it is only at all possible with sodium light provided the exact distance of a FIZEAU maximum is attained. The use of the light corresponding to the green mercury line, as recommended by PULFRICH, affords, fortunately however, excellent bands in the author's apparatus, even at distances considerably beyond 2 centims. At 12 millims., between glass surfaces, the bands are really admirable. A further difficulty, however, arose, owing to the greater volume of reflection from the polished plane surface of the platinum-iridium table than from the glass cover-wedge, the excess of light from the former illuminating the dark spaces and causing the bands to appear very faint. This difficulty was overcome by depositing upon the relevant glass surface, by means of a milk-sugar ammoniacal silver solution, a thin film of silver of the right density to afford the necessary increased reflection from the glass and diminished reflection from the platinum-iridium. After numerous attempts a film was obtained which caused the reflections to be almost exactly equal, and consequently enabled excellent bands to be obtained. The reference mark of the disc was a minute central ring, where the silver had been scraped off by a needle point whilst rotating the disc on a microscopist's turn-table. It should be remarked that this cover-wedge, employed for the purpose of the determination of the expansion of the tripod, was a duplicate of the ordinary one carrying the silver ring previously referred to. A corresponding duplicate smaller disc was employed in connection with it to close the upper aperture of the interference chamber, both being cut from the same slightly wedge-shaped slab of glass, as in the case of the ordinary pair. The two pairs, being of slightly different angle, were provided with distinguishing marks on their edges to prevent the use of a wrong combination.

The screws of the tripod were adjusted so that about 12 millims. length projected beyond that side of the table which carried no points, and the tripod was placed in the interference chamber with this side uppermost. The glass floor of the chamber had previously been made exactly horizontal with the aid of a circular spirit level and the adjusting screws. In order to prevent reflection of light from the marginal portion of the platinum-iridium surface, a lens stop of 15 millims. aperture and convenient diameter was laid upon it as a mask. The cover-wedge was then laid upon the screws with the silvered side downwards, and the engraved chord, marking the direction of the wedge, arranged perpendicular to an imaginary line at this height joining the pedestals of the expansion and the observing apparatus, that is, running right and left of the observer in front of the latter. The counteracting upper wedge was arranged with its chord in a parallel direction and on the same side of the centre to that of the cover-wedge, but below, while that of the cover-wedge was uppermost, on the non-silvered side.

The expansion apparatus was arranged close up to the observing apparatus for the adjustment, and carried the large single reflecting prism in position at its summit.

The heating bath had been temporarily removed. The telescope was fitted with the simple eyepiece and was arranged for parallel rays, the eyepiece focussing the vertical line of division of the iris aperture formed by the edge of the little reflecting prism. An incandescent electric lamp with ground glass globe, and provided with an opaque shade pierced by a circular aperture the same size as the illuminating lens, was brought close to the end of the side illuminating tube, the Geissler tube fitting having been removed. The telescope was then directed at the reflecting prism of the expansion apparatus, and the prism of the latter adjusted so that the images of the rectangular signal stop in front of the small reflecting prism of the telescope, reflected from the four glass surfaces of the two wedges and from the platinum-iridium table, could be viewed and any one brought into the centre of the clear half of the iris aperture by slight movement of the telescope. The iris aperture should be open to the full during this operation. By use of the rack and pinion of the telescope any slight want of sharpness of the images can immediately be corrected.

There are, in general, five such images from the interference chamber, two due to the two surfaces of the upper wedge, two to the cover-wedge, and one from the object, in this case the platinum-iridium table. They are easily recognisable. By slightly lifting and replacing the cover-wedge, the two due to that are distinguished by their temporary movement or disappearance. The angle of the wedge is such that the two images are so far apart in a horizontal line as to be just incapable of being simultaneously visible in the semicircular field of view. The pair of images due to the upper correcting wedge are equally distant from each other, but they should be about the same distance below the other pair as they are apart from each other, and a little to one side, owing to the suitable tilt given to the wedge by the two small tilting screws. These four images are thus arranged at the four corners of a rhombus.

The first thing is to distinguish which of the two cover-wedge images is afforded by the lower surface. The author always works with the thickest side of the wedge to the right, and this causes the image from the lower surface to appear to the right of the other. Hence, the image in question forms the right top corner of the rhombus. Moreover, if the lengths of the platinum-iridium screws had been adjusted to approximate equality, the image due to the object or tripod table would lie very near to this. The next thing is to adjust the image from the lower surface of the cover-wedge so as to lie symmetrically to the horizontal diameter of the field, and almost in contact with the vertical edge which divides the light half from the dark half of the field. This is achieved by movement of the telescope by the fine azimuth adjustment screw and the tilting screw. The next operation is to bring the object image exactly to the same height as the one just adjusted, that is, also symmetrical to the horizontal diameter. If this is not done exactly, the interference bands will be oblique to the vertical diameter of the field instead of parallel thereto, as is desired. To achieve it,



one of the screws of the tripod is manipulated, and if the tripod has been conveniently placed so that one screw is to the left side, in the diameter of the chamber floor at right angles to the imaginary line at this height joining the pedestals of the expansion apparatus and the telescope, the screw to rotate may be either the front or back one, at pleasure. It then only remains to manipulate the screw to the left, in order to bring the object image so as to more or less cover the adjusted image, according to the size of signal stop employed and the width of interference band desired. If the two images were absolutely coincident, the conditions would be those for the production of alternate fields of light and darkness, rather than bands, during alteration of temperature and the resultant alteration of the thickness of the air space between the two reflecting surfaces, which, under these conditions, would be precisely parallel. As parallelism is more and more deviated from, by manipulating the left screw so as to bring the object image to cover less and less of the cover-wedge image, the conditions are those for the production of bands, at first few and very wide, but becoming narrower and narrower, and proportionately more numerous, as the process of uncovering the first image proceeds. If an object were supported on the tripod, and the air wedge consequently thin, the bands would be immediately visible if the iris diaphragm were so far closed as to exclude all images but the double one, the simple eyepiece replaced by the micrometer combination, and the white source of light exchanged for a sodium flame.

In order to employ mercury light for the determination of the expansion of the screws of the tripod, or as in ordinary determinations, red or greenish blue hydrogen light, the single reflecting prism of the expansion apparatus is replaced by the pair of refracting prisms. These have previously been set for minimum deviation for the green mercury line, which occupies a convenient mean position in the spectrum, with the aid of the special slit signal stop. The source of white light is removed, and the Geissler tube and its fitting attached, and connected to the induction coil. The telescope is raised to the height of the upper refracting prism, and the induction coil set in action. Instead of the four simple images, the right top one a double one, there are now visible four corresponding vertical spectra, each showing sharp brilliant images of the signal stop in the red and greenish blue, corresponding to C and F hydrogen light, and a fainter one, so long as the tube is at the ordinary temperature, in the bright green, corresponding to the green mercury line. The reason for arranging the white light images as a rhombus, rather than as a square, is now apparent; for if the double image in white light had another image vertically below it in the field, in the Geissler tube light the spectrum of the lower image would partially overlap the double one due to the surfaces whose reflections are to produce bands, and the interference bands would be blurred by the illumination due to the undesired radiations. The angle of the rhombus is such that when the iris is arranged so as just to permit of the complete passage of the desired double image in any of the three colours, the other spectrum is entirely excluded.



The telescope is now to be so arranged for height, tilt, and azimuth that the double image in the desired colour is symmetrical to the horizontal diameter of the field, and almost in contact with the vertical line of demarcation between the light and dark halves of the field. The paths of the incident and reflected light are then almost identical, and the condition for maximum brilliancy of bands is attained. The micrometer combination, fitted with the movable lens carried in its shorter tube, is now attached in place of the simple eyepiece, and its eyepiece is withdrawn, to enable the observer to inspect the double image after the change and ascertain that it is still in position. For it conveniently happens that when the eyepiece of the micrometer is withdrawn, the movable lens left in position acts as a simple eyepiece, although not quite so well as the simple eyepiece supplied for the purpose, being further from the eye and not quite at the focus of the vertical edge of the semicircular field; it affords, however, images which are sufficiently good for the purpose of adjustment of their proper position with respect to the iris aperture, or of the relative position of the two overlapping images required to produce the bands.

On replacing the eyepiece in front of the micrometer, the interference bands are at once seen in the colour adjusted for. If an object is being used, and the air wedge is thin, the bands are exceedingly brilliant in red hydrogen C light, and only slightly less so in greenish blue F hydrogen light. In green mercury light they are faint at the ordinary temperature of the Geissler tube, but most brilliant if the capillary and the limb containing the mercury globule are gently warmed. The observer no longer sees a semicircular field, as with the simple eyepiece, but a full circular field, completely covered with bands if the object is as large as the diaphragm containing the upper glass wedge of the interference chamber. The semicircular opening, which is really very small when the iris is arranged as has been described, now simply acts as a diaphragm to cut off undesired light. The disturbing reflection from the inner surface of the objective, which would otherwise be very serious, is also thrown behind either the iris diaphragm, or the micrometer serrated diaphragm, by the device for slight tilting of the objective which has been described in the earlier portion of the memoir (p. 328).

The bands can be adjusted to the desired width by manipulation of the left-hand screw of the tripod, and any inclination from the vertical, as observed by reference to the pair of vertical spider-lines, can be corrected by slight rotation of the tripod, or, if preferred, of the micrometer itself within its outer carrying tube. The most suitable width is dependent upon the nature of the object. If its reflecting surface is truly plane, as proved by the severe test imposed by its use for the purpose under discussion, namely, by its generation of rectilinear bands at regular distances, the width of band chosen may be more considerable than when the bands are slightly curved or the distance is not quite regular, as the irregularities largely disappear when the bands are closer together, produced by an air wedge of greater angle. Usually a breadth of about 100 drum divisions, one complete rotation, is a con-

venient one. The two vertical spider-lines are then to be arranged at such a mutual distance as best facilitates the setting of a band to their centre, and if the width mentioned is chosen, the spider-lines are then also at the most suitable distance apart to enable the centre of the reference mark—the inner edge of the silver ring of the ordinary cover-wedge, or of the clear ring of the silvered cover-wedge used in the observations of the tripod expansion—to be located; in either case the desirable small segment of the circle is visible outside each spider-line when the ring is set for a drum-reading of its position. The horizontal spider-line is to be arranged as a tangent to the ring at the lower end of its vertical diameter. This is achieved by use of the rack and pinion of the pedestal.

The surface of the author's platinum-iridium tripod, truly plane to all ordinary tests, such as reflection of goniometer signals, proves to be very slightly concave, the bands in green mercury light being slightly curved, that is, arcs of large circles. They are perfectly regular, however, and move parallel (concentrically) to themselves when the temperature is altered, demonstrating the fortunate perfect equality of expansion of the whole of the parts of the tripod, and particularly of the screws, and enabling excellent determinations of the expansion to be made by observations of the number of bands passing the centre of the reference circle. A width of band of about 60 drum divisions was found most suitable in these determinations of the expansion of the tripod itself.

The preliminary adjustment having been carried out, the screws of the tripod are rigidly fixed by tightening the little clamping screws. The bent thermometer is then to be carefully attached. It is held near its upper end by a pair of loops at the ends of a circular brass spring collar, fitting just under the flange at the base of the prisms. This mode of support admits of sufficient play to enable the half of the lid, through a hole in which the thermometer is to pass, to be slid over the bent lower end and up the stem into position, and the half lid is supported by the upper end of the interference chamber while the bulb of the thermometer is carefully arranged in proper contact with the upper surface of the tripod table and one of the screws; the bulb may then be secured in this position by means of silk thread. After ascertaining that this operation has not impaired the adjustment, or correcting any slight derangement, the expansion apparatus is smoothly removed, by means of the handles of the cloth-lined slab upon which it is mounted, to its marked place at the further end of the slate table. The micrometer is detached, and its movable lens extracted, unscrewed from its short tube, and fitted to its longer tube, which is then pushed home in the tube of the micrometer until the ends are flush; the micrometer is then placed in position, being now in a condition to sharply focus the bands and reference mark for the new position of the expansion apparatus. The height of the telescope will require slight alteration, and a little further adjustment for azimuth may be necessary; removal of the front eyepiece will, as before, enable the double image to be properly adjusted. The height



adjustment is best achieved while the eyepiece is in position and the bands and reference circle in view, the rack and pinion of the pedestal being manipulated so as to again bring the lower margin of the little circle into apparent contact with the horizontal spider-line.

The heating bath may now be placed in position upon its annular support, which latter is always left *in situ*. To enable this to be done, the interference tube is gently raised, by the rigidly moving rack and pinion, to the necessary height, indicated by a mark on the sliding column. A large circular sheet of asbestos millboard, pierced by the necessary apertures, is intended to be laid on the top, and a large semi-cylindrical screen of the same material is placed between the bath and the pedestal for the full height of the latter. It is most convenient that the circular sheet should be arranged while fixing the bent thermometer, above the two halves of the lid, which are readily supported on the upper end of the interference chamber. Several circular sheets of asbestos are already laid beneath the Fletcher ring gas-burner, alternated with porcelain tiles. The interference tube is then lowered into the bath, another mark on the sliding column indicating the proper position, which, as adjustment had designedly been made with the column at this same height as indicated by the mark, will be the same as before. The two halves of the lid and the asbestos circle are left in position by the sinking of the interference tube, and the wide aperture necessarily left in the asbestos for the passage of the interference chamber is covered with two additional overlapping semicircles of asbestos. The thermostat and the bath thermometers may next be placed in position and the gas connections made. In order to protect the dispersing apparatus as much as possible during the heating a further circular screen of mica, arranged in two halves, attached by studs at each end and perforated by suitable apertures at the overlapping inner edges, is fixed about the height of the top of the porcelain tube. It is conveniently supported by the top of the semi-cylindrical screen and a couple of asbestos supports laid on each side of the front part of the asbestos circle, leaving ample space between for a clear view of the thermometers. By this means the temperature at the dispersion apparatus is only appreciably higher than that of the room when the thermometers are indicating the highest limit of  $120^{\circ}$ .

It should be here stated that the author was much troubled at first by the absorption or condensation of moisture in or on the copper walls of the inner bath. It is quite unnoticeable at the ordinary temperature, but when the temperature is raised and the transit of bands commences, the latter become suddenly obscured and the observation is lost, owing to the condensation of a film of distilled moisture on the glass wedges in the upper part of the interference chamber. This awkward circumstance necessitates the abandonment of the work for the day, as the apparatus requires to stand at least twelve hours to re-take the atmospheric temperature. It can be completely avoided, however, by placing on the floor of the bath, after each determination, so as to desiccate it before the next, a shallow dish of vitriol, and



leaving the interference tube immersed in the bath, so as also to share in the desiccation. It is a great advantage if the adjustment can be carried out the evening before a determination, as not only is there then ample time for the temperature of the room and apparatus to attain equilibrium, but also vitriol can then be left in the bath during the night, and removed just before the observations commence in the morning. For if the bath and interference apparatus are long exposed to moist air during adjusting operations, and the tripod subjected to much manipulation with the fingers, condensation again occurs to a greater or less extent which it is desirable to remove. The fixing of the mica screen should be left till this is carried out. The momentary raising of the interference apparatus from the bath in order to remove the vitriol dish serves, moreover, to enable the observer to note the reading of the inner bent thermometer, whose bulb is in contact with the tripod. This temperature is, of course, taken as the starting temperature in preference to that indicated by the bath thermometers, in case any slight difference is apparent.

Finally, before commencing actual observations, the bands are closely examined, and any slight alteration of the height of the reference ring above the horizontal spider-line corrected; also the eyepiece should be momentarily removed to satisfy oneself that the double image is still correctly placed. The observation of the position of the bands for the starting temperature can then be proceeded with.

The method adopted for determining the position of the bands, with reference to the centre of the silver ring, is that recommended by PULFRICH. Two things require to be ascertained. First, the distance between the centres of two adjacent dark bands, which may be termed the width of the bands; and second, the distance between the centre of the nearest dark band and the centre of the reference ring. The quotient of the second by the first is evidently the fraction of a band by which the centre of reference is distant from the nearest band. When the temperature is raised, and the bands move past the vertical spider-lines, the first dark band which comes between the pair of spider-lines is to be counted, not as one band, but as the fraction of a band thus determined, in case the movement of the nearest band occurs in the direction of the centre; and in the contrary case as the complementary fraction, that is, unity minus the determined fraction. The whole observation is completed by counting or otherwise determining the number of succeeding dark bands which pass during the interval until the temperature again becomes constant at the higher limit, and by then determining in a similar manner as for the lower limit the fraction of a band to be added, beyond the last which has passed the centre. Hence, the essentials of the determination are the whole number of bands which pass, and the initial and final fractions of a band to be added to this whole number; and the determination of each of these fractions involves the two operations previously referred to.

To determine the width of the bands the author measures the five whose middle one is immediately to the left of the centre of the little reference ring. Duplicate

readings of the micrometer drum, the one which moves both spider-lines simultaneously, are taken, corresponding to the setting of each of these five bands in turn, precisely midway between the two spider-lines. It is convenient to call these values  $a, b, c, d, e$ , letting  $a$  be the one most to the left; the values will then be in ascending order of drum divisions, each complete revolution of the drum being 100. In addition to the four direct values for the width of a band obtained by taking the differences between successive readings, a considerable number of other combinations can be obtained for the width from the five readings, and PULFRICH has shown that the whole of the values reduce eventually, converting his symbols into the author's, to the simple formula :

$$w = \frac{1}{8}(d + e) - (a + b).$$

To determine the distance of the middle band,  $c$ , from the reference point, it is only necessary to determine the true position of  $c$  by taking the arithmetical mean of the five readings, and to determine the drum reading of the reference point by taking the mean of two or three independent readings corresponding to the adjustment of the silver ring so that equal small arcs are visible outside each of the vertical spider-lines. The difference between these mean readings for the band  $c$  and the centre of the reference ring is evidently the distance  $\delta$  required. The fraction of a band corresponding to this temperature is, then, either  $\delta/w$  or  $\frac{w - \delta}{w}$ , according to the direction in which the bands are found to move on raising the temperature.

The Fletcher ring gas-burner is then ignited, the gas supply to it being regulated by the graduated lever tap, so that the temperature of the bath may not rise too rapidly. As has already been explained, the author prefers to actually observe the passage of the bands, and to count their number with the aid of the recording apparatus, rather than to rely exclusively upon the ABBE method of calculation from observation with two wave-lengths. Moreover, in the case of the determination of the expansion of the tripod, the bands are so very much clearer with green mercury light than with either C or F hydrogen light, or sodium light, at the necessarily long distance apart of the two reflecting surfaces, that the author prefers to base his determinations entirely upon observations in green mercury light. The recorder is placed upon the little cloth-covered accessory table to the observer's right, and as each dark band (after the first, which has the determined fractional value) advances to the middle of the two vertical spider-lines, the knob is pressed and the puncture made in the tape.

For a reason which will be fully discussed in considering the mode of calculation of the experimental results, the author makes two series of observations, the upper temperature limits of which are in the neighbourhood respectively of  $70^\circ$  and  $120^\circ$ .

The rise of temperature is so regulated, that fully an hour is occupied in attaining the neighbourhood of the first higher limit, about  $70^\circ$ , the transit of the bands occurring with more or less regularity during this interval. A further valuable



control of the number of bands passing the spider-lines is afforded if the observer notes the passage of each quarter of a band in his note-book. The author calls the quarters respectively 1, 2, 3, 4, and as the four successively make their transit, puts down their number on the same line; after 4 is so put down, the puncture is immediately made on the recorder, and a "p" added after the 4 in the note-book, to signify that the puncture has been made. If, also, the number of this particular band is added, an additional check is obtained. It may be mentioned that, in the case of the tripod determination, and all other cases where the bands are not so numerous as to pass with great rapidity, it is not necessary to keep the induction coil in action during the whole time; it is sufficient to run it intermittently in such a manner as to observe the passage of the four quarters of each band. This prevents overheating of the Geissler tube, and considerably prolongs its efficiency. The hydrogen spectrum rapidly deteriorates if the coil is worked continuously. A further period of two to three hours is then allowed for the attainment of equilibrium at this temperature, during the last hour and a half, at least, of which the temperature is maintained constant with the aid of the thermostat and the graduated gas-tap. It is found that if constancy to within  $0^{\circ}2$  is attained in the inner bath, as indicated by the outer two thermometers, a degree of constancy fairly easy to maintain, the actual temperature of the interference apparatus, as indicated by the inner bent thermometer whose bulb is in direct contact with the platinum-iridium tripod, remains absolutely constant. For the last ten degrees or so of the rise of temperature, the bands move precisely with the mercury column of this thermometer, and their motion ceases exactly with the attainment of constancy by it, a circumstance which is eminently satisfactory. When the last band to pass the vertical spider-lines for this temperature interval has been recorded, and no further movement of the bands is observable, the tape is drawn forward some distance, in order to interpose an interval between the last puncture for this temperature interval and the first of the next, and a series of measurements similar to those at the starting temperature are made, in order to determine the fraction of a band which has passed since the last transit.

When these measurements are completed, the gas supply is again increased so as to gradually raise the temperature to the highest limit, about  $120^{\circ}$ . The transit of bands is again observed and recorded, the same interval as before is allowed for the attainment of equilibrium and constancy, and finally, the determination of the last fraction of a band is made in the same manner as the two former determinations. The gas supply can then be turned off, and the apparatus allowed to cool. The tape is again drawn forward, and as the determination is finished, cut off. It is convenient to leave ample tape on each side of the punctures for a brief description of the determination to which it refers; also to add at each end of the row of punctures corresponding to each of the two temperature intervals the values of the fractions to be added. In the case of the first puncture of the second interval, this only



represents the complementary fraction to that to be added to the punctures of the first interval, so this fraction is set, by exception, over the top of this particular puncture rather than alongside it. It is convenient to actually make this puncture, because the total number of punctures made for the two sets of observations then represents the actual number of dark bands which have made their transit between the starting temperature and the highest limit.

Three quantities are the result of these observations :

(*a.*) The number of whole bands and fraction of a band which have passed the point of reference between the starting temperature and the first higher limit (about  $70^{\circ}$ ). This is given by the number of punctures for the first interval plus the initial and final fractions for that interval.

(*b.*) The number of whole bands and fraction of a band which have passed the reference point between the first higher limit and the highest limit (about  $120^{\circ}$ ). This is equal to the number of punctures for the second interval minus one, plus the initial and final fractions; or, which is the same thing, the number of punctures plus the final fraction for this interval minus the final fraction for the first interval.

(*c.*) The number of bands and fraction of a band which have made their transit past the reference point between the starting temperature and the highest limit. This is given by the sum of the total number of punctures and the initial and last fractions. Naturally,  $c = a + b$ .

If we retain FIZEAU's symbol,  $f$ , for the number of bands and fraction of a band passing the reference point during a given interval of temperature, the alteration of thickness of the air film denoted by the observations for that interval is approximately equal to  $f\frac{1}{2}\lambda$ , where  $\lambda$  is the wave-length of the particular light employed to generate the bands. The wave-length of the green mercury line is 0.0005460 millim. The word approximate is introduced because there is a correction to apply for the alteration of the refractive index of air, and consequent variation of the wave-length involved, due to change of temperature and pressure. The result of this alteration in the wave-length is that the number of bands actually observed to pass the fixed point is slightly different to the number which would have been observed if no change in the wave-length had occurred. This correction is very small, only a small fraction of a band, when the air film is thin, as is usually the case; but it is very considerable, amounting to a tenth of the whole value, when the thickness is relatively great, as it is during the determination of the expansion of the tripod. The nature and extent of this correction have been fully discussed by both BENOIT and PULFRICH (*loc. cit.*), and the formulæ arrived at by each of these observers lead to the same result. After employing both, the author finds the form of expression given by PULFRICH ('Zeits. für Instrumentenkunde,' 1893, p. 456), rather more convenient, as the values of several of the factors can be taken directly from LANDOLT's, 'Physikalisch-Chemische Tabellen.' The formula is as follows :—

$$f' = f + d(t_2 - t_1) \cdot \frac{b_1}{760} \cdot \frac{1}{1 + \alpha t_1} \cdot \frac{1}{1 + \alpha t_2} \cdot 2 \frac{(n-1)\alpha}{\lambda} - d(b_2 - b_1) \cdot \frac{1}{1 + \alpha t_2} \cdot 2 \frac{n-1}{760\lambda},$$

where  $f'$  is the corrected number of bands,  $f$  the observed number,  $d$  the thickness of the air layer,  $t_1$  and  $t_2$  the limiting temperatures, and  $b_1$  and  $b_2$  the corresponding barometric pressures;  $\alpha$  is the coefficient of expansion of air 0.00367, and  $n$  the refractive index of air for the wave-length  $\lambda$  of the light employed. The logarithmic values of  $2 \frac{(n-1)\alpha}{\lambda}$  and  $2 \frac{n-1}{760\lambda}$  can be found once for all; they are respectively 3.59901 and 3.15353. The logarithms of the factors  $\frac{b_1}{760}$ ,  $\frac{1}{1 + \alpha t_1}$ , and  $\frac{1}{1 + \alpha t_2}$  can be extracted directly from LANDOLT'S tables. Hence the expression, although apparently long and troublesome, lends itself to very easy computation. For the purpose of the calculation of this correction it is necessary to take the reading of the barometer at the time of making the observation of the exact position of the bands at the initial temperature, and again when they have attained their condition of rest at each of the two higher limits. These barometric observations are of particular importance in the case of the determination of the expansion of the tripod, as the share of the variation of pressure in the correction is then an appreciable one. The author employed a standard barometer suspended in the same room.

With regard to the sign of the correction, the signs given in the above formula for the temperature and pressure portions are the correct ones for use in all cases where the result of the increase of temperature is to effect an increase in the separation of the two reflecting surfaces, as in the case in question of the determination of the expansion of the tripod. In such cases the effect of change of temperature is to cause the number of observed bands to be less than it would be if such a change did not occur. The contrary is the case where the thickness of the air-layer diminishes, as in the cases of the determination of the expansion of an object supported on the tripod table whose expansion is greater than that of the screws, and in all determinations of expansion involving the use of the compensator. In such cases the signs of the temperature and pressure portions of the above formula should be respectively  $-$  and  $+$ .

The temperature portion of the correction is usually much the larger, and its sign governs that of the total correction. Indeed, in all cases when the air-layer is very thin, the share of pressure can be neglected. Moreover, as will be evident from the formula, the sign of the pressure portion varies according as  $b_1$  is less or greater than  $b_2$ . With regard to the sign of the more important temperature portion, the fact that it follows the rule just indicated will be evident from the following considerations. Rise of temperature causes a diminution in the refractive index of the air, and a corresponding increase in the wave-length. The same amount of separation of the surfaces will, therefore, contain fewer wave-lengths, and if no expansion of the tripod and object (if one is being used) occurred, the



observer would perceive the transit of the bands to the extent indicated by the correction, and in the same direction as the bands move when the two surfaces approach each other; so that the effect of rise of temperature on the refraction of the air is similar to that produced by approach of the two surfaces. Hence the correction requires to be added when the effect of rise of temperature is to cause recession of the surfaces, and to be subtracted when the surfaces approach each other.

After the conclusion of the observations of the number of bands which effect their transit for the two intervals of temperature, it is the author's practice to leave the apparatus untouched for about sixteen hours, during which time it regains the ordinary temperature of the air, and subsequently to repeat the whole series of operations and observations for two similar temperature intervals on the following day, and on as many succeeding days as it is desired to make independent determinations. After the completion of the series, and the final slow cooling of the apparatus during the night, the measurement of the exact length of the screws of the tripod above the reflecting surface of the table is made, with the aid of the thickness measurer. In the case of the determination of the expansion of the tripod, this is the only additional quantity required before being able to proceed with the calculation of the results.

The full data now available for this determination are as follows:—

$Lt_1$ , length of platinum-iridium screws at the initial temperature,  $t_1$ ; in this case  
 $= d$  thickness of air-layer.

$t_1$ , initial temperature;  $t_2$ , first higher limit, about  $70^\circ$ ;  $t_3$ , highest limit, about  $120^\circ$ .

$b_1, b_2, b_3$ , corresponding barometric readings.

$f_2$ , observed number of bands for interval  $t_2 - t_1$ .

$f_3$ , observed bands for interval  $t_3 - t_1$ .

By means of the correction formula and these data we first calculate  $f'_2$  and  $f'_3$ , the corrected number of bands for the two intervals. These quantities, when multiplied by  $\frac{1}{2}\lambda$ , in this case, for green mercury light, 0.000273 millim., afford the amounts of expansion of the screws, or alteration of thickness of the air-layer for the two intervals of temperature; and therefrom, by addition to the measured initial length,  $Lt_1$ , we obtain the lengths  $Lt_2$  and  $Lt_3$ , at the first and second higher limits (near  $70^\circ$  and  $120^\circ$ ) respectively. That is to say, the length of the platinum-iridium screws is now known at three temperatures in the neighbourhood of  $10^\circ$ ,  $70^\circ$ , and  $120^\circ$ .

The author's object in making observations for two different temperature intervals is to be able to determine not only the mean coefficient of expansion between two limits of temperature, but also the absolute coefficient of expansion at any given temperature, and the variation of the coefficient with change of temperature. For it is well known that, in general, the linear coefficient of expansion is not a constant quantity but varies slightly with the temperature; the increment of the coefficient per degree, however, remains practically constant. If, therefore, the



length of the solid substance at  $0^\circ$  is represented by  $L_0$ , and the length at  $t^\circ$  by  $Lt$ , the nature of the change of length is adequately represented by the formula

$$Lt = L_0 (1 + at + bt^2).$$

If  $\alpha$  be the absolute coefficient of linear expansion, then the constant increment per degree is  $\Delta\alpha/\Delta t$ , and their relations to the constants  $a$  and  $b$  of the above formula can be ascertained at once by successive differentiation with respect to the temperature.

$$\alpha = a + 2bt, \quad \Delta\alpha/\Delta t = 2b.$$

The mean coefficient of expansion between  $0^\circ$  and  $t^\circ$  is therefore  $a + bt$ ; but the true coefficient at any particular temperature  $t$ , and also the mean coefficient between any two temperatures whose mean is  $t$ , is  $a + 2bt$ .

Hence in order to be able to determine the true and mean coefficients, and the increment per degree, which together afford full information as to the nature of the expansion, it is only necessary to ascertain the constants  $a$  and  $b$  in the general expression above quoted for  $Lt$ .

In this formula  $Lt$  and  $t$  are known, and there are three unknown quantities,  $L_0$ ,  $a$ , and  $b$ . To determine them three equations, for three different temperatures, are required. The data derived from the observations at the ordinary temperature,  $t_1$  (about  $10^\circ$ ), at the first higher limit,  $t_2$  (about  $70^\circ$ ), and at the highest limit,  $t_3$  (about  $120^\circ$ ), enable the required three equations to be compiled. For  $Lt_1$  is the length measured at the ordinary temperature by means of the thickness measurer;  $Lt_2$  is  $Lt_1 + f'_2 \frac{1}{2} \lambda$ ; and  $Lt_3$  is  $Lt_1 + f'_3 \frac{1}{2} \lambda$ . We have then the three equations

$$Lt_1 = L_0 (1 + at_1 + bt_1^2),$$

$$Lt_2 = L_0 (1 + at_2 + bt_2^2),$$

$$Lt_3 = L_0 (1 + at_3 + bt_3^2).$$

By subtracting respectively the first from the second, and first from the third equations, we obtain a pair of equations which, by complementary multiplication with the coefficients of  $a$  and  $b$  respectively, enable each of these constants to be eliminated in turn and the other to be obtained in terms only of  $L_0$  and the known quantities. On substituting these values of  $a$  and  $b$  in any of the three fundamental equations  $L_0$  is at once obtained, and its substitution in the expressions for  $a$  and  $b$ , just referred to, affords the desired numerical values of these constants.

The three expressions for  $a$ ,  $b$ , and  $L_0$ , in the form actually employed by the author in the reductions, are as follows:

$$a = \frac{1}{L_0} \left[ \frac{(t_1 + t_3)(Lt_2 - Lt_1)}{(t_2 - t_1)(t_3 - t_2)} - \frac{(t_1 + t_2)(Lt_3 - Lt_1)}{(t_3 - t_1)(t_3 - t_2)} \right] = \frac{\theta}{L_0},$$

$$b = \frac{1}{L_0} \left[ \frac{Lt_3 - Lt_1}{(t_3 - t_1)(t_3 - t_2)} - \frac{Lt_2 - Lt_1}{(t_2 - t_1)(t_3 - t_2)} \right] = \frac{\phi}{L_0},$$

$$L_0 = Lt_1 - \theta t_1 - \phi t_1^2.$$

Although the expressions  $\theta$  and  $\phi$  appear long they are really very readily computed from the experimental data, as there are only three sets of quantities involved, the differences of the temperatures, the sums of the temperatures, and the differences of the lengths at these temperatures; also the two pairs of denominators are inversely identical.

The results of the observations connected with the determination of the expansion of the platinum-iridium tripod will now be given. Five independent series were carried out, in order that this fundamental constant might be ascertained with the utmost precision. With regard to the length of the screws of the tripod, given in the tables of results, it should be remarked that the actual lengths of the three screws are necessarily very slightly different, in order to produce the slightly wedge-shaped air-layer required for the generation of the bands. The length given is of course the mean length, and is the thickness of the air-layer at the centre, where the passage of bands is observed. In order to determine this mean length of the screws, besides direct measurement of each screw, which is somewhat difficult on account of the more or less pointed character of the screw-ends and of the agate end of the rod of the thickness measurer, recourse was had to the use of a plane parallel disc of thick glass, whose thickness was accurately known, and which was laid on the screws.

### *Expansion of Platinum-iridium Tripod.*

#### SERIES 1.

Length of screws above table,  $Lt_1 = 12.369$  millims.

Thickness of air-layer,  $d = 12.369$  millims.

Temperatures of observations,  $t_1 = 11^{\circ}.1$  C.,  $t_2 = 68^{\circ}.5$ ,  $t_3 = 117^{\circ}.8$ .

$$t_2 - t_1 = 57.4. \quad t_3 - t_1 = 106.7.$$

Barometric pressures,  $b_1 = 763.9$ ,  $b_2 = 765.2$ ,  $b_3 = 766.4$  millims.

Number of transited interference bands observed during the two intervals,

$$f_2 = 20.63, f_3 = 39.40.$$

Correction for alteration of refraction of air,  $+ 2.17$  and  $+ 3.52$ .

Corrected number of bands,  $f'_2 = 22.80$ ,  $f'_3 = 42.92$ .

Wave-length of light employed (green mercury line),  $\lambda = 0.0005460$  millim.

$$\frac{1}{2}\lambda = 0.000273 \text{ millim.}$$

Elongation of screws,  $Lt_2 - Lt_1 = f'_2 \frac{1}{2} \lambda = 0.0062243$  millim.

$Lt_3 - Lt_1 = f'_3 \frac{1}{2} \lambda = 0.011717$  millim.

$\theta = 0.000\ 106\ 22.$   $\phi = 0.000\ 000\ 027\ 918.$

$L_0 = 12.3678$  millims.

$a = 0.000\ 008\ 588\ 2.$   $b = 0.000\ 000\ 002\ 257\ 3.$

#### SERIES 2.

Length of screws above table,  $Lt_1 = 12.369 =$  thickness of air-layer,  $d.$

Temperatures,  $t_1 = 12^\circ.0$ ,  $t_2 = 70^\circ.0$ ,  $t_3 = 118^\circ.5.$

$t_2 - t_1 = 58.0.$   $t_3 - t_1 = 106.5.$

Pressures,  $b_1 = 763$ ,  $b_2 = 762.1$ ,  $b_3 = 761.6$  millims.

Number of transited bands,  $f_2 = 20.91$ ,  $f_3 = 39.44.$

Correction,  $+ 2.19$ ,  $+ 3.52.$

Corrected number of bands,  $f'_2 = 23.10$ ,  $f'_3 = 42.96.$

Wave-length of light employed,  $\lambda = 0.000546.$   $\frac{1}{2} \lambda = 0.000273.$

Elongation of screws,  $Lt_2 - Lt_1 = f'_2 \frac{1}{2} \lambda = 0.0063062.$

$Lt_3 - Lt_1 = f'_3 \frac{1}{2} \lambda = 0.011728.$

$\theta = 0.000\ 106\ 37.$   $\phi = 0.000\ 000\ 028\ 739.$

$L_0 = 12.3677.$

$a = 0.000\ 008\ 600\ 6.$   $b = 0.000\ 000\ 002\ 323\ 7.$

#### SERIES 3.

Length of screws above table,  $Lt_1 = 12.369 =$  thickness of air-layer,  $d.$

Temperatures,  $t_1 = 12^\circ.0$ ,  $t_2 = 70^\circ.3$ ,  $t_3 = 122^\circ.2.$

$t_2 - t_1 = 58.3.$   $t_3 - t_1 = 110.2.$

Pressures,  $b_1 = 761.0$ ,  $b_2 = 761.2$ ,  $b_3 = 761.9$  millims.

Number of transited bands,  $f_2 = 21.04$ ,  $f_3 = 40.97.$

Correction,  $+ 2.18$ ,  $+ 3.57.$

Corrected number of bands,  $f'_2 = 23.22$ ,  $f'_3 = 44.54.$

Wave-length of light employed,  $\lambda = 0.000546.$   $\frac{1}{2} \lambda = 0.000273.$

Elongation of screws,  $Lt_2 - Lt_1 = 0.006339.$   $Lt_3 - Lt_1 = 0.012159.$

$\theta = 0.000\ 106\ 18.$   $\phi = 0.000\ 000\ 031\ 011.$

$L_0 = 12.3677.$

$a = 0.000\ 008\ 585\ 2.$   $b = 0.000\ 000\ 002\ 507\ 4.$



## SERIES 4.

Length of screws above table,  $Lt_1 = 12.369 =$  thickness of air-layer,  $d$ .

Temperatures,  $t_1 = 10^{\circ}.8$ ,  $t_2 = 70^{\circ}.7$ ,  $t_3 = 121^{\circ}.1$ .

$$t_2 - t_1 = 59.9. \quad t_3 - t_1 = 110.3.$$

Pressures,  $b_1 = 761.5$ ,  $b_2 = 761.0$ ,  $b_3 = 759.7$  millims.

Number of transited bands,  $f_2 = 21.61$ ,  $f_3 = 40.78$ .

Correction,  $+ 2.26$ ,  $+ 3.65$ .

Corrected number of bands,  $f_2' = 23.87$ ,  $f_3' = 44.43$ .

Wave-length of light employed,  $\lambda = 0.000546$ .  $\frac{1}{2}\lambda = 0.000273$ .

Elongation of screws,  $Lt_2 - Lt_1 = 0.0065164$ .  $Lt_3 - Lt_1 = 0.012129$ .

$$\theta = 0.00010688. \quad \phi = 0.00000023394.$$

$$L_0 = 12.3678.$$

$$a = 0.0000086421. \quad b = 0.000000018915.$$

## SERIES 5.

Length of screws above table,  $Lt_1 = 12.369 =$  thickness of air-layer,  $d$ .

Temperatures,  $t_1 = 10^{\circ}.3$ ,  $t_2 = 68^{\circ}.1$ ,  $t_3 = 118^{\circ}.8$ .

$$t_2 - t_1 = 57.8. \quad t_3 - t_1 = 108.5.$$

Pressures,  $b_1 = 765.3$ ,  $b_2 = 766.0$ ,  $b_3 = 766.8$  millims.

Number of transited bands,  $f_2 = 20.78$ ,  $f_3 = 40.14$ .

Correction,  $+ 2.19$ ,  $+ 3.58$ .

Corrected number of bands,  $f_2' = 22.97$ ,  $f_3' = 43.72$ .

Wave-length of light employed,  $\lambda = 0.000546$ .  $\frac{1}{2}\lambda = 0.000273$ .

Elongation of screws,  $Lt_2 - Lt_1 = 0.0062708$ .  $Lt_3 - Lt_1 = 0.011935$ .

$$\theta = 0.00010615. \quad \phi = 0.00000029846.$$

$$L_0 = 12.3678.$$

$$a = 0.0000085828. \quad b = 0.000000024132.$$

The results derived from the five series are compared in the following table, and the mean values for  $a$  and  $b$  extracted:—

	$a.$	$b.$
Series 1 . . . .	0.0000085882	0.000000022573
" 2 . . . .	86006	23237
" 3 . . . .	85852	25074
" 4 . . . .	86421	18915
" 5 . . . .	85828	24132

2 z 2

Mean values,  $a = 0.000\,008\,599\,8$ ,  $b = 0.000\,000\,002\,278\,6$ .

The final result, therefore, for the mean coefficient of expansion  $a + bt$  of the platinum-iridium of the tripod, between  $0^\circ$  and  $t^\circ$ , is

$$0.000\,008\,600 + 0.000\,000\,002\,28t, \text{ or } 10^{-9}(8600 + 2.28t).$$

The true coefficient of expansion at  $t^\circ$ , or mean coefficient between any two temperatures whose mean is  $t$ ,  $\alpha = a + 2bt$ , is

$$\alpha = 0.000\,008\,600 + 0.000\,000\,004\,56t, \text{ or } = 10^{-9}(8600 + 4.56t).$$

It will be interesting to compare the value for the expansion of the specimen of platinum-iridium composing the author's tripod with the values found by BENOIT for various specimens of JOHNSON and MATTHEY 10 per cent. alloys of iridium with platinum. BENOIT's values are as follows; they refer to the mean coefficient between  $0^\circ$  and  $t^\circ$  :—

Specimen 1,	$10^{-9}(8615 + 2.21t)$ .
„ 2,	$10^{-9}(8593 + 2.40t)$ .
„ 3,	$10^{-9}(8598 + 2.22t)$ .
„ 4,	$10^{-9}(8575 + 2.38t)$ .
„ 5,	$10^{-9}(8575 + 2.54t)$ .

It will be observed that the author's value agrees most satisfactorily with these values of BENOIT. As the coefficient of iridium is much lower, namely,  $0.0000065$ , than that of platinum,  $0.0000089$ , a slight difference in the proportions of the two metals present in the alloy might be expected to produce just such slight variations in the coefficient as are observed.

#### DETERMINATION OF THE EXPANSION OF THE ALUMINIUM COMPENSATOR.

The aluminium compensating blocks were cut from the same casting of the purest obtainable aluminium. The thickest block of the 25 millim. series, slightly over 12 millims. thick, was used for the purpose of determining the expansion by the differential Fizeau method. The platinum-iridium tripod screws were preliminarily set to about 12.5 millims., on that side of the table which was furnished with raised points. The aluminium block was placed upon the table, resting on the three outer highest points. The tripod and block were then placed in the interference chamber, and the large glass cover-wedge provided with the minute central silver ring (not the cover-wedge silvered all over, which is used only for the tripod determinations) was placed over the screws. The corresponding counter-balancing wedge had previously been exchanged for the former one in the upper aperture of the chamber, and the engraved direction chords were, as before, arranged

parallel, but that of the cover-wedge above and that of the counteracting wedge below.

The adjustment of the upper surface of the aluminium, and of the lower surface of the cover-wedge, was carried out precisely as has been described for the tripod table and cover-wedge; the two parts of the apparatus were arranged at close quarters, and the images of the signal-stop obtained in white light with the aid of the single reflecting prism were brought to the desirable partially overlapping position, and isolated by the iris diaphragm from all other radiations. The single prism was then exchanged for the train of refracting prisms, the white light was replaced by red hydrogen light from the Geissler tube, and the simple eyepiece was replaced by the micrometer combination, when very brilliant bands were at once observed. After adjustment of the width of the bands and their parallelism to the vertical spider-lines, the platinum-iridium screws were fixed, the bent thermometer was arranged in position, and the expansion apparatus removed to its proper distant position. The interference tube was then raised, while the bath, containing its dish of vitriol, was arranged in position; the interference apparatus was then lowered into the bath, the thermometers fixed, and the whole left overnight to attain equilibrium of temperature.

Next morning the vitriol was removed, and the inner thermometer read while the interference apparatus was raised for this purpose; the barometric pressure was also taken, the adjustment of the bands for height effected, the silver ring being brought into contact at its lower limb with the horizontal spider-line, and the measurements of the positions of the centre of the silver ring and of the five adjacent bands were carried out in red hydrogen light. This light is particularly suitable, on account of the bands being separated at a greater distance than with the other available wavelengths, and also by reason of the particular brilliancy of this radiation afforded by the Riedel-Geissler tube at the ordinary temperature of the tube. The better of the two parallel surfaces of the aluminium block, which was placed uppermost for use, afforded bands which were strictly regular and almost perfectly rectilinear, yielding excellent measurements. The temperature was then raised to the neighbourhood of  $70^{\circ}$ , recording the transit of bands on the tape, and after a couple of hours' constancy measurements were made at this temperature. After the attainment of another interval, the passage of bands being recorded as before, a final set of measurements were made for a constant temperature in the neighbourhood of  $120^{\circ}$ .

After cooling overnight, a duplicate series of determinations for three similar temperatures were made next day.

Lastly, after cooling during another night, the exact lengths of the platinum-iridium screws projecting above the height of the points were measured by the aid of the thickness measurer; similarly, the known thickness of the aluminium block was verified.



After these duplicate determinations with the same screw-length, an independent series was taken on another day with a different setting of the screws.

Following are the results obtained :—

*Expansion of Aluminium.*

SERIES 1.

Thickness of aluminium block,  $Lt_1 = 12.188$ .

Length of screws above table points,  $l = 12.400$ .

Thickness of air-layer,  $d = 0.212$ .

Temperatures,  $t_1 = 10^\circ.1$ ,  $t_2 = 68^\circ.9$ ,  $t_3 = 123^\circ.4$ .

$$t_2 - t_1 = 58.8. \quad t_3 - t_1 = 113.3.$$

Pressures,  $b_1 = 754$ ,  $b_2 = 754.8$ ,  $b_3 = 756$  millims.

Number of transited bands,  $f_2 = 30.35$ ,  $f_3 = 60.32$ .

Correction for air refraction,  $-0.04$ ,  $-0.06$ .

Corrected number of bands,  $f'_2 = 30.31$ ,  $f'_3 = 60.26$ .

Wave-length of light employed (C hydrogen),  $\lambda = 0.0006562$ .  $\frac{1}{2}\lambda = 0.0003281$ .

Diminution of thickness of air-layer,  $f'_2 \frac{1}{2}\lambda = 0.0099449$ .  $f'_3 \frac{1}{2}\lambda = 0.019772$ .

Elongation of tripod screws = measured length of screws  $\times (a + 2bt$  for tripod alloy)  $\times$  temperature interval,

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0064074.$$

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.012510.$$

Expansion of aluminium block = diminution of thickness of air-layer + elongation of screws,

$$Lt_2 - Lt_1 = 0.0099449 + 0.0064074 = 0.0163523.$$

$$Lt_3 - Lt_1 = 0.019772 + 0.012510 = 0.032282.$$

$$\theta = 0.000\ 268\ 23. \quad \phi = 0.000\ 000\ 125\ 31.$$

$$L_0 = 12.1853.$$

$$a = 0.000\ 022\ 013. \quad b = 0.000\ 000\ 010\ 284.$$

SERIES 2.

Thickness of aluminium block,  $Lt_1 = 12.188$ .

Length of screws above table points,  $l = 12.400$ .

Thickness of air-layer,  $d = 0.212$ .

Temperatures,  $t_1 = 8^\circ.0$ ,  $t_2 = 70^\circ.0$ ,  $t_3 = 120^\circ.6$ .

$$t_2 - t_1 = 62.0. \quad t_3 - t_1 = 112.6.$$

Pressures,  $b_1 = 755.1$ ,  $b_2 = 753.7$ ,  $b_3 = 752.7$  millims.

Number of transited bands,  $f_2 = 32.20$ ,  $f_3 = 60.22$ .

Correction,  $-0.04$ ,  $-0.06$ .

Corrected number of bands,  $f_2' = 32.16$ ,  $f_3' = 60.16$ .

Wave-length of light employed,  $\lambda = 0.0006562$ .  $\frac{1}{2}\lambda = 0.0003281$ .

Diminution of thickness of air-layer,  $f_2' \frac{1}{2}\lambda = 0.0105520$ .  $f_3' \frac{1}{2}\lambda = 0.019739$ .

Elongation of tripod screws,

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0067481.$$

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.012417.$$

Expansion of aluminium block,

$$Lt_2 - Lt_1 = 0.010552 + 0.0067481 = 0.0173001.$$

$$Lt_3 - Lt_1 = 0.019739 + 0.012417 = 0.032156.$$

$\theta = 0.000\ 268\ 94$ .  $\phi = 0.000\ 000\ 129\ 37$ .

$L_0 = 12.1858$ .

$a = 0.000\ 022\ 070$ .  $b = 0.000\ 000\ 010\ 617$ .

### SERIES 3.

Thickness of aluminium block,  $Lt_1 = 12.188$ .

Length of screws above table points,  $l = 12.812$ .

Thickness of air-layer,  $d = 0.624$ .

Temperatures,  $t_1 = 9^{\circ}.8$ ,  $t_2 = 67^{\circ}.6$ ,  $t_3 = 118^{\circ}.7$ .

$$t_2 - t_1 = 57.8. \quad t_3 - t_1 = 108.9.$$

Pressures,  $b_1 = 772.6$ ,  $b_2 = 771.7$ ,  $b_3 = 771.9$  millims.

Number of transited bands,  $f_2 = 29.43$ ,  $f_3 = 57.18$ .

Correction,  $-0.11$ ,  $-0.18$ .

Corrected number of bands,  $f_2' = 29.32$ ,  $f_3' = 57.00$ .

Wave-length of light employed,  $\lambda = 0.0006562$ .  $\frac{1}{2}\lambda = 0.0003281$ .

Diminution of thickness of air-layer,  $f_2' \frac{1}{2}\lambda = 0.0096199$ .  $f_3' \frac{1}{2}\lambda = 0.018702$ .

Elongation of tripod screws,

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0064992.$$

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.012407.$$

Expansion of aluminium block,

$$Lt_2 - Lt_1 = 0.0096199 + 0.0064992 = 0.0161191.$$

$$Lt_3 - Lt_1 = 0.018702 + 0.012407 = 0.031109.$$

$\theta = 0.000\ 268\ 58$ .  $\phi = 0.000\ 000\ 132\ 75$ .

$L_0 = 12.1854$ .

$a = 0.000\ 022\ 041$ .  $b = 0.000\ 000\ 010\ 895$ .

The three results for aluminium are collated below :—

	<i>a.</i>	<i>b.</i>
Series 1 . . . .	0·000 022 013	0·000 000 010 284
„ 2 . . . .	22 070	10 617
„ 3 . . . .	22 041	10 895

Mean values,  $a = 0\cdot000\ 022\ 041$ .  $b = 0\cdot000\ 000\ 010\ 599$ .

The agreement of the results is very gratifying, and leaves no doubt as to the accuracy of the observations. The final result may be expressed as follows :

The mean coefficient of expansion  $a + bt$  of the pure aluminium of the compensators, between  $0^\circ$  and  $t^\circ$ , is

$$0\cdot000\ 022\ 04 + 0\cdot000\ 000\ 010\ 6t, \text{ or } 10^{-8} (2204 + 1\cdot06t).$$

The true coefficient of expansion at  $t^\circ$ , or the mean coefficient between any two temperatures whose mean is  $t$ ,  $\alpha = a + 2bt$ , is

$$\alpha = 0\cdot000\ 022\ 04 + 0\cdot000\ 000\ 021\ 2t, \text{ or } = 10^{-8} (2204 + 2\cdot12t).$$

The value given by FIZEAU for aluminium for  $40^\circ$  ('Compt. Rend.,' vol. 68, p. 1125; and 'Ann. Chim. Phys.,' [4], vol. 8, p. 335), is  $0\cdot000\ 023\ 13$ ; he further gives for the increment,  $\Delta\alpha/\Delta t = 2b$ , the value  $2\cdot29$ . This increment agrees fairly well with the author's value of  $2\cdot12$ ; and if the value for  $0^\circ$  is calculated by diminishing  $0\cdot000\ 023\ 13$  by 40 times the increment  $2\cdot29$ , the number  $0\cdot000\ 022\ 21$  is obtained, a figure which also agrees satisfactorily with the author's constant  $a$ . The coefficient for  $40^\circ$ , calculated from the author's formula, is  $0\cdot000\ 022\ 89$ .

#### DETERMINATION OF THE EXPANSION OF THE CRYSTAL-COVERING-GLASSES.

For the purpose of determining the expansion of the black glass of the covering-glasses employed with crystals whose surfaces cannot be made to take a polish equal to that of glass, a block of the same glass, 13 millims. thick, was procured. The mode of adjustment and observation was precisely similar to that for the aluminium block. Two series of determinations were made, employing red hydrogen light. In each case the platinum-iridium screws were adjusted so as to leave an air-layer about a quarter of a millimetre thick between the upper black glass surface and the lower surface of the large cover-wedge.

This glass proved to have an expansion coefficient slightly less than that of platinum-iridium, so that the converse was observed of what happens in the case of aluminium; that is to say, instead of a reduction of the thickness of the air-layer by



the approach of the two reflecting surfaces, a slight recession was found to occur. Moreover, as the metallic tripod was affected more rapidly by the rising temperature of the air bath, a recession for several bands was first observed, due to the expansion of the screws; the movement of the bands subsequently became arrested for a considerable interval of time, until, when the badly conducting glass eventually approached the same temperature as the tripod, one or two of the transited bands retraced their steps past the spider-lines, so that the actual number of bands eventually found to have effected their permanent transit was very small, corresponding to close similarity of expansion between the glass and the tripod.

The results obtained were as follows :

*Expansion of Black Glass.*

SERIES 1.

Thickness of black glass block,  $Lt_1 = 12.956$ .

Length of screws above table points,  $l = 13.120$ .

Thickness of air-layer,  $d = 0.164$ .

Temperatures,  $t_1 = 8^\circ.9$ ,  $t_2 = 68^\circ.9$ ,  $t_3 = 122^\circ.6$ .

$$t_2 - t_1 = 60.0. \quad t_3 - t_1 = 113.7.$$

Pressures,  $b_1 = 739.2$ ,  $b_2 = 739.2$ ,  $b_3 = 739.4$ .

Number of transited bands,  $f_2 = 2.83$ ,  $f_3 = 4.66$ .

Correction for air refraction,  $+ 0.03$ ,  $+ 0.05$ .

Corrected number of bands,  $f'_2 = 2.86$ ,  $f'_3 = 4.71$ .

Wave-length of light employed,  $\lambda = 0.0006562$ ,  $\frac{1}{2}\lambda = 0.0003281$ .

Increase of thickness of air-layer,  $f'_2 \frac{1}{2}\lambda = 0.00093838$ .  $f'_3 \frac{1}{2}\lambda = 0.0015454$ .

Elongation of tripod screws,

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0069092.$$

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.013275.$$

Expansion of black glass block,

$$Lt_2 - Lt_1 = 0.0069092 - 0.00093838 = 0.00597082.$$

$$Lt_3 - Lt_1 = 0.013275 - 0.0015454 = 0.0117296.$$

$$\theta = 0.000\ 094\ 226. \quad \phi = 0.000\ 000\ 067\ 939.$$

$$L_0 = 12.95514.$$

$$a = 0.000\ 007\ 273\ 3. \quad b = 0.000\ 000\ 005\ 244\ 2.$$

## SERIES 2.

Thickness of black glass block,  $Lt_1 = 12.956$ .

Length of screws above table points,  $l = 13.221$ .

Thickness of air-layer,  $d = 0.265$ .

Temperatures,  $t_1 = 6^\circ.0$ ,  $t_2 = 69^\circ.0$ ,  $t_3 = 119^\circ.1$ .

$$t_2 - t_1 = 63. \quad t_3 - t_1 = 113.1.$$

Pressures,  $b_1 = 743$ ,  $b_2 = 745$ ,  $b_3 = 746.5$ .

Number of transited bands,  $f_2 = 3.24$ ,  $f_3 = 5.19$ .

Correction,  $+ 0.05$ ,  $+ 0.08$ .

Corrected number of bands,  $f_2' = 3.29$ ,  $f_3' = 5.27$ .

Wave-length of light employed,  $\lambda = 0.0006562$ ,  $\frac{1}{2}\lambda = 0.0003281$ .

Increase of thickness of air-layer,  $f_2' \frac{1}{2}\lambda = 0.0010795$ .  $f_3' \frac{1}{2}\lambda = 0.0017291$ .

Elongation of screws,

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_2}{2} \right) \right] (t_2 - t_1) = 0.0073052.$$

$$l \left[ 10^{-9} \left( 8600 + 4.56 \frac{t_1 + t_3}{2} \right) \right] (t_3 - t_1) = 0.0132850.$$

Expansion of black glass block,

$$Lt_2 - Lt_1 = 0.0073052 - 0.0010795 = 0.0062257.$$

$$Lt_3 - Lt_1 = 0.0132850 - 0.0017291 = 0.0115559.$$

$\theta = 0.000\ 093\ 804$ .  $\phi = 0.000\ 000\ 066\ 894$ .

$L_0 = 12.95514$ .

$a = 0.000\ 007\ 240\ 7$ .  $b = 0.000\ 000\ 005\ 163\ 6$ .

The two results are compared below, and their mean extracted :—

	<i>a.</i>	<i>b.</i>
Series 1 . . . .	0 000 007 273 3	0.000 000 005 244 2
„ 2 . . . .	7 240 7	5 163 6

Mean values,  $a = 0.000\ 007\ 257\ 0$ .  $b = 0.000\ 000\ 005\ 203\ 9$ .

Hence the mean coefficient of expansion  $\alpha + bt$  of the black glass of the crystal-covering-discs, between  $0^\circ$  and  $t^\circ$  is

$$0.000\ 007\ 257 + 0.000\ 000\ 005\ 20t,$$

$$\text{or } 10^{-9}(7257 + 5.2t).$$

The true coefficient of expansion at  $t^\circ$ , or the mean coefficient between any two temperatures whose mean is  $t$ ,  $\alpha = a + 2bt$ , is

$$\alpha = 0.000\ 007\ 257 + 0.000\ 000\ 010\ 4t, \text{ or}$$

$$= 10^{-9}(7257 + 10.4t).$$

## THE DETERMINATION OF THE EXPANSION OF A CRYSTAL.

It only remains now to briefly indicate the mode of employing the compensator for the determination of the expansion of a parallel-faced block of a crystal or any other substance, whose thickness between the parallel plane surfaces need not exceed 5 millims.

In case the method is followed of employing the compensator below the crystal, one of the two thinnest aluminium compensating blocks, respectively 4 and 6 millims. thick, is employed, together with one of the 2 millims. thick black glass discs of 10 millims. diameter. The exact length of platinum-iridium screw which expands to the same amount as each of these, is readily found from the knowledge now available of the coefficient of expansion of each. Knowing, therefore, the length of screw whose expansion would be exactly counterbalanced by the added expansion of the particular aluminium compensator and crystal-covering-glass, the screws are to be set to this length above the raised points by the aid of the thickness measurer. As regards choice of compensator, that one is selected whose thickness, added to that of the covering-glass, will be less than the corresponding screw-length by an amount which is just slightly greater than the thickness of the crystal block, slightly greater in order to leave room for the necessary air film between the two reflecting surfaces.

The compensator is placed on the raised points of the tripod. A masking disc (lens stop), similar to that referred to in connection with the determinations of the tripod expansion, but of smaller aperture, is next laid on the compensator. Its aperture is slightly less than the diameter of the black covering-glass, so that it arrests all light other than that from the covering-glass. The crystal block is then laid with one of its two plane parallel surfaces resting on the centre of the aluminium block within the aperture of the mask, and the black crystal-covering-glass, polished face upwards, is placed on the top of the crystal. On then laying the large cover-wedge over the screws, its lower surface should be found to be separated by the desired air-layer from the polished surface of the black glass. The whole is then transferred to the interference chamber and the adjustment of the images from the reflecting surfaces proceeded with.

If the method of employing the compensator above the crystal is chosen, the latter is laid first on the three points of the tripod table, and on it is then laid one of the smaller compensators furnished with three points, these latter resting directly on the crystal. No covering-glass is required, the aluminium surface reflecting light adequately. The screw-length should be that calculated for complete compensation by the compensator chosen.

In effecting the adjustment of the lower reflecting surface by slight variation of the screws, it is preferable to do so in such a manner that any difference of screw-length introduced, from that which is compensated, should be an excess. This can



readily be done by making the adjustment by screwing further out for the required minute distance one, or, if necessary, two of the screws, rather than by slightly withdrawing the opposite screw. The final measurement of the actual screw-length, after the completion of the observation, affords the information required for the calculation of the exact amount uncompensated. This minute amount is most conveniently taken into account in the calculations in the form of a correction to the change of thickness of the air-layer, or, which is the same thing, amount of expansion or contraction of the crystal. If, as usual, the crystal expands with rise of temperature, the correction is positive; the effect of the slight apparent expansion of the screws being to reduce the observed amount of approach of the two reflecting surfaces; if the crystal is a contracting one along this particular direction, the correction requires to be subtracted, as the effect is to enhance the apparent contraction. The amount of the correction is the difference between the calculated amounts of expansion of the screws and the compensator; these are obtained by multiplying the measured screw-length and the thickness of the aluminium block respectively, by the now-ascertained mean coefficient of expansion of the tripod alloy and aluminium, for each of the two intervals of temperature, and by the number of degrees of rise of temperature, in each case.

The mode of carrying out the determinations of the number of bands which make their transit during the two intervals of temperature is precisely similar to the procedure which has been described for the cases of the determination of the expansion of the aluminium compensator and of the glass of the covering-discs.

In a subsequent memoir, the author hopes to present to the Royal Society the results of a series of determinations, carried out in this manner, of the thermal expansion of the crystals of the sulphates and selenates of potassium, rubidium, and caesium, an investigation which is now engaging his attention. Further details, therefore, as to the mode of carrying out determinations with crystals, and concerning the preparation of suitable crystal blocks, will be left for consideration in that memoir.

The dilatometer and its accessories have been constructed for the author in an altogether admirable manner by Messrs. TROUGHTON and SIMMS.

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d.

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properties of nitric acid corresponding to any particular concentration, adopting in

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outline the methods and arguments used by PICKERING\* in his investigations on sulphuric acid.

Pure acids and clean vessels are glib phrases, which represent unattainable ideals; we only trust that we have adopted reasonable precautions of purification, and that even if our determinations are not of absolute exactitude, yet the arguments deducible from them are not seriously affected.

Lastly, we desire to express our obligations to the Rev. F. JERVIS-SMITH, of Trinity College, Oxford; Mr. E. H. GRIFFITHS, of Sydney Sussex College, Cambridge, for assistance in the experimental portion; to Mr. R. T. GUNTHER, for photographs; to the Chemical Department of the University of Oxford, for loan of apparatus; to Mrs. V. H. VELEY, for certain illustrations; and to the Government Grant Committee of the Royal Society for moneys to defray a moiety of the expenses. The work was conducted partly in the Chemical Department of the University Museum, and partly in the laboratory of Magdalen College.

#### FORMER INVESTIGATIONS ON ELECTRIC CONDUCTIVITIES.

KOHLRAUSCH and GROTRIAN† determined the conductivity of dilute nitric acid of concentrations varying from 6.32 to 62.07 per cent., seven points in all, and at three different temperatures for each concentration, in order to deduce the values of the constants  $\alpha$  and  $\beta$  in the equation

$$K_t = k_0 (1 - \alpha_t + \beta_t^2).$$

These writers adopted their well-known method of alternating currents, but allowed that the acid used was impure, as they wrote:—"Die concentrirte Salpetersäure, ursprünglich farblos, hatte nach längerem Stehen eine hellgelbe Färbung angenommen, offenbar durch Bildung eines Spurs von salpetriger Säure unter dem Einfluss des Lichtes." Their investigations, however, brought out an interesting detail, namely, that the conductivity increases with the percentage concentration up to 30 to 33 per cent., and subsequently decreases; in the sequel this point will be more fully discussed.

OSTWALD,‡ adopting KOHLRAUSCH's telephone method with slight modifications, made a number of determinations of the molecular conductivity of nitric acid, diluted so as to contain a gram-molecular weight in volumes of water varying from 2 to 8,192 litres.

Finally, BOUTY§ studied the changes in conductivity caused by the solution in

\* 'Journal of the Chemical Society,' 1890, Trans., 64-184.

† 'Ann. Phys. Chem.,' vol. 154, pp. 215-239.

‡ 'Journ. Prakt. Chem.,' vol. 30, p. 228, and vol. 31, p. 437; also 'Zeits. Physikal. Chem.,' vol. 1, p. 75.

§ 'Compt. Rend.,' vol. 106, p. 595.



nitric acid of small quantities of metallic nitrates. This work has, however, only an indirect bearing upon the present investigation.

#### PRELIMINARY EXPERIMENTS.

In consequence of certain initial difficulties, experiments were made upon the effects produced by (1) an added impurity of nitrous acid; (2) sunlight; and (3) imperfect insulation of the electrolytic cell. The results obtained may serve to explain certain discrepancies between our values and those of previous workers.

(1.) *Added Impurity of Nitrous Acid.*—The following experiment serves to illustrate this point, to which reference was made in the introductory section:—The resistance of a mixture of 1 vol. of acid (1.48 sp. gr.) and 4 vols. of water was found to be 6.97 ohms; a few bubbles of nitric oxide gas passed in so as to form a trace of nitrous acid caused an increase of .07 ohm, or 1 per cent. in round figures.

(2.) *Sunlight.*—It is a common matter of observation that the space above the liquid in a partially-filled bottle of concentrated nitric acid contains coloured fumes, though the liquid itself may be almost colourless. This would show that the vapour rather than the liquid was decomposed by the action of light. In order to decide this point a comparative experiment was made by taking two stoppered flasks of the same capacity, and presumably of the same glass. One of these was completely, and the other partially filled with acid of 85 per cent. concentration, which contained one part per million of nitrous acid. Both flasks were exposed to the direct sunlight of an April day for two hours thirty minutes. Though the amount of nitrous acid in the former sample was unaltered, that in the latter was increased to 21 parts per million, and it had become bright yellow in colour.

As a further illustration, the following experiment may also be quoted:—The resistance of a certain length of acid (1 : 4 as above) was found to be 7.18 ohms, but after exposure for six hours to the diffuse daylight of a dull and cloudy day in October, the resistance had altered to 7.31 ohms, or an increase of 2 per cent. in round numbers, both experiments being made under the same conditions of temperature.

(3.) *Imperfect Insulation of Electrolytic Cell.*—As it appears from the diagrams given in certain standard text-books on physical chemistry that the electrolytic cells may be kept rigid by metallic clamps, it was thought desirable to ascertain if such an arrangement might lead to incorrect determinations. Our experiment showed that (1) winding the ends of a copper wire around the cell in the neighbourhood of the electrodes, (2) wrapping tin-foil round the whole cell, produced a diminution of resistance amounting to 0.3 and 1.6 per cent. respectively.

## PURIFICATION OF WATER.

Ordinary distilled water was purified by redistillation with a few crystals of potassium permanganate and a few cubic centimetres of concentrated sulphuric acid instead of potassium hydrogen sulphate, as recommended by STAS. A slow current of purified hydrogen was passed through the water during the operation, which was conducted in a laboratory used for no other purpose. The condensation was purposely kept somewhat imperfect to prevent any ammonia gas from being redissolved in the cool water; the first and last fifths of the distillate were rejected, and the intermediate portion again distilled over barium hydrate to remove any sulphuric acid. On the day before any conductivity experiments were made the water was distilled a third time in a platinum still fitted with a condensing tube and receiver of the same material, and again also with imperfect condensation. It was observed that after several litres of the water had been distilled a deposit of silica, distinctly crystalline, was formed on the still. Finally, a few hours previous to the commencement of a series of experiments, the water contained in small bottles was partially frozen (by immersion in a freezing mixture) so as to obtain a hollow tube of ice; the central core of water was poured off, the ice quickly melted and mixed with a quantity of nitric acid necessary for the concentration required. It is not, of course, presumed that, by the method adopted, water was obtained of the degree of purity arrived at by KOHLRAUSCH and HEYDWEILER;\* indeed, it was not considered necessary to use such precautions, when the solutions to be measured would possess a conductivity many million times greater than that of the purified water.

## PURIFICATION OF NITRIC ACID.

Two different methods were adopted for this purpose according as the concentration required was less or greater than 68 per cent. In the former case, acid of sp. gr. = 1.4, sold as pure, and nearly colourless, was distilled continuously under a pressure of a few millimetres in an apparatus constructed entirely of glass, as to the essential parts, namely, the still, condenser and receiver. The acid could be drawn up into the still, and syphoned off from the receiver by glass tubes provided with stop-cocks; it was stored in tubes of hard glass (a precaution adopted by STAS in his investigations upon atomic weights) the ends of which were drawn off but not sealed, as it was found that the vapour of the acid given off at ordinary temperatures decomposed to produce nitrous fumes in the operation of sealing.

When acids of concentration greater than 68 per cent. were required, a proportion of the water was removed by distillation of such an acid with an equal bulk of sulphuric acid; the pale yellow acid thus obtained was then redistilled with barium

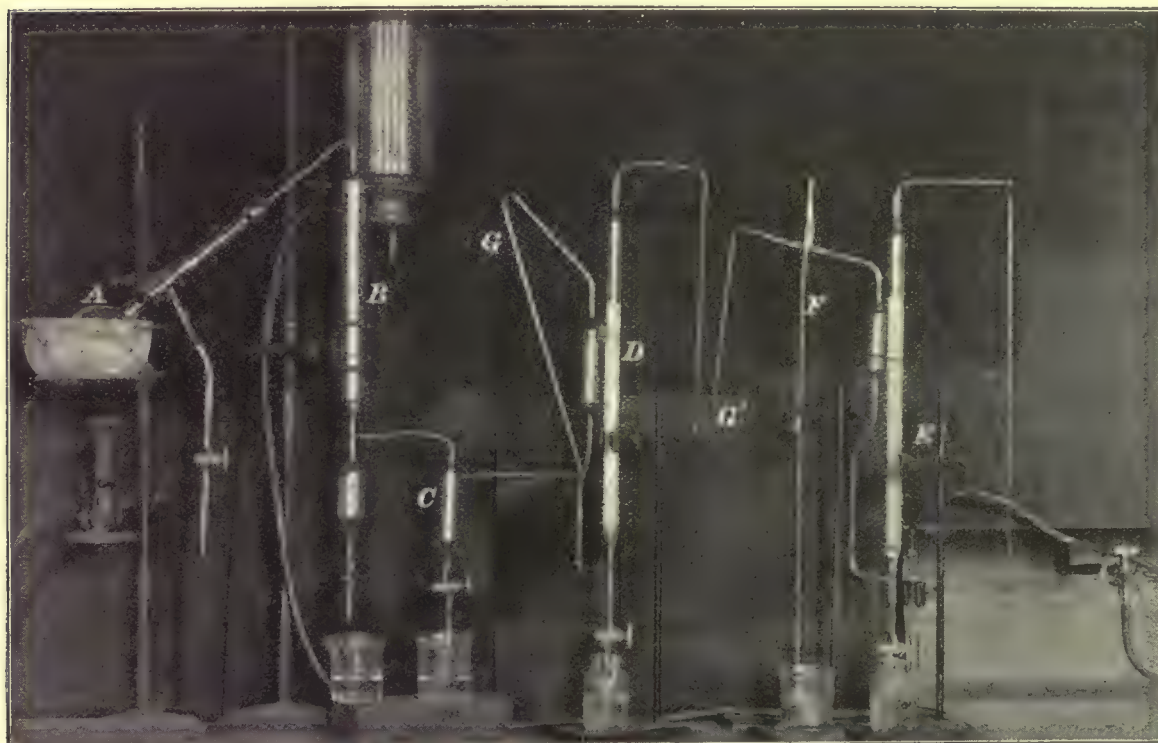
\* 'Zeits. Physikal Chem.,' vol. 14, p. 317.



and silver nitrates dissolved in it, the former to remove any sulphuric acid which might have been mechanically carried over, the latter to remove haloid acids remaining dissolved.\* The acid thus obtained was subjected to a third process of fractional distillation † in an apparatus constructed all of glass as follows.

A (fig. 1) is a distillation flask of about 400 cub. centims. capacity, the neck of which, considerably lengthened out, was sealed at its upper end to the top of a spiral condenser, B, kept cool by a current of ice-cold water; to the lower end of the latter the receiver was attached, which was provided with a stop-cock for drawing off the

Fig. 1.



condensed acid from time to time. A side trap, C, was sealed on to the upper end of the receiver, in order to retain any accidental spirting from the sulphuric acid towers, D and E; these latter were about half a metre high, and filled for three-quarters of their length with glass beads moistened with sulphuric acid, which could be renewed by means of stop-cocks sealed at their lower extremities. This process of cleansing and renewal was repeated not only previous to, but also during each distillation. It was found by experience that these towers were sufficient to prevent any moisture from passing from the water pump to the anterior portion of the apparatus. The

\* This process of purification was previously adopted by KOLB, 'Ann. Chim. Phys.' [4] vol. 10, p. 140.

† Attempts were made to remove the water present by means of phosphorus pentoxide, but were unsuccessful owing to the rapid decomposition of the 99 per cent. acid by this substance.



gauge, F, introduced between the two towers served to indicate the pressure, generally a few millimetres, under which the distillation was conducted, while the glass springs, G and G', decreased the rigidity of the apparatus. The process of distillation consisted in drawing up the acid into the flask by means of the stop-cock; the whole apparatus was then exhausted, and the water-bath surrounding the distillation flask gently warmed. The nitrous fumes present in the acid were for the most part immediately swept through the apparatus, and the remainder collected in the first portions of the nitric acid condensed in the receiver. This was drawn off and the whole process repeated until a small test sample of the acid produced no decoloration of a drop of standard potassium permanganate solution after the lapse of thirty minutes. The most effective method, according to our experience, for removing the last trace of nitrous fumes without either diluting the acid or adding any other impurity, consists in passing a stream of dried ozonised oxygen through the acid, and then subsequently distilling it in the above vacuum apparatus to eliminate any dissolved gas.

#### DETECTION AND ESTIMATION OF IMPURITIES.

The principal impurities likely to be present in any sample are (i.), nitrous acid, formed from the decomposition of the liquid itself, or its superincumbent vapour, (ii.), sulphuric acid, (iii.), the haloid acids, namely, hydrochloric and hydriodic, derived from the previous manufacture of the acid from "caliche," and, (iv.), alkaline silicates or silica from the glass of the containing vessels. The methods used to estimate the three first-mentioned impurities are described in detail.

##### (i.) *Nitrous Acid.*

For this purpose the metaphenylene-diamine method was employed, by means of which 1 part in 20 millions (according to GRIESS, its author) can be detected, and according to our own observation a quantity less than 1 in 15, but greater than 1 in 20 millions is revealed. The standard test adopted was that of 1 part nitrous acid in 1 million parts of water, produced either by the liberation of that amount of nitrous acid by the acidification of sodium nitrite (obtained from silver nitrite) or by an imitation tint formed from Bismarck-brown dissolved in dilute alcohol; the comparisons were made by the tintometer described by one of us in a previous communication.

##### (ii.) *Halogen Acids.*

The solubility of silver chloride was fully investigated by STAS\* in the course of his investigations upon the atomic weights. According to this writer, the solubility

\* 'Ann. Chim. Phys.' [4], vol. 25, p. 22, and [5], vol. 3, p. 145.

varies according as the chloride is in the pulverulent or caseous condition. More recently the same subject has been investigated by J. P. COOKE,\* who states the solubility is not affected by nitric acid. The method of estimating the traces of halogen acids in the samples was as follows :—

10 cub. centims. of the sample to be examined were placed in a miniature flask and 1 cub. centim. of a silver nitrate solution containing .05 gram of the salt was added; the turbidity, if any, was compared with that produced by the same quantities of purified water, the silver nitrate solution and a fractional part of 1 cub. centim. of a solution of hydrochloric acid of known strength. It was found in the course of experiment that a sample of nitric acid of 20 per cent. concentration gave no turbidity with the silver nitrate, but on addition of .5 per million of hydrochloric acid a turbidity was produced which appeared to be equal in amount to a blank experiment, made at the same time, with water only; but after standing for some hours the turbidity in the former case was less than that in the latter, and remained nearly white, instead of changing to a violet tint on exposure to diffuse daylight. It was probable, therefore, that the chloride was being slowly decomposed by the nitric acid. As a result of this observation, acids of concentration greater than 20 per cent. were diluted down to this strength, and the comparison made with the blank water experiment after the interval of one hour. It is not presumed that the estimations are of absolute accuracy, especially as the greatest quantity of haloid acids found in any sample was 3.8 parts per million, the average being less than 2.0 parts. The correction to be applied for these amounts of impurity is of the second order, and may reasonably be neglected.

### (iii.) *Sulphuric Acid.*

The observations upon the solubility of barium sulphate in water and in dilute nitric acid appear to be very discordant. According to FRESSENIUS and HINTZ† one part of barium sulphate in 400,000 water can be detected provided that a slight excess of barium chloride is added, but only 1 in 100,000 parts if no such excess is present; under the same conditions, in presence of nitric acid of 7.8 per cent. concentration, only 1 part of barium sulphate in 33,000 can be detected.

These estimations were made with a view of examining the accuracy of the quantitative determination of sulphuric acid by barium chloride, as the precipitant, in the presence of various salts and acids which might be present in the course of analytical experience; otherwise the use of barium chloride rather than the nitrate in the presence of nitric acid might be open to objection. Furthermore, it is evident that the delicacy of the reaction between a barium salt and sulphuric acid will depend *ceteris paribus* on the relative masses of the reacting substances. The work

\* 'American Journal of Science,' vol. 21, p. 220.

† 'Zeits. Analyt. Chem.,' 1896, vol. 33, p. 170.



of the above writers has been criticised by KÜSLER,\* who found by an electric resistance method that a saturated solution of barium sulphate in water contains 1 part of salt in 425,000 water (equals 2.35 parts per million), a result in accordance with the former observations of ROSE, KOHLRAUSCH, and HOLLEMANN.

In the estimation of the amount of impurity of sulphuric acid in the samples used, 1 cub. centim. of a 10 per cent. solution of barium nitrate was added to 10 cub. centims. of the nitric acid, which had been neutralised previously by a solution of ammonia; the quantity of the barium nitrate was more than a hundredfold in excess of that required to precipitate the greatest amount of sulphuric acid likely to be present. In a blank experiment with water which had been distilled over baryta it was found that 2.3 parts of sulphuric acid per million water produced a turbidity with the barium nitrate; but in the case of a sample of nitric acid of 50 per cent. concentration, after neutralisation with ammonia, 2.8 parts of sulphuric acid per million produced a turbidity. The mean of these two numbers, namely 2.5, may reasonably be taken as the limit of delicacy of the reaction within the limit of .0650 per cent.; from the results calculated upon this basis the greatest quantity of sulphuric acid found was 4.3 parts per million. A selection is given below (p. 385) of various acids used in order to show the degree of purity attained.

#### DETERMINATION OF THE CONCENTRATION OF THE SOLUTIONS.

For this purpose standard solutions of sodium hydrate were made from metallic sodium, purified by melting it in vacuo and filtering the molten metal through iron gauze; it was then placed in a silver dish and allowed to hydrate slowly in the vapour of water from a dilute soda solution; the method adopted has previously been fully described.

The solution thus obtained was made up to a definite volume and standardised as against (i.) sulphuric and (ii.) hydrochloric acid; the acidity of the former was determined volumetrically by sodium carbonate, and gravimetrically as barium sulphate, that of the latter (i.) volumetrically by pure metallic silver, by the method of STAS, and (ii.) volumetrically by sodium carbonate. Three different methods and two different indicators, namely, litmus and methyl-orange, were used by two different individuals. The degree of concordance obtained for one such standard solution of soda is shown by the following table:—

\* 'Zeits. Anorgan. Chem.,' vol. 112, p. 261.



TABLE I.

Acid.	Method.	Sodium hydrate in 1 cub. centim.
Sulphuric	Barium sulphate	gram. 0.03912
"	Sodium carbonate	0.03912
"	"	0.03914
Hydrochloric	Silver	0.03914
"	"	0.03912
"	Sodium carbonate	0.03912
"	"	0.03912

For the determination of the nitric acid solutions a weighed quantity was expelled from a pycnometer, neutralised by a slight excess of the standard sodium hydrate, and the amount of such excess added determined by a solution of hydrochloric acid of  $\frac{1}{100}$  equivalent strength of the soda.

#### STANDARDISATION OF THERMOMETERS.

The thermometers employed were graduated to  $0.1^{\circ}$  C., and were compared with (i.) a standard Kew thermometer which could be read to  $0.05^{\circ}$  F., and (ii.), within certain limits, with another standard graduated to  $0.05^{\circ}$  C., constructed by BAUDIN and formerly in the collection of the 8th Duke of MARLBOROUGH.

The zero displacements of all the instruments were determined by immersing them in a cylinder filled with washed ice, and surrounded by an outer jacket packed with the same material. The readings were made with a telescope, furnished with cross wires and mounted on glass plates, after the manner of QUINCKE's reading microscope. For the estimation of the errors at higher temperatures, the thermometers were suspended in a deep tank (containing several gallons of water), so that the thread of mercury at the point to be observed was just visible above the surface; the instruments were arranged in a line with the standard in the middle, so that by a slight rotatory movement of the telescope stand each of them could be brought into the field of view in rapid succession. The water was warmed about  $2^{\circ}$  above the point required, allowed to cool slowly, while thoroughly stirred, and a first set of readings taken; the process of stirring was repeated, and a second set taken; in case of any variation, the mean value was accepted. At the commencement and conclusion of each set of observations, the standards were read, and their constancy throughout was regarded as a safe criterion that the temperature of the large mass of water had not appreciably altered. The whole series of observations was repeated for every  $5^{\circ}$  within the range required, and in the case of two thermometers found to be less uniform in bore than the others, readings were taken at an interval of each degree.

This method of standardisation, though tedious, probably eliminates the errors due to a lag of the mercury thread to a greater degree than a more rapid process. It was not thought necessary to apply the corrections for variations of atmospheric pressure, or others adopted by the Bureau International, as the methods of determining the electric conductivity of a highly decomposable electrolyte are not susceptible of such a degree of accuracy that such corrections would materially affect the result.

#### CALIBRATIONS OF ELECTROLYTIC CELL.

In order to determine the mean cross-sectional area of this apparatus, two methods were adopted: (i.) that described by BUNSEN (Gasometrische Methode), namely, by pouring in equal volumes of mercury and reading the different levels by a telescope, the absolute volume of the measuring tube being subsequently ascertained by two independent weighings of mercury at the temperature of the observation; (ii.) by pouring in from a funnel such a weight of mercury as would fill the burette for every tenth graduation in cubic centimetres, the temperature being noted at the time. The following results are given to illustrate the concordance obtained by the two methods:—

TABLE II.

Method I.	Method II.	Difference.
cub. centims.	cub. centims.	cub. centims.
10.12	10.04	+ .08
10.02	10.02	Nil
10.02	9.99	+ .03
10.33	10.36	— .03
10.02	10.04	— .02
9.82	9.82	Nil
10.02	9.96	+ .06
9.82	9.88	— .06
10.02	10.06	— .04
10.02	10.05	— .03
10.22	10.25	— .03

The mean values were accepted as the true volumes.

In order to measure the different lengths so as to deduce the cross sectional areas, a pair of slide callipers was used, reading to .01 centim. and compared with a standard metre. The highest and lowest value for the area within the limits of observation were 4.4732 and 4.3839 sq. centims. respectively, so that the burette was of a fairly uniform bore.

#### RESISTANCE COILS.

The resistance coils, supplied by Messrs. ELLIOT BROS., and marked as true ohms at 16° were standardised by Mr. E. H. GRIFFITHS as against his coils, which had been

compared with the standards in the possession of the Cavendish Laboratory, Cambridge. We desire to express our appreciation of Mr. GRIFFITHS' courtesy and kindness in this matter. Two sets of determinations were made at different times and at slightly different temperatures. The results are given as true ohms, when the thermometer used always for the box registered a temperature of 16°.

TABLE III.

Coils as marked.	Series I.	Series II.	Mean.
ohms.	ohms.	ohms.	ohms.
50	49.9994	49.9927	49.9961
20	19.9999	20.0003	20.0001
10 A	10.0026	10.0026	10.0026
10 B	10.0031	10.0028	10.0029
5	4.9995	4.9996	4.9996
2	2.0022	2.0034	2.0028
1 A	1.0026	1.0018	1.0022
1 B	1.0022	1.0018	1.002
.5	.5027	.5017	.5022
.2 A	.2005	.2007	.2006
.2 B	.2001	.2007	.2004
.1	.1005	.1007	.1006

With the exception of the 50-ohm coil the results are concordant to within .001 ohm, and in the majority of cases to .0005 ohm, which is within the limit of experimental error.

The values of the higher coils were referred to the 50-ohm coil by reducing the bridge arm to  $\frac{1}{100}$ ; they are not, therefore, of such a high degree of accuracy, but these coils were seldom required.

Coils as marked.	Values.
ohms.	ohms.
100 A	99.9895
100 B	99.9956
200	200.0117
500	500.1618

#### TEMPERATURE COEFFICIENT OF BRIDGE WIRE.

The bridge wire, supplied by Messrs. JOHNSON and MATTHEY, was of platinum-(90 per cent.) iridium (10 per cent.) alloy; its mean diameter was 1.5 millim.

In order to determine its temperature coefficient, a small portion was drawn out to .005 millim. diameter, and the wire introduced into a Callendar-Griffiths pyrometer. Its resistance at the three temperatures of steam, ice, and 20.15 was determined by one of their platinum resistance boxes, in which the compensating leads are



thrown into one arm of the bridge, while the coil and its leads are in another arm thus, changes in resistance of the leads produce no effect on the readings. The values for  $R$  in steam and ice were ascertained by different arrangements of plugs, and hence different bridge wire readings. The coils used are indicated by letters.

## STEAM.

Coils.	Readings.	Temp. of resistance box.	Bar.	R. con.
C.F.H.	+ 6.04	19.95	millims. 748.8	191.351
C.F.H.	+ 6.04	20.03	748.76	191.355
C.F.G'.	+ 1.11	20.01	748.76	191.359
C.F.G.H.	- 4.05	20.12	748.78	191.353
Mean . . . .			748.78	191.353

The boiling point of water for  $p = 748.78$  is  $99.59$ ; hence the value for  $R$  under this condition is  $191.355$ .

## ICE.

Coils.	Readings.	Temp. of resistance box.	R. con.
C.G.	+ 2.04	20.38	172.288
C.G.H.	- 3.095	20.4	172.286
C.G'.	+ 2.04	20.4	172.289
Mean . . . .			172.288

Hence  $R = 172.288$ , if  $t = 0$ ; similarly, for  $t = 20.15$   $R$  was found in the same manner to be  $176.149$ . Taking  $R$  at  $0$  as  $1$ ,  $R$  at  $20.15 = 1.02241$ , and at  $99.59 = 1.11067$ , the curvature is very slight, and is expressible by the equation

$$R_t = 1 + .0011123 t;$$

but if  $R_{15} = 1$ , then  $R = .0010911$ , and the equation for the temperature coefficient is

$$\delta R_t = R_{15} (1 + (t - 15) \times .001094),$$

with a probable error of 1 in 1000.

## CONSTRUCTION OF THE BRIDGE.

In preliminary experiments with the ordinary form of metre bridge used with a telephone and small induction coil as described by KOHLRAUSCH and others, difficulty

was experienced in obtaining silence in the telephone within the limits desired. This was eventually traced to imperfect insulation of the wooden board, even though well seasoned and previously soaked with paraffin wax (figs. 2 and 3). It was there-

Fig. 2.

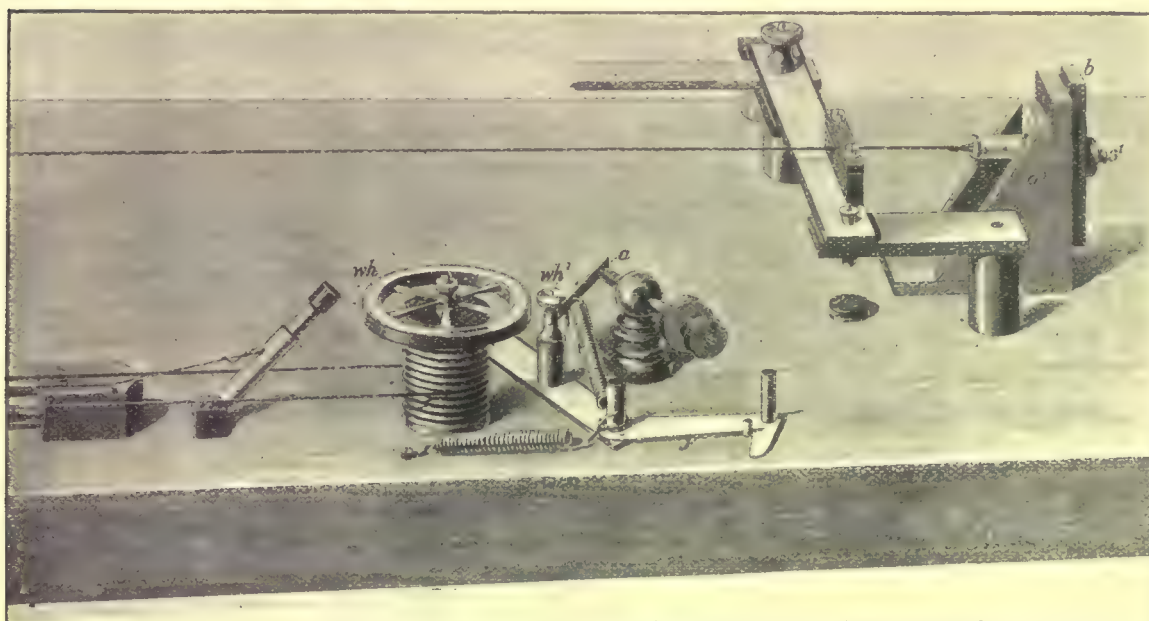
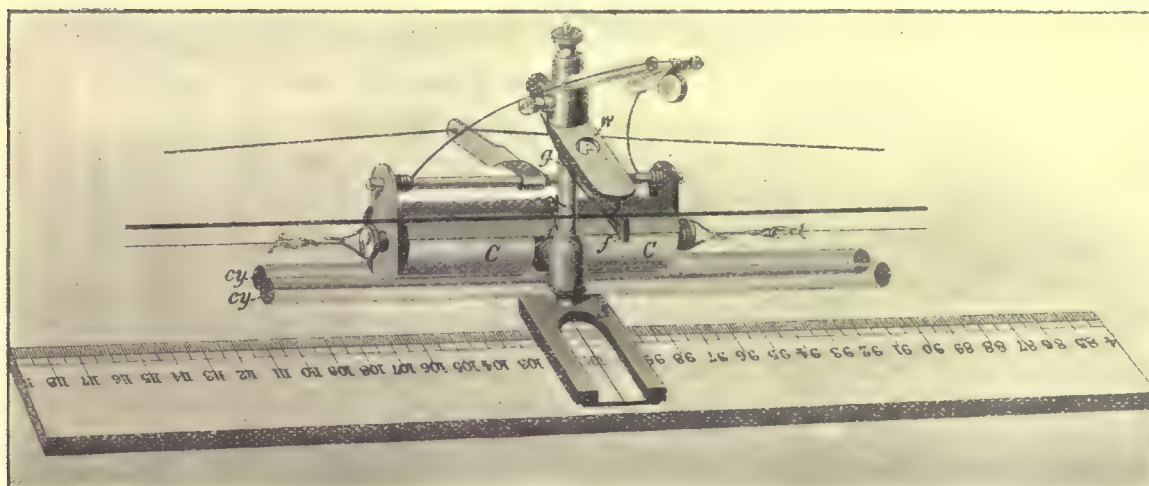


Fig. 3.



fore thought desirable to construct a bridge in which the wire was an air line, and all the strappings were mounted upon ebonite pillars. The wire was stretched between two massive brass castings, *aa*, and insulated from them by ebonite plates, *bb*; at one end it was fixed to a brass pin, and at the other to a stretching screw, *s'*, so arranged

as to tighten but not twist the wire ; its effective length was 2 metres. As this arrangement obviously required a special form of slider to tap, without sagging the wire, the following plan was adopted : two weighted opposed cones, *cc*, moved upon two uniform glass cylinders, *cy*, placed side by side and set parallel to the wire, according to the principle of Lord KELVIN, that a body provided with four points will always move parallel to itself along a cylinder. A brass shoe, *s*, carrying a glass plate projected from the centre of the cones at a right angle and rested upon the millimetre scale of the bridge ; the position of the slider was indicated by a line etched upon the lower surface of the glass plate, and observed through a little window, *w*, situated vertically above it. A platinum contact piece, *f*, mounted upon an ebonite rod, *g*, was adjusted to within a fraction of a millimetre of the wire. When it was desired to make contact, an insulated arm, *H*, was brought down upon the wire from above by tilting the lever, *I*. In order to move the slider a wheel, *wh*, was attached to one of two pulleys, over which passed a cord kept stretched by a spiral spring. For the purpose of fine adjustment, the wheel, *wh*, was furnished with a rubber band, cemented at its edge, on which a second and smaller wheel, *wh'*, could be brought to bear by releasing the lever, *J* ; from the latter wheel projected an arm, *A*, which rested upon the end of the milled head screw, *S*. By this arrangement the slider could be moved by the observer from the extremity of the bridge, and all thermo-currents due to his proximity and direct handling were avoided.

#### CALIBRATION OF BRIDGE WIRE.

The wire was placed in the bridge, stretched to the requisite extent, and allowed to rest undisturbed for nine weeks ; then it was calibrated by CAREY FOSTER'S method, using as a gauge a short length of the same wire soldered on to massive copper terminals and a copper connector of the same length, both of which were joined to the bridge by mercury cups furnished with stout copper tags. The process of calibration was conducted on two different dates. The wire was found to be very uniform throughout, the highest and lowest values for the resistance of 1 millim. being '0001516 and '0001510 ohm respectively.

The total resistance of the 851 millims. in the centre of the bridge, as also of the ends of the bridge, were determined by a fall of potential method, using one Daniell cell working through a resistance of 200 ohms, a standard coil of '1 ohm value, and a high resistance galvanometer ; the current was uniform within limits of time greatly in excess of those required for the observation ; the necessary corrections for the temperature coefficient were subsequently applied. Ten independent observations were made of the total resistance, one set of two on one date, and another set of eight after an interval of a week. For the interchange of the standard coils and electrolyte, a switch board, constructed upon S. P. THOMPSON'S pattern, was used. The mercury cups were mounted upon ebonite stems to secure better insulation,



and for the purpose of better contact between the copper connectors and tags the former were furnished with leaden weights instead of the more troublesome elastic bands generally used.

### THE COMMUTATOR.

In our earlier experiments a small induction coil was used according to the well-known KOHLRAUSCH method, but owing to the susceptibility of the nitric acid to polarisation, this apparatus was found to be unsatisfactory. A commutator of the disc pattern was therefore substituted, being driven by a small water-motor served through a Kelvin tap to secure a regular flow of water; the necessary electric current was obtained from a one-quart Daniell cell.\* The speed of the commutator disc was determined by a kymograph, a lever arrangement being fitted up so as to give one vibration for each revolution. A number of experiments showed that the revolutions varied between the extreme limits of 16 and 20 per second; as the disc carried nine commutators, there were for each revolution 18 makes and breaks and 9 reversals; this gives per second 288–360 of the former and 144–180 of the latter. In order to test the efficacy of this method, the alternating currents were led directly into nitric acid of various concentrations for one hour. No evidence of polarisation could be detected when the commutating arrangement was switched off and the electrodes simultaneously connected up with a very sensitive galvanometer, and, further, no trace of nitrous acid could be detected by the metaphenylenediamine test.

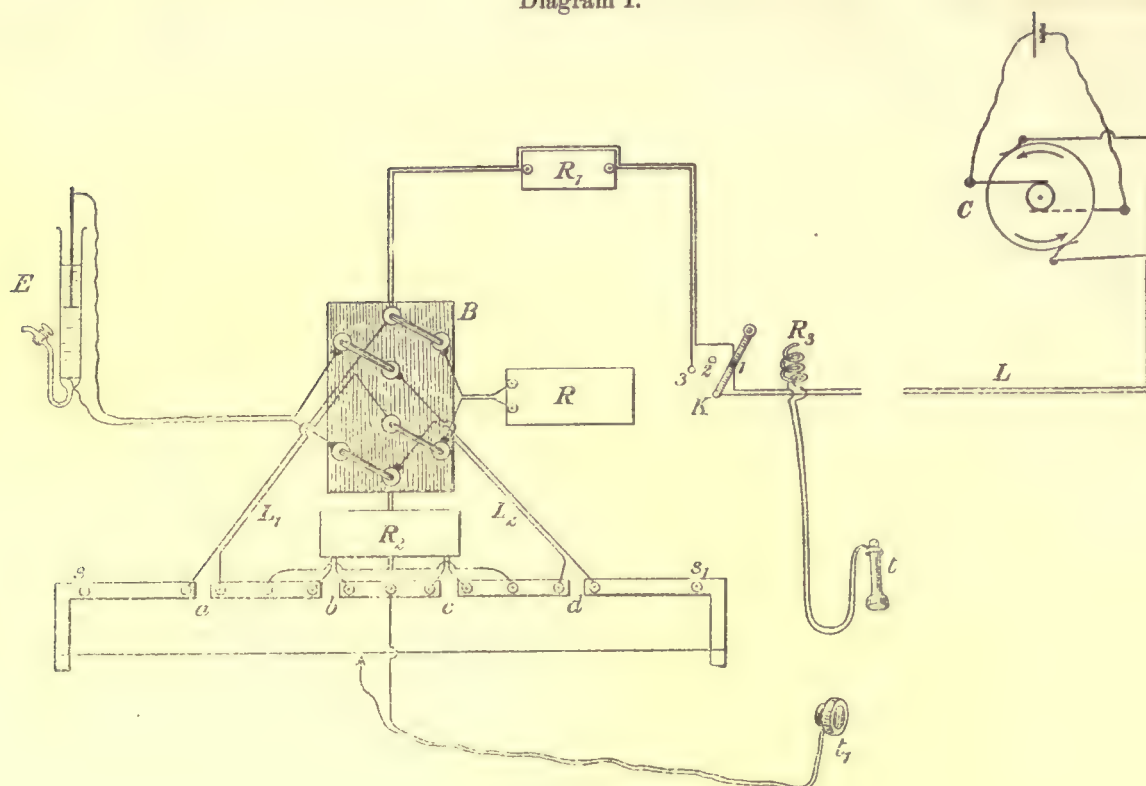
The commutator, working-cell, and water-motor were placed in a different wing of the building, the current being brought into the experimental room by leads twisted together.

### GENERAL ARRANGEMENT OF APPARATUS (Diagram I).

C represents the commutator, L its leads, K a key for either short-circuiting the current (position 1), or connecting up with the bridge and electrolytic cell E (position 3) or breaking connection altogether (position 2).  $R_3$  is a small bifilar coil to the extremities of which the leads of a call telephone,  $t$ , were attached, the latter serving to detect any alteration or failure of the commutator.  $R_1$  is a resistance box, the coils of which were adjusted so as to approximately compensate any alteration in the standard resistance,  $R$ , and therefore to maintain the current uniform through the whole series of experiments; B is a switchboard,  $R_2$  a box of auxiliary cells used in connection with CAREY FOSTER's method, and  $t_1$  a balancing telephone. The leads  $L_1$  and  $L_2$  connecting the switch-board with the bridge consisted of ten strands of

\* The current of this cell, measured by a standard tangent galvanometer, was found to be .22 ampere when flowing through the commutator at rest, but it would probably be rather less when the commutator was rapidly running, owing to a consequent increase of resistance in the contacts.

Diagram I.



Showing general arrangement of the apparatus.

No. 18 copper wire soldered on to massive copper tags; their resistances were determined in position and found to be as nearly as possible equal;  $L_3$  the leads (of similar construction) to the standard coils were of resistance  $\cdot 005$  ohm., a constant factor added to all values of  $R$ .

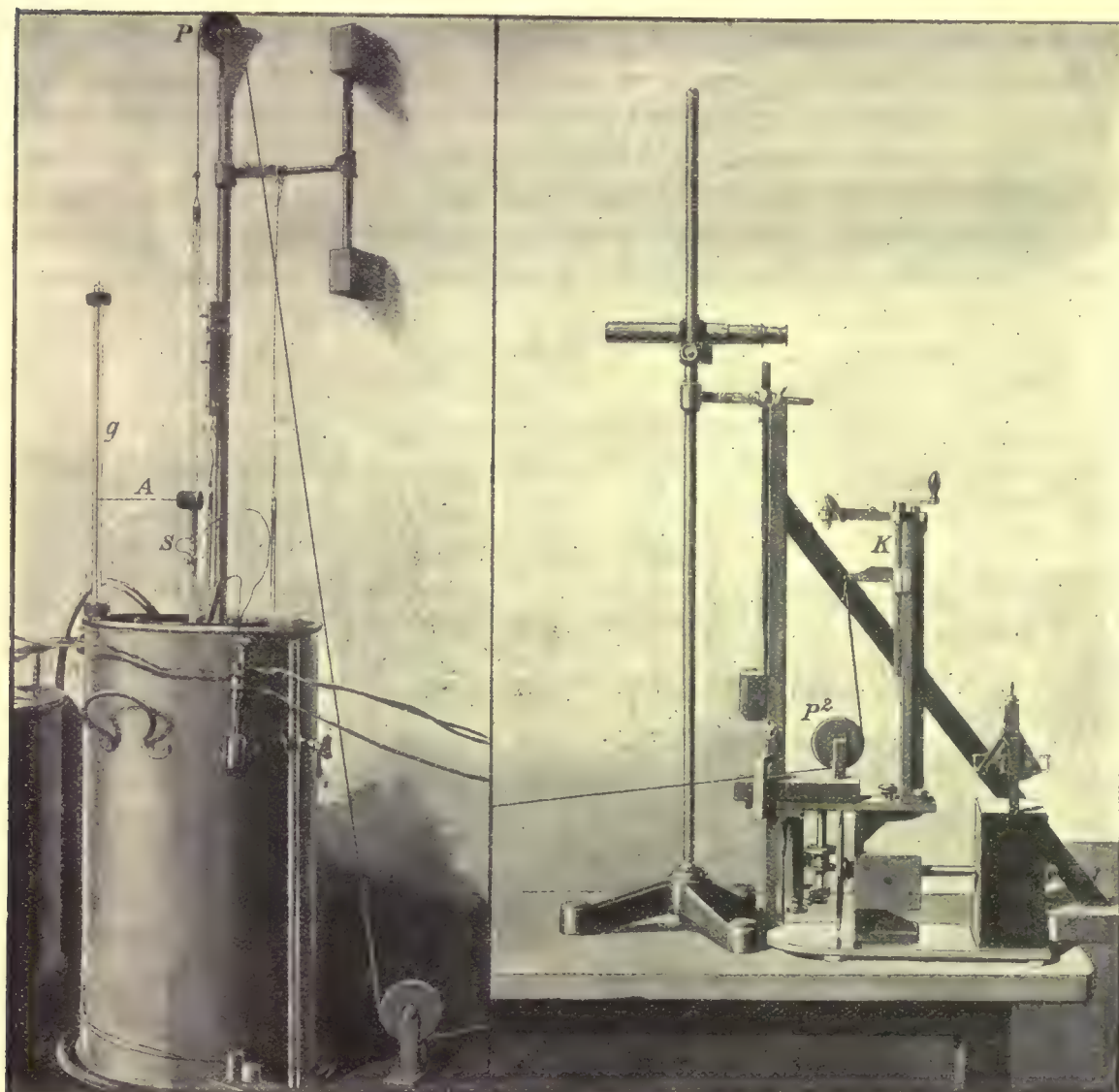
### THE ELECTROLYTIC CELLS.

#### (i.) *Standard Cell.*

Several forms of cell were adopted in the earlier experiments, but as all those of the **U** or bottle pattern, as described by KOHLRAUSCH, involve their calibration by a solution of an electrolyte of known conductivity, it was resolved to use a cell of a straight burette form, with one electrode fixed, and the other movable, so as to throw different lengths of the acid into circuit, and thus obtain independent measurements. For this purpose the movable electrode was welded on to a platinum wire passing through a glass tube to the binding-screw, **S** (fig. 4); above this a similar tube was fastened, upon which was etched a centimetre scale copied directly from the standard. In order to ensure that the whole arrangement should slide parallel to its axis, the graduated tube was allowed to run in **V** grooves

mounted upon a block of ebonite, and provided with a small spring to avoid accidental displacement; their rotation was prevented by a small arm, *A*, working in glass tube guides, *g*. A cord passing over the pulleys, *P*, *P*<sub>1</sub> and *P*<sub>2</sub>, was attached to the upper end of the graduated tube, and the other end to an arm; this cord could

Fig. 4.



be shortened by suitable loops, and the final adjustments effected by means of the kymograph screw, *K*. The divisions of the tube were observed by a telescope mounted on a rigid stand.

To secure perfect insulation the cell rested upon and against blocks of ebonite, *a*, *b* and *c*, the two latter being **V** grooves, against which it was kept pressed by elastic bands. Preliminary experiments already described showed that all fastenings of a

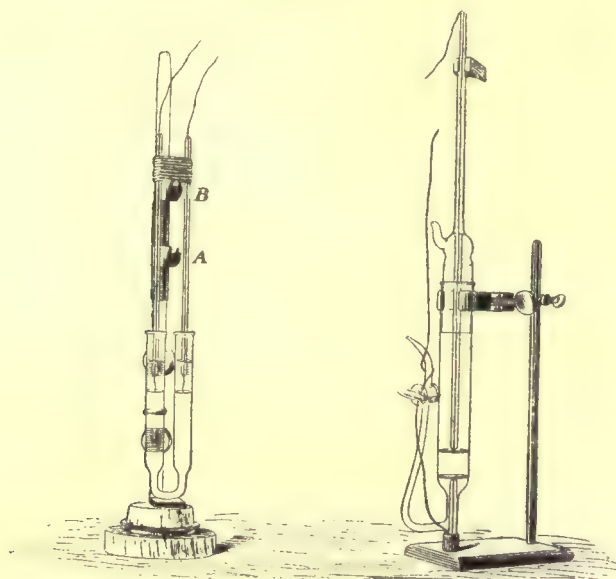


conducting nature should be avoided, especially if high tension currents are used. So far as can be judged by the description and diagrams of certain previous writers, sufficient attention does not seem to have been given to this essential point. Finally, to avoid contamination of the acid by dust, the cell was closed by a platinum disc.

(ii.) *Temperature Coefficient Cell* (fig. 5).

This was a U tube of 1.77 sq. centims., cross sectional area, rigidly attached to an insulating stand containing the electrodes, both movable in this case and welded on to platinum wires passing through capillary tubing. The upper ends of these were tied and cemented on to an ebonite block which could be made to rest upon either of two brackets furnished with a hole, slot and plane arrangement similar to that used for a KELVIN galvanometer. When the block rested on the lower bracket (position

Fig. 5.



Temperature coefficient cell.

Electrolytic cell No. 2.

A), measurements were made of the resistance of the leads and the acid in the bend of the U; when, on the upper bracket (position B), the total resistance was made up of these factors, together with the introduced length of acid, being double that of the height through which the electrodes had been lifted. To obtain, so far as possible, independent measurements the process of raising and lowering the electrodes was repeated during any one series of experiments. The top of each limb of the U tube was closed with platinum discs.

(iii.) *Cell for Acids from 70 to 99·8 per cent. Concentration.*

As the acid above 70 per cent. is very hygroscopic and its fumes are liable to attack the electric connections, a cell for use not only at a particular but also at any required temperature was devised, as follows: a cylinder of glass, with an etched centimetre scale, was drawn out at its lower end and a vacuum tap sealed on; the upper end was closed by an accurately ground glass cap carrying a tube on which were etched two lines at a distance of 10·24 centims. apart. Within this tube moved a piece of thermometer tubing so as to form a stuffing-box arrangement containing within it the platinum lead from the movable electrode; one line was etched upon it, which could be made to coincide with either line upon the outer tube. The method of measurement was similar to that described in (i.) (standard cell), and the whole was, as in previous cases, thoroughly insulated. It was found that acid could be preserved in this cell for a considerable time without any appreciable change of concentration.

The electrodes for all the above cells were coated with the velvet-like deposit of platinum, which, according to the experience of KOHLRAUSCH and successive writers, gives the most satisfactory results. After preparation they were kept immersed for some weeks in the prepared distilled water, which was frequently changed.

TANKS FOR ELECTROLYTIC CELLS.

Two zinc tanks, of many gallons capacity, and furnished with covers, were used in the course of the investigation; they were both similarly constructed, one of two, the other of three concentric chambers; the innermost, of small dimensions, containing the electrolytic cell with its accessories, stood on three legs, so as to raise its level about six inches; the other, of large dimensions, furnished with ring stirrers, was filled with water to a level above that of the top of the electrolytic cell. It was found that a difference of 20° between the tank and room temperatures caused only a fall in the former of 1° after two hours; there was, therefore, no appreciable alteration of temperature in the course of any set of experiments lasting only a few minutes. Standardised thermometers were placed both in the outer and inner chambers; when their corrected readings differed only by 0·1 it was assumed that equilibrium of temperature in the outer and inner chambers had been established.

When a temperature of 0° was required, the outer chamber of the triple-chambered tank contained air only, and the middle chamber was filled with finely-powdered ice, the water melting from which was continuously drained off. For other temperatures, water was run into the outer chamber, and steam blown in, if necessary.

## THE METHOD OF PERFORMING AN EXPERIMENT.

The electrolytic cell was charged with the required mixture of nitric acid and recently-melted ice as previously described, then placed in the inner chamber of the tank and its electrodes connected up with the bridge leads. When an hour had elapsed after equilibrium of temperature had been first observed, preliminary measurements were made in order to ascertain approximately the conductivity of the acid, and thus not only to facilitate the final measurement, but also to eliminate the risk of decomposition. The cell was then emptied, recharged with a second portion of the acid, and connections made as before; after an interval of some hours the final measurements were made as follows:—

The graduated glass tube was adjusted so that division 5 of its scale coincided with the point of intersection of the cross threads of the telescope; the standard coils were placed in the left gap (L) and the electrolytic cell in the right gap (R) of the bridge; if then the current from the commutator was found satisfactory, connection was made with the bridge and its accessories, and a first balance obtained.

The current was then switched off, the position of the coils and electrolytic cell interchanged, the current switched on, and a second balance obtained. The movable electrode was then raised to division 15 on the scale, and the value of the standard coils altered so that their combined resistances were approximately equal to those of the introduced length of acid and leads (as determined by the preliminary experiment), while simultaneously the resistances in the control box were also changed so as to maintain the current intensity uniform. Then balancings were obtained with the electrolytic cell successively in the R and L gaps as before. The whole series of operations were repeated for the introduction of the second length of acid caused by raising the graduated glass tube through another five divisions.

The thermometers in the standard resistance box, bridge and tanks were read before and after each set of observations, and any necessary corrections for alteration of temperature subsequently applied.

As an illustration, all the measurements and observations for two samples of acid, namely of 1.299 and 30.52 per cent. concentration are given; these particular values have been selected as those of maximum and minimum conductivity for each form of electrolytic cell.



TABLE IV.

Series.	Percentage of acid.	Impurities in parts per million.
I.	1.299	{ Halogen acids = 0.5 Sulphuric acid = 1.0 Nitrous acid = 0.07
II.	30.52	{ Halogen acids = 3.7 Sulphuric acid = 2.3 Nitrous acid = nil
Other samples used in the course of the investigation	45.01	{ Halogen acids = 0.29 Sulphuric acid = 2.3 Nitrous acid = 0.16
	58.2	{ Halogen acids = 2.7 Sulphuric acid = 3.6 Nitrous acid = 0.75
	73.82	{ Halogen acids = 1.1 Sulphuric acid = 3.16 Nitrous acid = nil.
	90.37	{ Halogen acids = nil Sulphuric acid = nil Nitrous acid = 0.83
	98.87	Nitrous acid = 1.1

TABLE V.

	Position of movable electrodes.	Temperature (corr.) before and after each observation.			Resistance (corr.) of coils + leads.	Resistance of acid + leads.	Mean cross sectional area of tube.	Resistance of acid.
		Tank.		Resistance coils.				
		Outer chamber.	Inner chamber.					
Series I.	5	15.7	15.6	16.5	ohms. 15.0106	ohms. 14.9658 = $R_1$	sq. centims. 4.4591	ohms. 33.0002 = $R_2 - R_1$
	15	15.7	15.6	16.5	48.0260	48.2660 = $R_2$	4.4560	50.0521 = $R_3 - R_1$
	20	15.7	15.6	16.5	65.0179	65.0179 = $R_3$		
Series II.	5	13.7	13.6	14.2	22.9031	22.9031 = $R_1$	4.4591	3.0474 = $R_2 - R_1$
	15	13.7	13.6	14.2	25.8988	25.9505 = $R_2$	4.4560	4.5779 = $R_3 - R_1$
	20	13.7	13.6	14.2	27.3989	27.4810 = $R_3$		

The following table serves to illustrate the method of determining the temperature coefficients, the acid used in Series II. being selected as an example :—

TABLE VI.

Temperature.	Resistance (corr.) of acid and leads.	Difference (B—A).
	ohms.	ohms.
0·2	17·2093 (A)	12·5037
	29·7130 (B)	
14·5	15·9365 (A)	9·8592
	25·7957 (B)	
27·9	15·1748 (A)	8·2322
	23·4070 (B)	

Hence, from these data, the values for the constants  $\alpha$  and  $\beta$  in the equation  $R_t = R_0(1 \pm \alpha t \pm \beta t^2)$  can be deduced.

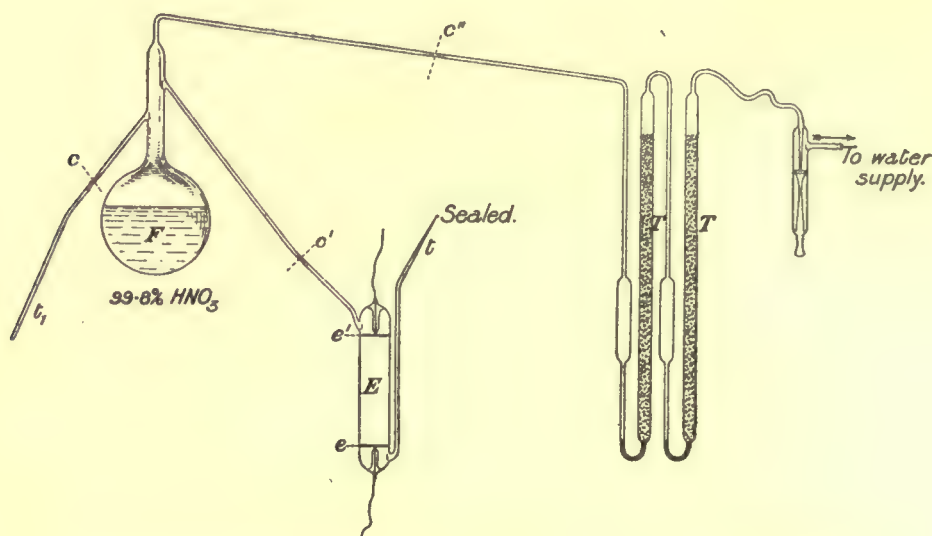
#### PREPARATION AND PROPERTIES OF ANHYDROUS ACID ( $\text{HNO}_3$ ).

By the vacuum apparatus described in the previous part of the paper it was not found possible to obtain acid of concentration greater than 99·88 per cent. With a view of eliminating the remainder of the water, another apparatus was constructed, all of glass, but without any glass taps, through which, however perfect, moist air may find a passage.

TT (Diagram II.) are drying towers similar to those in the first form of apparatus.  $c, c', c''$  are thick-walled capillary tubes, which can be sealed off in succession, F is the distillation flask of about 200 cub. centims. capacity, and E the electrolytic cell furnished with another capillary,  $t$ , sealed before distillation, but which subsequently served to withdraw a sample of acid for analysis after the determination of conductivity. The apparatus anterior to the towers was dried by exhausting through the capillary,  $t$ , and subsequently admitting dry air, the whole process being repeated a number of times, while the glass was kept continuously warmed by a flame;  $t$  is then sealed, acid of 99·8 per cent. drawn up into the distillation flask through the suction tube, which was then sealed off. The whole apparatus was then exhausted by the pump, and the flask gently warmed until the acid began to distil over into the towers; when this was observed the source of heat was removed, and the whole of the tube  $c''$

continuously heated to free it from any condensed acid ; then it was sealed off at the lowest possible temperature while the pump was kept in action. By immersing E in a freezing mixture acid passed over from F, and this process was allowed to continue until the upper electrode was well covered ; the electrolytic cell was then detached by cutting off at  $c'$ , plugging the opening immediately by a sealed capillary, and covering the whole with a glass cap. Though it is true that a trace of moisture might be introduced here, yet the error due to this would be less than that caused

Diagram II.

Apparatus employed for the preparation of the anhydrous acid ( $\text{HNO}_3$ ).

by the decomposition of the vapour of acid if  $c'$  had been sealed in a flame ; thus, of two evils, the less was chosen.

The acid thus obtained was perfectly colourless. Analyses made after measurement of conductivity gave the following values :—

$$\text{Acid, per cent.} \left\{ \begin{array}{l} (1) 99.97 \\ (2) 99.98 \end{array} \right.$$


---

99.975 mean.

Determinations were made of its density at  $4^\circ$ ,  $14.2^\circ$ , and  $24.2^\circ$  (*vide infra*), and a further pair of analyses made, with the result :

$$\text{Acid, per cent.} \left\{ \begin{array}{l} (1) 99.97 \\ (2) 99.97 \end{array} \right.$$

3 D 2



The acid may therefore be regarded as anhydrous ( $\text{HNO}_3$ ) within the limits of experimental error. Hitherto the highest recorded values appear to be 99.45 per cent. (PERKIN),\* 99.67 per cent. (LUNGE and REY),† and 99.84 per cent. (KOLB),‡ but doubts have been expressed as to the correctness of the determinations of the last observer.

#### CALIBRATION OF CELL AND WIRES USED FOR ANHYDROUS ACID.

As the method originally proposed by KOHLRAUSCH and adopted by subsequent workers for determining the "resistance capacity" of their cells appeared in this particular case to involve manipulative difficulties and errors consequent therefrom, the following method was used. The distance between the inner surface of the electrodes was determined along nine different lengths; the mean value found was 9.77 centims.; the volume of the portion of the cell between the electrodes was measured by water delivered from a calibrated burette connected with the cell by pressure-tubing, readings being taken for the levels of the upper surface of the lower electrode and lower surface of the upper electrode. Five or six series of experiments were conducted, different burette levels being taken in each such series; the mean of all the values was 35.2 cub. centims., with an error of  $\pm 1$  per cent. From these data the mean cross sectional area was calculated to be 3.603 sq. centims. In order to determine the resistance of the platinum leads sealed into the cell, their lengths were measured and the resistance of a known length of the same wire determined by CAREY FOSTER'S method; the value thus calculated was added to that of the copper wire leads which joined the cells to the bridge, due allowance being made in each case for its coefficient at the temperature of the experiment. Such indirect measurements cannot, of course, lay claim to absolute accuracy, but appeared to be the best attainable after the cell had been fitted up and used for the experimental determinations.

#### (i.) *Chemical Properties of Anhydrous Acid.*

As a considerable quantity of the anhydrous acid was prepared in the course of the enquiry some of its properties were examined. It has no action upon the following metals (i.) copper, (ii.) silver, (iii.) cadmium, (iv.) mercury, all of a high degree of purity, and (v.) commercial magnesium at ordinary temperatures; purified iron and commercial granulated tin were unaffected by the acid even when boiling. Zinc purified by frequent distillation in vacuo was slightly acted upon at ordinary temperatures, but sodium immediately caught fire. The acid has no action whatever

\* 'Journal of the Chemical Society,' 1893 (Trans.), p. 65.

† 'Journal of the Society of Chemical Industry,' 1891, p. 543.

‡ 'Ann. Chim. Phys.' [4], vol. 10, p. 140.

on finely-powdered calcium carbonate (Iceland spar and marble) either at ordinary temperatures or the boiling point. Vitreous phosphorus at first melts in the acid and then catches fire, while a considerable quantity of a dark-red sublimate (probably phosphorus suboxide  $P_4O$ ) condenses in the walls of the tube above the surface of the liquid; flowers of sulphur, as also iron pyrites, dissolve quickly and completely in the gently-warmed acid. Nitration of aromatic hydrocarbons is readily effected by the acid, whilst such a substance as chloral hydrate is decomposed, with complete conversion of the chlorine into hydrochloric acid.

(ii.) *Determination of Density.*

For this purpose the form of Sprengel pyknometer, as modified by DITTMAR, was used, and of capacity about 54 cub. centims.; the ends of both limbs were closed by caps, and on account of the high coefficient of expansion a series of bulbs were blown on one limb of the U tube.

The necessary corrections were made for reducing the observations to weighings in vacuo.

The following results were obtained :—

Density 4/4 = 1.54216 (i.)	Density 14.2/4 = 1.52236 (i.)	Density 24.2/4 = 1.50390 (i.)
= 1.54209 (ii.)	= 1.52231 (ii.)	= 1.50398 (ii.)
Mean . . 1.54212	Mean . . 1.52234	Mean . . 1.50394

The values obtained by former observers are given for the sake of comparison.

Observer.	Percentage of acid.	Values.
KOLB	99.84	$\left\{ \begin{array}{l} D_0 = 1.559 \\ D_{15} = 1.530 \end{array} \right.$
LUNGE and NEF	99.65	$D_{15.4} = 1.520$
PERKIN	99.45	$\left\{ \begin{array}{l} D_{15/15} = 1.5191 \\ D_{25, 25} = 1.5043 \end{array} \right.$

The results of LUNGE and NEF, as also of PERKIN, do not differ materially from those obtained by us after allowance is made for the slightly higher percentage strength and lower temperature, while those of KOLB are considerably higher; the method adopted by the last for the determination of acidity was, however, open to considerable objection, though his preliminary methods of purification were similar to those described above. It is proposed to publish shortly an account of the determination of densities and of other physical properties of the acid to confirm the conclusions drawn from the present work.

SUPPLEMENTARY NOTE ON THE APPLICATION OF CAREY FOSTER'S METHOD FOR  
COMPARISON OF RESISTANCES.

It was thought advisable to obtain certain measurements by CAREY FOSTER'S method for the comparison of resistances, not only to serve as a check upon the rest of the work, but also to ascertain if this method could readily be applied in the case of a decomposable electrolyte.

For this purpose the auxiliary coils were wound upon glass tube frames and soaked in paraffin wax; the frame was then enclosed in a box which was placed in a larger one, the space of two inches between them being packed with cotton wool. Two thermometers were introduced through the lids, so that their bulbs were in the centre. The maximum change registered during an hour was never more than one-tenth of a degree.

The process of working was as follows :—The ordinary method was first employed to obtain the value for  $K \times 10^8$ , then the auxiliary coils introduced into circuit and the determinations repeated. It will be seen from the following table that the results obtained by the two methods are concordant when due allowance is made for the slight variation of temperature.

TABLE VII.

Strength of acid per cent.	$K \times 10^8$ by C. FOSTER'S method.	Temperature.	$K \times 10^8$ by KOHLEAUSCH'S method.	Temperature.	Difference.
25.23	7094	17.6	7079	17.3	—15
45.34	5984	12.6	5980	12.6	— 4
51.77	5294	12.0	5277	11.8	—17
56.07	4402	6.6	4398	6.6	— 4
61.60	3829	5.5	3841	5.6	+12
64.37	3521	4.7	3514	4.7	— 7

As a matter of experience it was found that for CAREY FOSTER'S method the electrodes should be continually recoated with the platinum deposit, otherwise it was difficult to ascertain exactly the point of minimum sound in the telephone.

EXPERIMENTAL RESULTS.

The approximate temperatures selected for observation were  $0^\circ$ ,  $15^\circ$ , and  $30^\circ$  for acids of percentage concentration from 1.3 to 50, but  $0^\circ$ ,  $10^\circ$ , and  $20^\circ$  above this strength, as the more concentrated acids are not only extremely volatile, but also liable to decomposition.

In Table VIII. the values are given for the specific resistances in true ohms, corrected



at 0°, 15°, and 30° for temperature coefficients; these of the first and second order, multiplied by  $10^4$  and  $10^6$  respectively, are shown in the fifth and sixth columns.

TABLE VIII.

Percentage concentration.	Specific resistance in true ohms.			Temperature coefficients.	
	At 0.	At 15.	At 30.	$\alpha 10^4$ .	$\beta 10^6$ .
1.30	19.924	14.883	11.974	-204	+238
3.12	8.739	6.53	5.279	-205	+244
5.99	4.767	3.593	2.927	-200	+237
10.13	3.022	2.306	1.895	-192	+224
15.32	2.236	1.723	1.416	-184	+206
20.11	1.910	1.478	1.224	-182	+206
25.96	1.746	1.368	1.120	-169	+165
30.42	1.722	1.342	1.105	-174	+183
33.81	1.740	1.351	1.109	-177	+188
35.90	1.771	1.372	1.195	-178	+184
39.48	1.837	1.415	1.146	-181	+186
45.01	1.969	1.511	1.222	-183	+190
51.78	2.248	1.694	1.366	-198	+225
53.03	2.316	1.726	1.427	-211	+275
58.20	2.571	1.916	1.576	-210	+277
61.20	2.755	2.016	1.375	-190	+77
65.77	3.074	2.317	1.963	-208	+292
69.53	3.227	2.523	2.150	-177	+228
73.82	4.153	3.270	2.831	-174	+237
76.59	4.7584	3.599	2.302	-153	-6
78.90	6.239	4.836	4.493	-206	+377
84.08	10.188	8.164	7.177	-166	+226
86.18	12.535	10.106	8.874	-161	+212
87.72	16.624	13.320	12.220	-177	+294
89.92	24.012	19.929	15.847	-120	+46
91.87	37.618	31.077	27.967	-146	+202
94.32	53.847	45.270	41.108	-133	+182
96.12	73.663	66.069	63.517	-192	+152
98.50	54.733	56.057	60.730	-4	+136
98.85	50.182	51.586	53.966	+122	+43
99.27	26.554	25.075	26.164	-69	+215
99.97	25.688	24.286	23.713	-47	+71

In Table IX., the values for  $K_{15} \times 10^8$  for the different concentrations obtained in Cell No. I. represent for each length of acid the mean of two, and in some cases three, four, and six, independent observations; the values are concordant in the majority of cases to .05 per cent., which is probably within the limit of experimental error; the values for Cells II. and III. also represent the mean of several experiments.

In Table X., the values for  $K_0 \times 10^8$  and  $K_{30} \times 10^8$  are given for the sake of comparison of our results with those of other observers. Throughout the conductivity of mercury at 0° is taken as unity, and its specific resistance as 94.07 microhms per 1 cub. centim.

TABLE IX.

## Measurements with Cell No. I.

Percentage concentration.  I.	Values of $K_{15} \times 10^8$ .		
	II. Length of acid =15 centims.	III. Length of acid =10 centims.	IV. Mean.
1.30	631.86	632.25	632.06
3.12	1440.2	1441	1440.6
5.99	2617.3	2618.9	2618.1
10.13	4077.9	4080.4	4079.2
15.32	5458.9	5462.2	5460.5
20.11	6361.8	6365.7	6363.8
25.96	6873.6	6877.8	6875.7
30.42	7006.1	7010.4	7008.2
33.81	6960.7	6964.9	6962.8
35.90	6855.5	6859.7	6857.6
39.48	6646.5	6650.5	6648.5
45.01	6224.1	6227.8	6225.9
51.78	5552.5	5555.9	5554.2
53.03	5448.2	5451.5	5449.8
58.20	4907.3	4910.3	4908.8
61.20	4663.7	4666.6	4665.1
65.77	4059	4061.5	4060.3

## Measurements with Cell No. II.

69.53	3728.7		
73.82	2877		
76.59	2613.5		
78.96	1945.1		
84.08	1152.2		
86.18	930.86		
87.72	706.23		
89.92	472.02		
91.97	302.7		
94.32	207.8		
96.12	142.38		
98.50	167.81		
98.85	182.36		
99.27	375.15		

## Measurement with Cell No. III.

99.97	387.34		
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TABLE X.

Percentage concentration.	Values of $K_0 \times 10^8$ .	Values of $K_{30} \times 10^8$ .
1.30	492.15	785.63
3.12	1076.4	1781.9
5.99	1973.3	3213.8
10.13	3112.7	4963.6
15.32	4206.5	6641.0
20.11	4925.4	7686.1
25.96	5386.2	8399.1
30.42	5464.4	8515.4
33.81	5405.7	8480.1
35.9	5312.3	8402.9
39.48	5119.5	8205.7
45.01	4777.1	7701.2
51.78	4183.9	6884.5
53.03	4061.4	6594.5
58.20	3658.6	5967.8
61.20	3414.9	6842.5
65.77	3059.8	4792.1
69.53	2915.3	4375.9
73.82	2265.2	3323.2
76.59	1976.9	4085.8
78.90	1507.75	2093.5
84.08	923.3	1310.6
86.18	750.5	1060
87.72	555.9	769.8
89.92	391.8	593.6
91.87	250.1	336.4
94.32	174.7	228.8
96.12	127.7	148.1
98.5	171.9	154.9
98.85	187.5	174.3
99.27	354.3	359.5
99.97	366.2	396.7

From the data given in Table VIII., it is evident that the specific resistance of nitric acid decreases for percentage concentrations from 1.3 to 30, at first more, then less rapidly; from this point the resistance increases slowly up to 76 per cent., thence more rapidly, until a maximum is reached at 96.12 per cent., when a sudden reversal takes place.

KOHLRAUSCH (*vide supra*), by interpolation of his results, gives the points of maximum conductivity at 29 per cent. at  $0^\circ$ , 29.7 per cent. at  $18^\circ$ , and 30.2 per cent. at  $40^\circ$ , data which are in general accordance with our results, though it is evident that the exact minimum point could only be ascertained to within one-tenth per cent., by a number of experimental determinations made on either side of the point in question, whereas KOHLRAUSCH gives data only for 23, 30.95, and 37.36 per cent., respectively.

As the reversal between 96.12 and 98.5 was wholly unexpected, and at first sight appeared anomalous, the determinations were continued within small percentage limits, and the results thereby confirmed.



It will also be observed that whereas nitric acid behaves as other electrolytes, namely, in possessing a positive temperature coefficient of conductivity for percentage concentrations from 1·3 up to 96·12, yet from this point up to 99·27 per cent. it behaves as a metallic conductor in possessing a negative temperature coefficient.

Formerly, negative and positive coefficients were regarded as the essential point of differentiation between metallic and electrolytic conductors respectively, until ARRHENIUS,\* led by theoretical considerations, based upon the ionic dissociation hypothesis, observed negative coefficients in the cases of hypo-phosphorous acid (of concentration = 1·014 normal) and phosphoric acid (of concentration =  $3 \times \cdot 95$  normal). Upon this matter ARRHENIUS expressed himself as follows:—"Es ist wohl kaum denkbar dass man nach den bisherigen Ansichten ohne Zuhilfenahme der Dissociationstheorie die Existenz von negativen Temperaturkoeffizienten erklären könnte." OSTWALD, and other writers of the same school, when quoting the results of ARRHENIUS, have adopted the same view, with greater or less modification. But in the case under discussion, nitric acid of concentration 96–99·97 per cent. would, *ex hypothesi*, contain few, if any, free ions, and, therefore, the theory would lead to a totally opposite conclusion. For the present it is not proposed to offer any explanation of the phenomenon observed, but merely to call attention to the somewhat analogous example of the density of sulphuric acid, which attains its maximum at 98·78 per cent. (PICKERING, LUNGE, NEF, and ISLER) and thence decreases up to 99·86 per cent. (PICKERING).†

In the paper quoted above, ARRHENIUS calculates, by interpolation, the temperature of maximum conductivity of nitric acid as at 668°, but to our minds it is idle to speculate upon its properties at a temperature far above that at which it could have any existence.

#### HYDRATES OF NITRIC ACID.

The existence of definite hydrates of nitric acid has been discussed by various writers‡ from the standpoints of (i.) direct experimental observations; (ii.) analogy of the composition of the metallic nitrates and of the phosphoric acids; (iii.) differentiations of observed values; cumulative evidence from the first of the methods appears the most cogent. The following *résumé* gives briefly the conclusions of the several writers:—

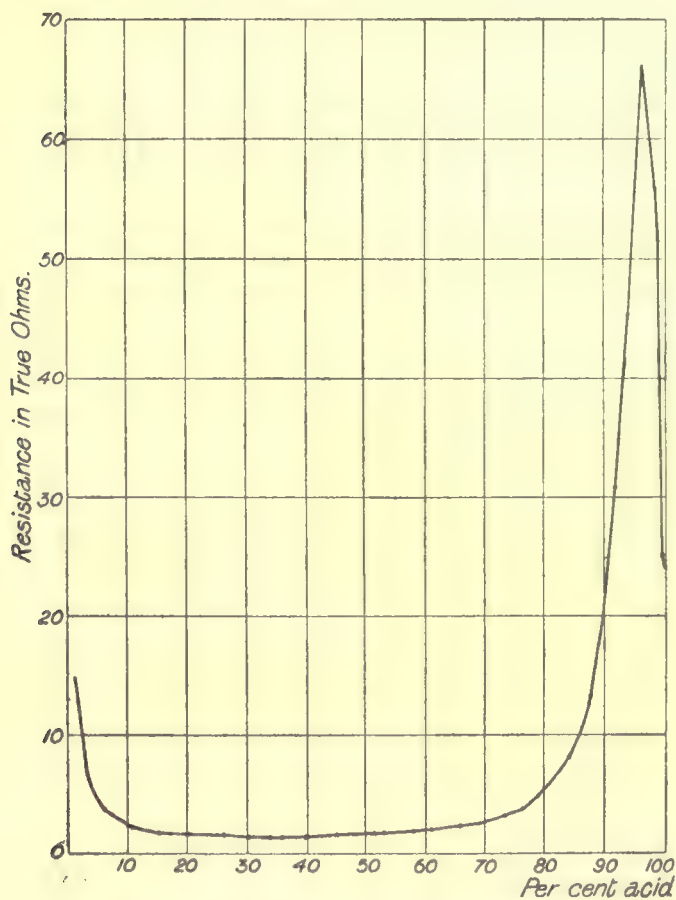
\* 'Zeits. f. Physikal. Chem.,' vol. 4, p. 96.

† Note added June 17, 1898.—Since the above was written, the suggestion has been put forward that in the case of nitric acid of 96–99·7 per cent., there is an initial decomposition into water and the anhydride  $N_2O_5$ , precisely as sulphuric acid to a greater degree at a higher temperature, and to a less degree at a lower temperature, decomposes into water and its anhydride  $SO_3$ .

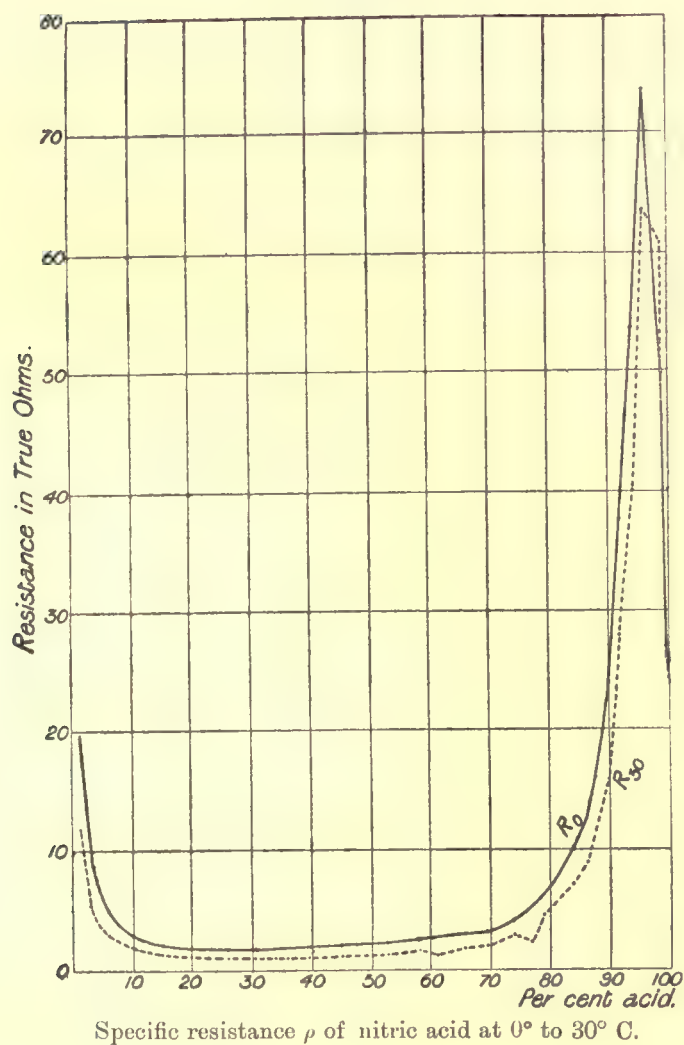
‡ MENDELEEFF, "Principles of Chemistry"; W. H. PERKIN, Sen., 'Journal of Chemical Society,' Trans., 1889, p. 724, and 1893, p. 65; KOLB (*vide supra*); BERTHELOT, 'Mécanique Chimique,' pp. 1, 397; PICKERING, 'Journal of Chemical Society,' Trans., 1893, p. 436; CROMPTON, *ibid.*, 1888, p. 121; VELEY, 'Ber. Deutsch. Chem. Ges.,' vol. 28, p. 928.

$\text{HNO}_3$ 10 $\text{H}_2\text{O}$	25.5	per cent. $\text{HNO}_3$	Differentiation of density curve (PICKERING).
$\text{HNO}_3$ 7 $\text{H}_2\text{O}$	33.3	„ „	{ Specific magnetic rotation (PERKIN). Freezing of solution (PICKERING). Chemical activity towards metals (VELEY).
$\text{HNO}_3$ 5 $\text{H}_2\text{O}$	41.2	„ „	Density curve (MENDELEEFF).
$\text{HNO}_3$ 3 $\text{H}_2\text{O}$	53.85	„ „	{ Isolation in crystals (PICKERING). Metallic nitrates (MENDELEEFF).
$\text{HNO}_3$ 2 $\text{H}_2\text{O}$	63.63	„ „	{ Vapour pressure (MENDELEEFF). Heat of dilution (BERTHELOT).
$\text{HNO}_3$ $\text{H}_2\text{O}$	78.02	„ „	{ Density determinations (KOLB). Isolation in crystals (PICKERING). Specific magnetic rotatory power, metallic nitrates and analogy with $\text{H}_3\text{PO}_4$ (PERKIN).

Curve 1.

Specific resistance  $\rho$  of nitric acid at 15° C.

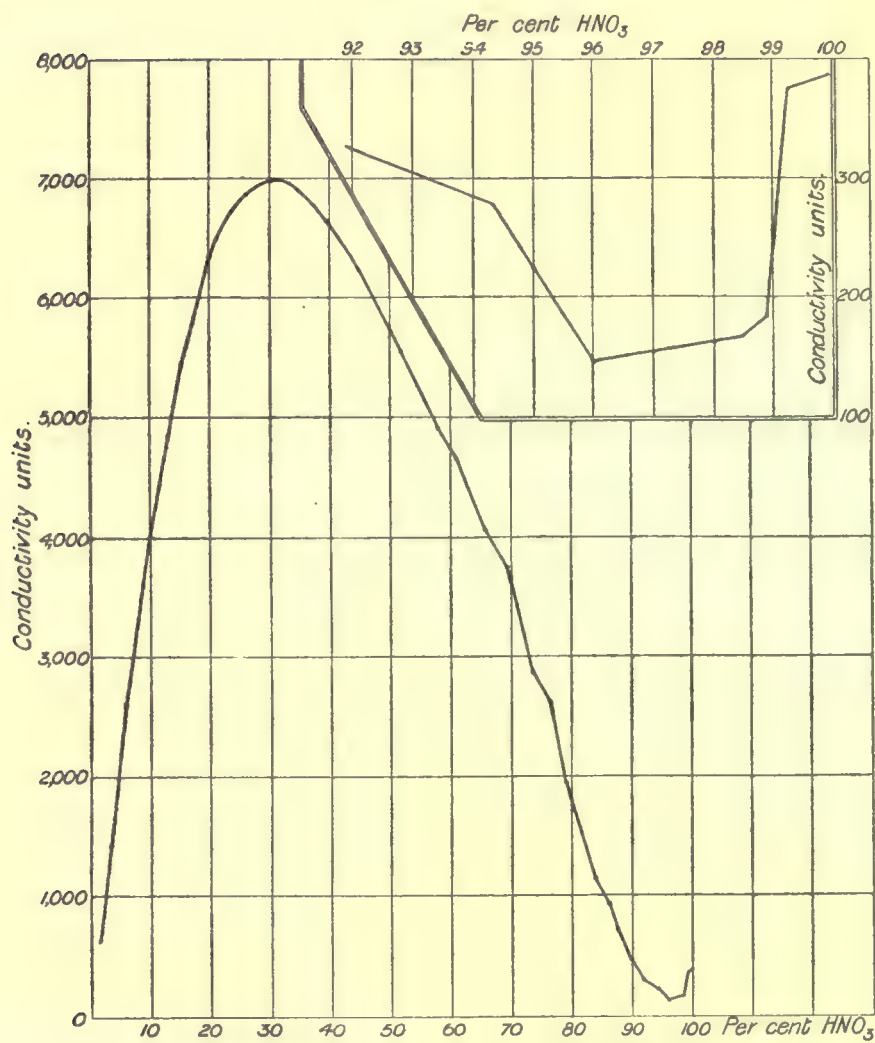
Curve 2.



In Table XI. the values for  $\alpha \times 10^4$  and  $\beta \times 10^6$  are given in relation to the *molecular* proportions of water from 3.07 to .07, as it would seem that the method adopted by comparatively few writers on such a subject as the present is the more rational, as nature deals with molecules, but art with percentages.



Curve 3.



Conductivity in mercury units of acid at 15° C.

TABLE XI.

Molecular proportions of water.	Value of $\alpha \times 10^4 (-)$ .	Value of $\beta \times 10^6 (+)$ .
3.07	211	275
2.43	210	277
2.25	190	77
1.84	208	292
1.59	177	228
1.30	174	236
1.09	153	6
0.99	206	377
0.78	166	226
0.65	161	212
0.58	177	294
0.47	120	46
0.39	146	202
0.27	133	182
0.18	192	152
0.07	4	136

From these figures it will be observed that the minima values for  $\alpha 10^4$  and maxima for  $\beta 10^6$  occur in the cases of 3.07, 1.84, .99, and .55 molecular proportions of water, or very approximately  $\text{HNO}_3.3\text{H}_2\text{O}$ ,  $\text{HNO}_3.2\text{H}_2\text{O}$ ,  $\text{HNO}_3.\text{H}_2\text{O}$ , and  $2\text{HNO}_3.\text{H}_2\text{O}$ , and the curve representing the conductivity units in terms of percentages is markedly discontinuous at points corresponding to the three last hydrates, the existence of which has also been confirmed by a series of experimental determinations of the densities and contractions of samples of acid of various concentrations. Further evidence is thus added by an independent method to that already accumulated as to the existence of definite combinations of nitric acid with water.

In conclusion, we trust that these observations may serve as an addition to our knowledge of the electric and chemical properties of dilute, and especially of the most concentrated nitric acid, purified, as we hope, with such methods as are available in the present state of chemical art. It appears that for this and similar investigations a material, which should at once be perfect in its transparency, its non-conductivity, and its unalterability by the strongest reagents, remains still an ideal.

X. *On the Thermal Conductivities of Single and Mixed Solids and Liquids  
and their Variation with Temperature.*

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Communicated by Professor ARTHUR SCHUSTER, F.R.S.

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PART I. CONDUCTIVITIES AND TEMPERATURE COEFFICIENTS OF SOLIDS.

*Sketch of Method.*

IN determining thermal conductivities\* of solids not very good conductors of heat, the method least open to objection on theoretical grounds is the one in which a spherical shell of the substance to be tested is filled with, and the exterior surrounded by, some good conductor of heat, the temperatures of the conductors, inside and out, being observed by means of thermometers or thermo-junctions, and maintained constant by heat supplied, *e.g.*, electrically, to the inner conductor at a measured rate (fig. 1). Difficulties, mainly of a mechanical kind, present themselves, however, in the carrying out of this method, which render it advisable to sacrifice some of the theoretical simplicity, in order to make the method more practicable.

Fig. 1.

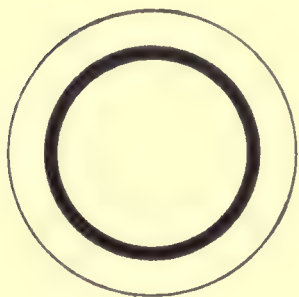


Fig. 2.

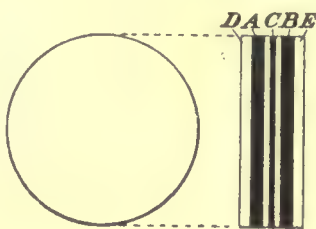
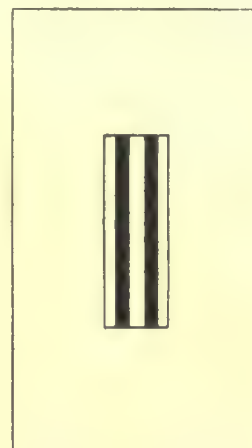


Fig. 3.



These difficulties are overcome most easily by having the material to be tested, the conductor to which heat is supplied, and the outside conductors, in the form of flat circular discs of the same diameter (fig. 2), the good-conducting disc, *C*, to which heat

\* The extremely good account of previous methods and results given by GRAETZ in WINKELMANN'S 'Handbuch der Physik,' vol. 2, pp. 273–314, renders an account of such work unnecessary here.



is supplied, being placed between two discs of the material to be tested, *A* and *B*, and two other good-conducting discs, *D* and *E*, placed in contact with the outer surfaces. By placing thermo-junctions in each good-conducting disc we can determine the relation between the heat supplied to the inner disc and the differences of temperature between the inner and outer discs.

If all the heat supplied to the inner disc flowed through the substance to be tested to the outer discs, the thermal conductivity of the material would be thus determined, but some heat is lost from the curved surfaces of the discs by conduction and radiation through the air to the enclosure in which the discs are placed. The amount thus lost can be determined by observing the temperature of the enclosure, if the loss per square centimetre per degree excess of temperature of the edges of the discs over the inside of the enclosure were known.

To enable this quantity to be determined directly by experiment, the whole of the heat supplied to the discs must be lost in the same way, *i.e.*, by conduction and radiation through air, and to satisfy this condition, the collection of discs should be placed in the centre of an air bath kept at constant temperature (fig. 3). The heat lost from the curved surfaces of the discs then follows with sufficient closeness the same law as that lost from the flat surfaces of the outer discs, and the relative amounts of the two can be determined from a knowledge of the areas and temperatures of the various surfaces, if the surfaces have the same "outer conductivity" or "emissivity," an equality which is easily secured by varnishing them.

Once the method of surrounding the discs by an enclosure at constant temperature has been adopted, a further simplification of the arrangement of discs is possible. If one of the discs of the substance under test and one of the outer discs were removed, the relative amounts of heat lost by the heating disc, *C* (fig. 4), by conduction and radiation directly to the air from its exposed surfaces, and by conduction through the disc, *B*, of material experimented on; partly to the air directly by conduction and radiation from the surface of *B*, and partly by conduction to the outer good-conducting disc, *E*, and radiation from its surfaces, could still be calculated, and, the total heat supplied being known, the conductivity of the material determined. On account of this method requiring only one disc of a substance the conductivity of which is to be determined, it has been adopted in the experiments to be described.

#### *Description of Apparatus.*

The materials to be tested were cut into discs of 4 centims. diameter, and of various thicknesses, in order to make the magnitude of the difference of temperature of the two flat surfaces, suitable for observation with the thermo-junctions employed. The discs between which the material was placed were of copper, and had the same diameter. On one side of the substance to be tested was placed a single copper disc, *M* (fig. 5), .320 centims. thick, on the other side a compound disc, consisting of

two discs of copper, *C*, .103 centim., and *U*, .312 centim. thick, with a flat heating coil of double silk-covered No. 25 platinoid wire, *P*, between them, well insulated by means of mica and shellac, the total thickness of the coil and mica being .110 centim. The contacts between the surfaces of the copper discs *U* and *M*, and the substances, were improved by smearing them with glycerine and sliding them together. The glycerine was generally sufficient to keep the discs together during an experiment, but to ensure the surfaces remaining properly in contact, a few turns of silk thread were wound round the discs.

At opposite extremities of a diameter of each disc, two holes, .7 millim. diameter and 3 millim. deep, were bored towards the centre of the disc, and into one hole of

Fig. 4.

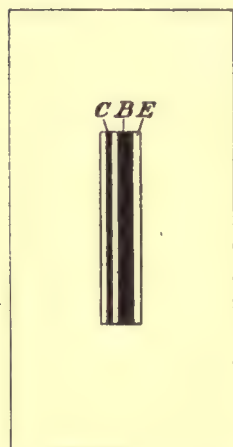


Fig. 5.



Fig. 6.

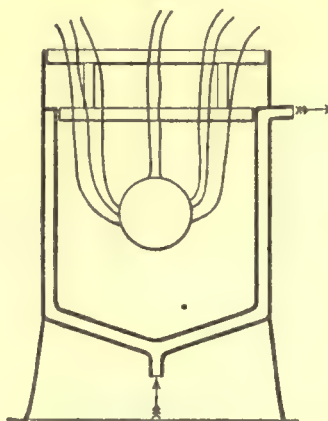
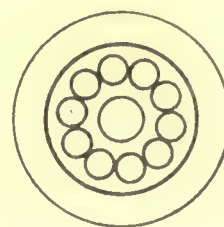


Fig. 7.



each disc a platinoid, and into the other hole a copper, wire was soldered. The wires were double silk covered, about 30 centims. long, and were all cut from the same two coils.

The surfaces of the pile of discs were varnished to give them the same emissivity, and the pile placed in the centre of a double-walled air-bath (fig. 6), of 17 centims. height, 15 centims. length, and 7 centims. breadth, supported on legs about 4 centims. above the table. The top of this bath consisted of a double wooden lid, cut down the centre to allow the wires from the discs to pass out. The edges of the lid were covered with green baize, in which the wires embedded themselves, and which closed the bath sufficiently to prevent convection currents between the inside and outside. On the outer walls of the air-bath a coil of insulated platinoid wire was wound, and an electric current could be sent through the wire to keep the bath at any required temperature. The inner walls were about 1 centim. within the outer, and between the two water could be circulated, or the space could be filled with air simply. A thermo-junction was soldered to the centre of an inner surface, and enabled the temperature of the enclosure to be determined. The wires from the heating coil, *P*, after passing out through the lid, dipped into mercury cups, from which wires passed to the cells supplying the heating current, the power absorbed being regulated by a rheostat and measured by a wattmeter.



Each of the thermo-junction wires was brought to a mercury cup, made by cutting 4 centim. lengths from a piece of thin glass tubing with a bore of about 3 millims., sealing one end of each length, and fastening it, with the sealed end down, in a brass tube 5 centims. long and 3 centims. in diameter, mounted in a block of wood and containing mercury. Ten of these glass tubes nearly full of mercury were arranged in a circle round the bulb of a thermometer graduated in  $\frac{1}{10}$  degrees, which was placed in the centre of the brass tube. To diminish loss or gain of heat, the brass tube was wrapped with green baize (fig. 7). The ends of the wires leading to the potentiometer arrangement for measuring the electromotive force in the thermo-electric circuits, could be placed in any two of these mercury cups, and by means of an additional short wire of platinoid, any two thermo-circuits could be placed in series, and the difference of the electromotive forces in the two determined.

*Theory of Flow of Heat in Discs.*

Let  $H$  = heat generated per second in the heating coil.

$h$  = heat lost per second, per square centimetre per degree excess of temperature of the discs over that of the enclosure.

$v$  = excess of temperature over that of enclosure.

$t$  = actual thickness of a disc, plus a small correction, p. (409) to allow for heat lost along thermo-wires.

$r$  = radius of discs.

Subscripts C, U, M, S refer to "cover" "upper" "middle" discs,\* and disc of substance respectively, the heating coil coming between C and U, and the substance to be tested between U and M (fig. 5).

When the "steady state" has been attained, the heat received per second by the copper disc M from the disc of substance S, and given up to the air

$$= \pi r^2 h \left( 1 + \frac{2t_M}{r} \right) v_M.$$

Similarly, the heat received by the disc S, of material tested and given up to the air and to the disc M

$$= \pi r^2 h \left( 1 + \frac{2t_M}{r} \right) v_M + \pi r^2 h \frac{2t_S}{r} \cdot \frac{v_M + v_U}{2}.$$

If  $k$  is the thermal conductivity of the material, the heat flowing through the disc tested

$$= \pi r^2 k \frac{v_U - v_M}{t_S}.$$

\* The reason for the choice of these terms will be understood from Part II.



Hence we have, to a very close degree of approximation,\*

$$k \frac{v_U - v_M}{t_S} = h \left\{ v_M + \frac{2}{r} \left( t_M + \frac{t_S}{4} \right) v_M + \frac{t_S}{4} v_U \right\}.$$

The total heat imparted to the discs and lost from their surfaces

$$= H = \pi r^2 h \left\{ v_M + v_C + \frac{2}{r} \left( t_M v_M + t_S \frac{v_M + v_U}{2} + t_U v_U + t_C v_C \right) \right\}.$$

Eliminating  $h$  between these equations we have

$$k = H \frac{\left( 1 + \frac{2}{r} t_M + \frac{t_S}{4} \right) v_M + \frac{t_S}{4} v_U}{\left( 1 + \frac{2}{r} t_M + \frac{t_S}{2} \right) v_M + \left( 1 + \frac{2}{r} t_C \right) v_C + \frac{2}{r} \left( t_U + \frac{t_S}{2} \right) v_U} \cdot \frac{t_S}{\pi r^2 (v_U - v_M)},$$

and from the second equation we have

$$h = \frac{H}{\pi r^2 \left\{ \left( 1 + \frac{2}{r} t_M + \frac{t_S}{2} \right) v_M + \left( 1 + \frac{2}{r} t_C \right) v_C + \frac{2}{r} \left( t_U + \frac{t_S}{2} \right) v_U \right\}},$$

which will enable us to determine  $k$  and  $h$  from observations of  $H$  and the temperatures of the discs.

In the foregoing theory it has been assumed that the steady state of temperature distribution had been attained. As, however, an infinite time is required before this condition is satisfied, it remains to determine at what previous period observations may be taken without the results deduced from these observations being in error by say  $\frac{1}{4}$  per cent.

If the temperature of the "middle disc" is increasing at a rate  $dv_M/dt$ , the expression for the heat it receives becomes

$$\pi r^2 h \left( 1 + \frac{2t_M}{r} \right) v_M + m_M c_M \frac{dv_M}{dt}$$

where  $m_M$  is the mass, and  $c_M$  the specific heat of the material, of the disc.

Similarly, the expression for the heat received by the disc of substance, since we may assume that the discs increase in temperature at the same rate  $dv/dt$ , becomes

$$\pi r^2 h \left( 1 + \frac{2t_M}{r} \right) v_M + \pi r^2 h \frac{2t_S}{r} \cdot \frac{v_M + v_U}{2} + (m_M c_M + m_S c_S) \frac{dv}{dt}.$$

\* It will be noticed that the heat flowing through the substance has been taken equal to the mean of the heats flowing into and out of it respectively. The closeness of this approximation may be tested by using the values of  $v$  in terms of  $x$ , obtained on the usual assumption of plane-isothermal surfaces, *i.e.*,

$$v = A \cosh ax + B \sinh ax.$$

Hence,

$$\pi r^2 k \frac{v_U - v_M}{t_S} = \pi r^2 h \left\{ v_M + \frac{2}{r} \left( t_M + \frac{t_S}{4} \right) v_M + \frac{t_S}{4} v_U \right\} + \left( m_M c_M + \frac{m_S c_S}{2} \right) \frac{dv}{dt},$$

and the total heat imparted to the discs

$$= H = \pi r^2 h \left\{ v_M + v_C + \frac{2}{r} \left( t_M v_M + t_S \frac{v_M + v_U}{2} + t_U v_U + t_C v_C \right) \right\} + \frac{dv}{dt} \Sigma mc.$$

Eliminating  $h$  between these equations, we have

$$\begin{aligned} \pi r^2 k \frac{v_U - v_M}{t_S} - \left( m_M c_M + \frac{m_S c_S}{2} \right) \frac{dv}{dt} \\ = \left( H - \frac{dv}{dt} \Sigma mc \right) \frac{\left( 1 + \frac{2}{r} t_M + \frac{t_S}{4} \right) v_M + \frac{t_S}{4} v_U}{\left( 1 + \frac{2}{r} t_M + \frac{t_S}{2} \right) v_M + \left( 1 + \frac{2}{r} t_C \right) v_C + \frac{2}{r} \left( t_U + \frac{t_S}{2} \right) v_U}. \end{aligned}$$

For the discs used  $m_M c_M = 3.2$ ,  $m_S c_S = 1.8$ ,  $\Sigma mc = 11.5$ , and during the experiments  $H$  was about .22, and the value of the fraction on the right-hand side of the equation, .4. Hence, for the error introduced by neglecting  $dv/dt$  to be less than  $\frac{1}{4}$  per cent., we must have

$$(-4.1 + 4.6) \frac{dv}{dt} < \frac{.09}{.400},$$

or

$$dv/dt < .00045,$$

so the change of temperature in 1 minute ought not to exceed .03° C.

No observations in which this rate was exceeded have been used in what follows.

### *Loss of Heat along Thermo-Wires.*

In the above theory, the heat conducted away from the discs by the wires of the thermo-junctions, has been taken into account by adding a small correction to the thickness of the discs. To determine the amount of this correction for any wire, we write

$$v = v_0 e^{-\sqrt{\frac{ph}{qk}} \cdot x}$$

where  $v$  is the temperature excess at a point of the wire distant  $x$  from the disc,

- $p$  = perimeter of section of the wire.
- $q$  = area                   "                   "
- $k$  = conductivity of the wire.
- $h$  = emissivity                   "

The heat conducted away by the wire per second

$$\begin{aligned} &= -qk \frac{dv}{dx}, \text{ for } x = 0, \\ &= \sqrt{qk \cdot ph} \cdot v_0. \end{aligned}$$

For the copper wire used (No. 28),  $p = \cdot 15$ , allowing for silk covering;  $h = \cdot 0003$ ,  $q = \cdot 0011$ , and  $k = 1\cdot 0$ , therefore

$$\sqrt{qk \cdot ph} = 2\cdot 2 \times 10^{-4}.$$

For the platinoid wire, which had the same diameter, we have  $k = \cdot 08$ , and

$$\sqrt{qk \cdot ph} = \cdot 6 \times 10^{-4},$$

therefore total loss of heat along thermo-wires from one disc per second

$$= 2\cdot 8 \times 10^{-4} \cdot v_0.$$

If  $w$  is the thickness of a disc which would lose the same amount of heat from its edges, we have, since the loss of heat from such a disc  $= 2\pi r w h v_0$ ,

$$w = \frac{2\cdot 8 \times 10^{-4}}{2\pi r h} = \frac{2\cdot 8 \times 10^{-4}}{12\cdot 6 \times 3 \times 10^{-4}} = \cdot 075 \text{ centims.},$$

and the loss along the thermo-wires may be taken into account in the theory of flow of heat in the discs, by taking as the thickness in the formula, p. 403, the measured thickness of the disc plus this quantity.

#### *Constants of the Discs.*

$r = 2$  centims., therefore  $\pi r^2 = 12\cdot 5$  sq. centims.

$w = \cdot 075$  centims.

$t_M = \cdot 32 + \cdot 075 = \cdot 395$  centims., therefore  $1 + \frac{2}{r} t_M + \frac{t_s}{4} = 1\cdot 395 + \frac{t_s}{4}$ .

$t_C = \cdot 103 + \cdot 075 + \cdot 055^* = \cdot 233$  centims., therefore  $1 + \frac{2}{r} t_C = 1\cdot 233$ .

$t_U = \cdot 312 + \cdot 075 + \cdot 055^* = \cdot 442$  centims., therefore  $\frac{2}{r} t_U + \frac{t_s}{2} = \cdot 442 + \frac{t_s}{2}$ .

\* Half the thickness of the heating coil and mica insulation is added to the upper disc and half to the cover.



Hence the formulæ, p. 403, become

$$k = H \cdot \frac{1.395 + \frac{t_s}{4} v_M + \frac{t_s}{4} v_U}{1.395 + \frac{t_s}{2} v_M + 1.233 v_C + .442 + \frac{t_s}{2} v_U} \cdot \frac{t_s}{12.5 v_U - v_M}.$$

$$h = \frac{H}{1.395 + \frac{t_s}{2} v_M + 1.233 v_C + .442 + \frac{t_s}{2} v_U} \cdot \frac{1}{12.5} *.$$

### *Measurement of Power.*

The power given to the heating coil of the discs was measured by means of a wattmeter, with its series coils in the main circuit, and the terminals of its shunt coil dipping into the same mercury cups as the terminals of the heating coil. The shunt coil had a resistance of 12 ohms, and was suspended bifilarly by means of the two leading wires. The instrument was standardised by sending a current through a 2-ohm resistance coil of thick platinoid wire, and regulating the strength of this current till the electromotive force at the terminals of the resistance was equal to that of a Clark cell. The wattmeter readings were taken by reversing the connections of the suspended coil to the heating coil in which power was to be measured. The series coils were placed parallel to the magnetic meridian, so that the effect of the earth field on the deflection would be small, and there would be no necessity for reversing the current in the heating circuit. For convenience in working out results approximately, the resistance in series with the suspended coil (320 ohms) was adjusted till the wattmeter deflection was nearly 1000 divisions per gram degree† of heat, generated per second in the heating coil.

If  $H$  is the heat generated per second in a coil of resistance  $R$ , and  $d$  is the deflection on the wattmeter, the shunt coil of which has a resistance of  $r$  ohms, we have

\* The value of  $h$  is only required in this work to make certain small corrections in the course of the calculation, and the above equation will not be further referred to. The mean values of  $h$  found during the course of the work, are :—

Mean temperature of discs and enclosure.	$h$ .
24 C.	·000276
54	316
72	339

† Water gram degree at 15° C. = 4·20 joules.

$$H = K \cdot d \cdot \frac{r}{R + r}$$

where  $K$  is a constant.

In one of the standardisation experiments the Clark cell had a temperature of  $19^{\circ}$  C. and an E.M.F. of 1.428 volts. The deflection on the wattmeter was found to be 245.5. Hence, since  $R = 2.018$ ,  $r = 332$  ohms, we have

$$K = \frac{(1.428)^2}{2.018 \times 4.2} \cdot \frac{1}{245.5} \cdot \frac{334}{332} = .000986.$$

When, therefore, the wattmeter was connected to the heating coil of the discs, which had a resistance of 3.3 ohms,

$$H = .000986 \frac{332}{335.3} \cdot d = .000976 \cdot d,$$

from which the heat generated in the coil could be found from the wattmeter deflection.

The power thus measured is that spent in the heating coil and the connections leading to it. A portion only of this power is imparted to the discs themselves, the rest is spent in the short lengths (1.6 centim.) of the platinoid wire projecting outside the discs, and in the No. 22 copper wires leading to the mercury cups outside the air-bath. The total length of the platinoid wire was 150 centims., and 3.2 centims. of it, and 50 centims. of No. 22 copper wire, were outside the discs. The resistance of the coil and connection was 3.3 ohms, .02 ohms being due to the copper, and .07 ohms to the projecting platinoid wire. Of the total power generated, therefore, only  $321/330 = .973$  is imparted directly to the discs.

There is, however, an indirect method by which some of the remaining power may be communicated to the discs, *i.e.*, by thermal conduction along the coil wire, and to determine the amount of this, we require to know the distribution of temperature along the wire.

Assuming that the isothermal surfaces in the wire are planes perpendicular to the axis, we have for the "steady state"

$$qk \frac{d^2v}{dx^2} + w = ph \cdot v,$$

where

$q$  = cross section of wire,

$p$  = perimeter of wire,

$k$  = conductivity of wire,

$h$  = emissivity of wire,

$w$  = heat developed in 1 centim. length of wire,

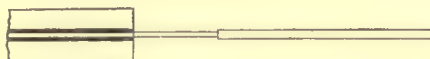
$v$  = temperature excess over air, at a point distant  $x$  from the junction of the platinoid and copper wires.

The solution of this equation is

$$v - \frac{w}{ph} = Ae^{\sqrt{\frac{ph}{qk}} \cdot x} + Be^{-\sqrt{\frac{ph}{qk}} \cdot x}$$

where A and B are constants of integration.

Fig. 8.



For the platinoid wire,  $w = \cdot 21/150 = \cdot 0014$ ,  $p = \cdot 027$ ,  $h = \cdot 0003$ ,  $q = \cdot 002$ ,  $k = \cdot 08$ , therefore

$$\frac{w}{ph} = 170 \text{ and } \sqrt{\frac{ph}{qk}} = \cdot 22.$$

Hence in the platinoid wire

$$v_1 - 170 = A_1 e^{\cdot 22x} + B_1 e^{-\cdot 22x}.$$

For the copper wire,  $w = \cdot 0014/40$ ,  $p = \cdot 029$ ,  $h = \cdot 0003$ ,  $q = \cdot 004$ ,  $k = 1$ , therefore,

$$\frac{w}{ph} = 4 \text{ and } \sqrt{\frac{ph}{qk}} = \cdot 047.$$

Since the temperature of the copper wire to the right is nowhere infinite, we have in the copper wire

$$v_2 - 4 = B_2 e^{-\cdot 047x}$$

where  $B_2$  is a constant of integration.

When  $x = -1\cdot 6$  in the expression for the temperature of the platinoid wire, the temperature excess  $v_1$  should be equal to that of the discs, which, on the average was  $17^\circ \text{C}$ .

Hence,

$$-153 = A_1 e^{-\cdot 35} + B_1 e^{+\cdot 35} = \cdot 705A_1 + 1\cdot 42B_1.$$

Also at  $x = 0$  the temperatures and fluxes in the two wires are alike.

Therefore,

$$\begin{aligned} 170 + A_1 + B_1 &= 4 + B_2 \\ \text{and } 35(A_1 - B_1) &= -190B_2. \end{aligned}$$

These three equations determine  $A_1$ ,  $B_1$ ,  $B_2$ , and give for the temperature in the platinoid wire,

$$v_1 - 170 = -100e^{\cdot 22x} - 57\cdot 8e^{-\cdot 22x}.$$

The heat flowing per second from the wire to the discs at  $x = -1\cdot 6$



$$\begin{aligned}
&= qk \cdot 22 (-100e^{-.35} + 57.8e^{+.35}) \\
&= .00016 \times .22 (-100 \times .705 + 57.8 \times 1.42) \\
&= .000035 (-70.5 + 82.1) \\
&= .0004 \text{ gram degrees.}^*
\end{aligned}$$

Hence, .0008 gram degrees of heat flow per second along the two wires to the discs.

Of the .21 gram degrees of heat imparted in the coil  $.21 \times .027 = .0057$  is expended outside the discs, and of this .0008 is conveyed to the discs by conduction, leaving .0049, or about 2.4 per cent., which does not reach the discs. The wattmeter readings must therefore be diminished by this amount to give the power actually imparted to the discs.

Combining this result with the one previously found, we have for the heat,  $H$ , imparted to the discs,

$$H = .000953 d,$$

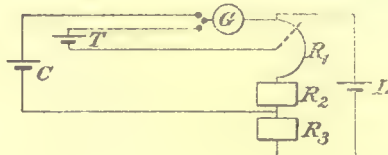
when  $d$  is the deflection on the wattmeter.

#### *The Thermo-electric Measurements.*

The principle adopted in measuring the electromotive forces in the various thermo-electric circuits, was that of balancing them against the fall of potential down a high resistance in series with a cell of known electromotive force.

The arrangements of the circuits will be understood from the accompanying diagram.

Fig. 9.



$R_1$ ,  $R_2$ , and  $R_3$  are resistances in series with the Leclanché cell,  $L$ .  $R_2$  and  $R_3$  are resistance boxes of platinoid, of 30,000 ohms and 10,000 ohms respectively.  $R_3$  is adjustable, and it was varied until the fall of potential down the 30,000 ohms was equal to that of a Clark cell,  $C$ . This adjustment was not absolutely necessary, but was carried out for convenience by the arrangement shown in the figure. The thermocouple,  $T$ , is balanced against the fall of potential down a part of  $R_1$ , which represents three dial resistances of copper wire, only one of which is shown in the figure. The movable arms of the dials consisted of short thick copper wires dipping into mercury cups. The dials read respectively, ohms, .1 ohms, and .01 ohms. They were made

\* More generally the heat conducted along one wire to the discs  $= .0077H - .000071v$ , where  $H$  is the heat generated in the coil, and  $v$  the temperature excess of the discs.

of copper wire to get rid of thermo-electric forces in the circuit,  $T$ , other than those to be measured.

The galvanometer enabled the determinations to be made to about  $\cdot 003$  ohm. By means of the mercury cups (fig. 7) any one of the thermo-electric circuits could be connected through the galvanometer to  $R_1$ , and the electromotive force in it determined, or by arranging two of the circuits in opposition, the difference of electromotive forces in any pair could be found.

### *Standardisation of Thermo-couples.*

To determine the constants of each of the thermo-couples, the various copper discs were bound together by tape, a thin sheet of mica being placed between each disc for insulation, and the bulb of a thermometer graduated in  $\frac{1}{10}$  degrees was placed in contact with, and bound to, the discs by a few turns of the same tape. The combination was then placed in its usual position in the air bath, and the bath heated by sending a current through the coil encircling it. When the temperature of the bath had been raised to about  $50^\circ \text{C}$ ., the current round it was diminished till the temperature ceased to rise, and further adjusted till the temperature remained constant. After about half-an-hour the indications of the thermometer in the bath, and of a similar thermometer in the mercury cups in which the other junctions were placed, were read, the electromotive forces in each thermo-circuit found, and the thermometers again read. After a few minutes' interval the process was repeated.

Since in the test of conductivity it is the difference of the temperatures of the discs which is required, observations of differences of the electromotive forces in the various circuits were in each case taken. As these differences were small, and could be observed with the same degree of accuracy as the actual electromotive forces, which were comparatively large, the calculations of conductivity have all been based on them, one of the observations of actual electromotive force having been used to fix the mean temperature of the disc experimented on.

The following table gives the results of the observations :—

$\theta$  = the temperature of the discs in the bath.

$\theta_0$  = that of the mercury cups.

$U$  = the number of ohms of the potentiometer necessary to give a balance in the circuit from the "upper disc." 1 ohm corresponds to  $1\cdot434/30,000 = \cdot 000044$  volt.

$U - M$  = the ohms giving a balance when "upper" and "middle" circuits were opposed.

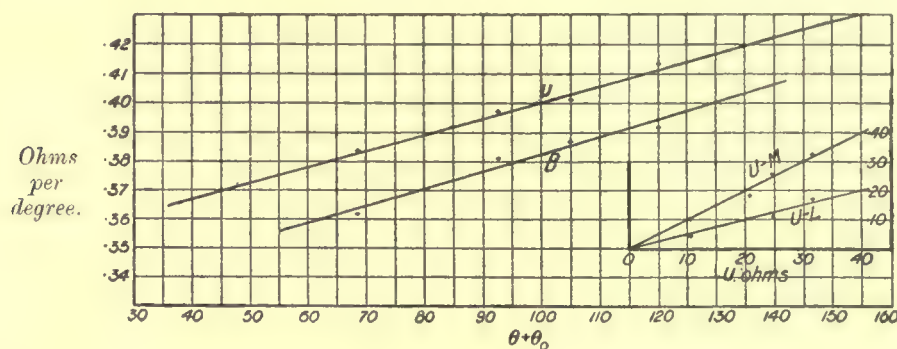
$U - L$  = ditto . . . ditto, when "upper" and "lower" opposed.

$B$  = ditto . . . ditto, when thermo-junction from inside surface of air bath was in circuit.

For convenience of interpolation the numbers headed "ohms per degree" are taken as ordinates, and  $\theta + \theta_0$  as abscissæ, of the curves which follow the table.

### CONSTANTS of Thermo-junctions.

$\theta$ .	$\theta_0$ .	Ohms.				Divisors. Ohms per degree.		$\theta + \theta_0$ .
		U.	U - M.	U - L.	B.	U.	B.	
47.68	20.80	10.31	0.10	0.05	9.72	0.384	0.362	68.5
47.57	20.91	10.21	0.10	0.04	9.67	0.383	0.362	68.5
82.63	21.75	24.46	0.26	0.12	..	0.402	..	104.9
83.13	21.87	24.67	0.24	0.10	23.73	0.402	0.387	104.9
97.63	21.87	31.34	0.31	0.16	..	0.414	..	119.5
97.63	21.88	31.34	0.32	0.17	29.70	0.414	0.392	119.5
72.60	19.92	20.91	0.18	..	20.05	0.397	0.381	92.5
72.66	19.41	21.15	0.19	..	20.26	0.397	0.381	92.1



It will be noticed from the table that although the wires used for the thermo-junctions were cut from the same bobbins and treated alike subsequently, the thermo-circuits had different constants, the difference in the case of the upper and middle circuits amounting to 1 per cent., and in the case of the upper and heater circuits to 6 per cent.

This prevents the difference of the temperatures of two junctions being determined directly by dividing the electromotive force in circuit when the junctions are placed in series, by a constant. The method to be used is indicated subsequently.

The "cover" and "upper" circuits were found to give identical results, and the upper junction was therefore adopted as the standard, and the differences between its electromotive force and those of the other junctions determined in each case.

To determine the temperature of the junction in the upper disc, given the temperature of the junction in the mercury cups, and the electromotive force in the circuit, we require to divide this electromotive force by a quantity which itself varies to a small extent with the temperature required. If, however, this tempe-



perature is known approximately, the value of the divisor may be found from the upper of the preceding curves, and the temperature found more accurately. If necessary this new value of the temperature may be used to find the divisor more accurately, and from it a second approximation to the temperature.

To obviate the necessity of determining the temperature of the disc in this way by two stages, a table of divisors was constructed having a double entry, one, the temperature of the junction at the mercury cups, the other the resistance of the potentiometer for a balance. The form of the table will be seen from the part of it given below :—

Resistance.	Temperature of mercury cups.			
	16°.	17°.	18°.	19°.
Ohms.				
1	·365	·366	·367	·368
2	·367	·368	·369	·370
3	·368	·369	·370	·371
..				
16	·386	·387	·388	·390
17	·387	·388	·390	·391
18	·389	·390	·391	·392

To determine the temperature of the upper junction when the junction in the mercury cups had a temperature of say 18° C., and the resistance for a balance was say 3 ohms, we find from the table that the divisor is ·388, and the temperature difference between the junctions is  $3/\cdot388 = 7\cdot73$ , and the temperature required therefore = 25·73° C.

The same plan was adopted for the junction in the air bath.

### *Theory of Observations with Thermo-junctions in Series.*

If two homogeneous wires of materials  $a$  and  $b$  have their ends soldered together, and the two junctions are kept at temperatures  $\theta_1$  and  $\theta_0$ , the electromotive force  $E_1$  in the circuit is given by an equation of the form

$$E_1 = (\theta_1 - \theta_0) e (1 + f \overline{\theta_1 + \theta_0})$$

where  $e$  and  $f$  are constants.

If the junctions of a second pair of materials,  $a'$  and  $b$ , where  $a'$  differs slightly from  $a$  in thermo-electric properties, are kept at temperatures  $\theta_2$  and  $\theta_0$ , we have

$$E_2 = (\theta_2 - \theta_0) e' (1 + f' \overline{\theta_2 + \theta_0})$$

where  $e'$  and  $f'$  are constants.

If at the junctions at temperature  $\theta_0$ , the two circuits are joined in series, the electromotive force in the circuit is the difference of the above, and if we write it  $E_{210}$  we have

$$E_{210} = (\theta_2 - \theta_0) e' (1 + f' \overline{\theta_2 + \theta_0}) - (\theta_1 - \theta_0) e (1 + f \overline{\theta_1 + \theta_0}).$$

Let  $\theta_2$  become identical with  $\theta_1$ , and let  $E_{110}$  now be the electromotive force in circuit. Then

$$E_{110} = (\theta_1 - \theta_0) e' (1 + f' \overline{\theta_1 + \theta_0}) - (\theta_1 - \theta_0) e (1 + f \overline{\theta_1 + \theta_0}).$$

Therefore

$$E_{210} - E_{110} = (\theta_2 - \theta_1) e' (1 + f' \overline{\theta_2 + \theta_1}),$$

or

$$\theta_2 - \theta_1 = (E_{210} - E_{110}) / e' (1 + f' \overline{\theta_2 + \theta_1}),$$

which furnishes a means of determining  $\theta_2 - \theta_1$  from observations of the electromotive force when the circuits are in series, and of the electromotive force in the same circuits when the temperatures  $\theta_2$  and  $\theta_1$  are identical. In the tables which follow the quantity  $E_{210}$  is indicated by the term "Observed," and  $E_{210} - E_{110}$  by the term "Reduced."

In the experiments,  $\theta_0$  is the temperature of the system of mercury cups (fig. 7, p. 401), indicated in what follows by J. The junctions in the discs are indicated by the letters U, M, C, B, for upper, middle, cover, and junction inside the air bath respectively.

### *Method of Experimenting.*

In carrying out the test of a disc of any substance, the copper discs, between which the disc to be tested was to be placed, were first brought together, and the total thickness of the combination measured by means of a micrometer wire gauge. The thickness of the disc to be tested was then measured, the disc placed in the proper position between the copper discs, glycerine being used to improve the contacts, and the total thickness measured. The differences of the observed thicknesses is the thickness of the disc of the material used, plus that of the two layers of glycerine, and we know, therefore, the thickness of the glycerine layers, and can calculate their effect on the flow of heat.

The combination was then suspended in the centre of the air bath, with the plane surfaces of the discs vertical. An electric current was then sent through the heating coil of the discs, which increased in temperature and began to lose heat to the bath.

If the test was to be carried out at a moderate temperature, current was also sent through the coil encircling the air bath for a short time, then stopped. Owing to the heat supplied by the coil to the discs, and given up by them to the bath, the temperature of the bath remains higher than that of the air, and would eventually become constant. The currents through the coil of the discs, and through the coil round the bath, were, however, adjusted from time to time, in order to attain the steady state of temperature as quickly as possible. When the temperatures of the discs and of the air bath had remained constant for 10 or 15 minutes, observations of the wattmeter, and of the electromotive forces in the various thermo-circuits, were made, an interval of 5 or 10 minutes allowed to elapse, and the observations repeated. If the second set did not show that the steady state had been reached, the temperature of the bath was varied suitably, and observations repeated. In this way it was insured that the observations were made during a state which approximated with sufficient closeness (p. 404) to the steady state.

If the test was to be carried out at a lower temperature, cold water was allowed to circulate between the inner and outer walls of the air bath, the observations being taken as before.

If the test was to be carried out at a higher temperature, the current through the heating coil was increased for a time, and current sent through the coil on the air bath till its temperature had been raised to the required amount. The current through the heating coil of the discs was then reduced to its normal amount, and the temperatures throughout allowed to become steady. Observations were taken as before. In cases where there was any suspicion of a change of the material under test having been brought about by the increase of temperature, for additional security the temperature of the bath was again lowered and a second test made at the original temperature.

The following tables and curves show the substances tested and the results obtained :

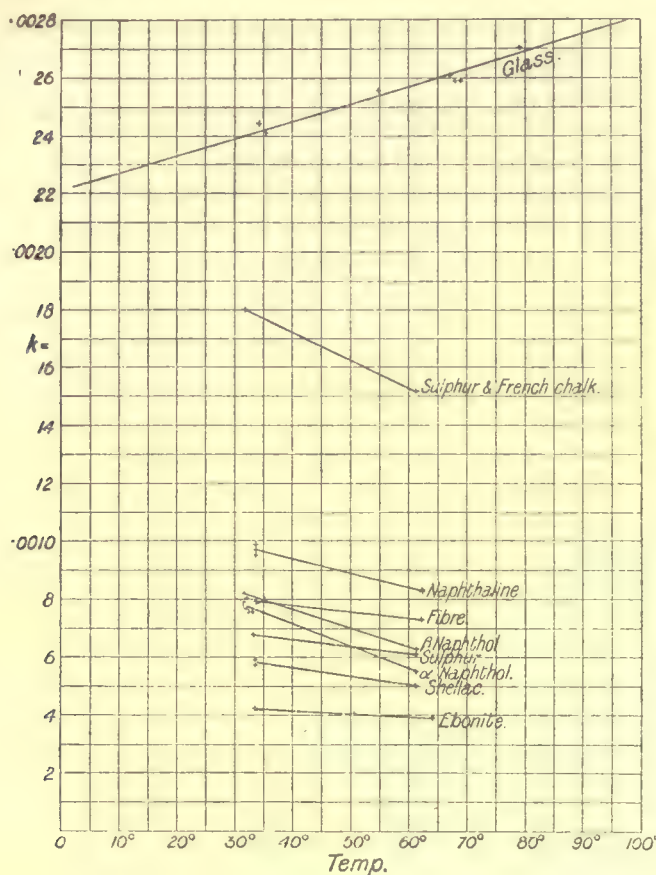


Substance.	$t_g$ centims.	$\theta_f$ .	C ohms.	U ohms.	Divisor.	$\theta - \theta_f$ .	$\theta$ .	U - M ohms.		Divisor.	$\theta - \theta_M$ .	$\theta_M$ .	B ohms.	Divisor.	$\theta_B - \theta_f$ .	$\theta$ .
Glass (July 31)	.281	17.92	6.44	6.21	.376	17.13	35.05	16.52	$\theta$ .							16.07
		20.27	5.93	5.70	.377	15.73	36.00	15.12	Observed.	Reduced.	.893	33.55	-0.64	.347	-1.85	16.45
		20.51	19.00	18.75	.395	48.10	68.61	47.47	.400	.342	.895	34.48	-1.33	.348	-3.82	16.45
Glass (Aug. 4)	.281	17.81	6.77	6.55	.376	18.01	35.82	17.42	.520	.340	.811	67.17	11.58	.367	31.55	52.06
		18.43	14.52	14.30	.387	37.52	55.95	36.95	.405	.345	.898	34.33	-0.58	.346	-1.68	16.13
		18.74	19.58	19.36	.394	49.70	68.44	49.14	.487	.347	.855	54.53	7.05	.357	19.75	38.18
		19.05	24.60	24.32	.401	61.35	80.40	60.65	.540	.350	.835	67.05	11.84	.366	32.35	51.09
									.590	.350	.802	78.90	16.81	.373	45.07	64.12

	$v_c$ .	$v_u$ .	$v_M$ .	Heat generated in gram degrees.	$(1.395 + \frac{1}{4} t_s) v_M$ .	$\frac{1}{4} t_s \cdot v_u$ .	Sum.	$(1.395 + \frac{1}{2} t_s) v_M$ .	$1.233 \cdot v_u$ .	$(.442 + \frac{1}{2} t_s) v_u$ .	Sum.	Ratio of sums.	Heat trans- mitted.	Heat trans- mitted per sq. centim.	Temperature slope.	Conductivity.	Temperature.
Glass (July 31) (continued)	18.98	18.37	17.48	.220	25.6	1.3	26.9	26.9	23.4	10.3	60.6	.444	.098	.00783	3.18	.00246	34
	19.55	18.94	18.03	.220	26.4	1.3	27.7	27.6	24.1	11.0	62.7	.442	.097	.00776	3.22	.00241	35
	16.55	15.92	15.11	.213	22.1	1.1	23.2	23.1	20.3	9.3	52.7	.410	.094	.00750	2.89	.00259	68
Glass (Aug. 4) (continued)	19.69	19.10	18.20	.220	26.7	1.3	28.0	28.0	24.4	11.1	63.5	.441	.0969	.00774	3.19	.00242	35
	17.77	17.20	16.35	.220	23.9	1.2	25.1	25.0	21.9	10.1	57.0	.440	.0971	.00776	3.04	.00256	55
	17.35	16.79	15.96	.220	23.4	1.2	24.6	24.6	21.4	9.9	55.9	.440	.0971	.00776	2.97	.00261	67
	16.28	15.58	14.78	.220	21.7	1.1	22.8	22.7	20.1	9.2	52.0	.438	.0962	.00770	2.85	.00270	79

Substance.	$t_g$ centims.	$\theta_j$	C ohms.	U ohms.	$\theta_v$	$\theta_v$	U—M ohms.	$\theta_v - \theta_w$	$\theta_w$	B ohms.	$\theta_\theta$	Heat generated.	$(1.395 + \frac{1}{2}t_g)v_m + \frac{1}{2}t_gv_v$	$(1.395 + \frac{1}{2}t_g)v_m + 1.233v_v$	Ratio of sums.	Heat transmitted.	Temperature slope.	Conductivity.	Temperature.
Naphthaline	.158	18.30 18.51 18.57 17.70 18.00	6.07 17.41 6.01 18.00 6.15	5.84 17.18 5.78 17.77 5.92	34.49 63.13 34.60 63.89 34.40	33.87 62.55 33.98 63.28 33.79	.440 .550 .447 .550 .457	1.15 1.33 1.17 1.33 1.20	32.72 61.22 32.81 61.95 32.59	-1.09 9.86 -1.20 10.52 -0.99	15.15 45.75 15.09 46.68 15.13	.209 .209 .209 .210 .209	26.0 22.9 26.2 22.6 25.9	60.8 54.3 61.4 53.6 60.6	.428 .422 .427 .422 .427	.0895 .0882 .0892 .0886 .0892	7.28 8.42 7.41 8.42 7.59	.00384 .000838 .000964 .000842 .000941	33 62 33 62 33
$\alpha$ Naphthol.	.152	17.30 17.53 17.52	6.08 17.58 6.17	5.88 17.38 5.97	33.55 62.61 33.97	33.02 62.09 33.44	.500 .773 .540	1.31 1.88 1.41	31.71 60.21 32.03	-0.90 10.30 -0.91	14.69 45.98 14.88	.205 .206 .206	25.1 21.0 25.3	59.5 51.2 60.0	.422 .410 .422	.0865 .0845 .0869	8.62 12.37 9.28	.000803 .000546 .000749	32 61 33
$\beta$ Naphthol.	.166	17.07 17.26 17.40	6.13 17.90 6.17	5.91 17.67 5.97	33.46 63.16 33.85	32.87 62.57 33.32	.535 .750 .540	1.41 1.82 1.41	31.46 60.75 31.91	-0.75 10.56 -0.86	14.90 46.43 14.91	.206 .206 .206	24.4 21.2 25.1	58.3 51.6 60.7	.419 .411 .414	.0863 .0847 .0853	8.49 11.0 8.49	.000813 .000616 .000803	32 61 33
Sulphur	.193	16.86 17.32 17.46	6.48 17.56 6.30	6.28 17.36 6.11	34.19 62.33 34.26	33.65 61.82 33.75	.745 .89 .74	1.95 2.17 1.94	31.70 59.65 31.81	-0.62 10.20 -0.80	15.06 45.58 15.14	.208 .208 .207	24.8 21.2 25.0	60.0 51.9 60.2	.413 .409 .415	.0859 .0851 .0857	10.1 11.2 10.1	.000680 .000608 .000680	33 61 33
Sulphur and French chalk	.3195	17.54 17.84 18.01	5.90 17.39 5.86	5.70 17.18 5.67	33.32 62.54 33.68	32.78 62.01 33.17	.48 .60 .48	1.26 1.46 1.26	31.52 60.55 31.91	-0.86 10.31 -0.99	15.05 46.32 15.14	.207 .207 .207	25.7 22.3 26.2	60.2 53.0 61.4	.427 .421 .427	.0884 .0871 .0884	3.95 4.58 3.95	.00179 .00152 .00179	32 61 32
Ebonite	.130	17.72 18.45 18.76 18.78	6.32 13.23 18.09 5.84	6.11 13.02 17.88 5.65	34.53 52.90 64.79 34.31	33.97 52.36 64.26 33.81	.797 .85 .89 .76	2.08 2.11 2.15 2.02	31.89 50.25 62.11 31.79	-0.81 6.08 -1.13 -1.14	15.38 35.53 48.51 15.48	.207 .205 .203 .198	24.1 21.4 19.9 23.8	58.7 52.8 49.3 57.8	.411 .405 .404 .412	.0851 .0830 .0820 .0816	16.0 16.2 16.5 15.5	.000426 .000410 .000398 .000421	33 51 63 33
Shellac	{.168} {.161}	17.80 18.43 18.60	6.23 17.48 6.02	6.03 17.27 5.82	34.41 63.14 34.61	33.88 62.60 34.08	.76 .88 .73	1.99 2.14 1.885	31.89 60.46 32.20	-0.78 10.44 -1.00	15.55 47.13 15.71	.208 .207 .207	24.1 19.7 24.3	58.7 49.1 58.9	.411 .401 .413	.0851 .0834 .0855	11.8 13.3 11.7	.000577 .000501 .000585	33 61 33
Fibre	.121	18.00 18.20 18.30	5.94 17.55 6.05	5.74 17.34 5.85	33.84 63.09 34.43	33.31 62.55 33.90	.41 .47 .41	1.07 1.14 1.07	32.24 61.41 32.83	-0.96 10.29 -1.04	15.22 46.55 15.29	.207 .207 .207	24.8 21.7 25.5	58.5 51.5 60.0	.424 .421 .425	.0878 .0871 .0880	8.85 9.42 8.85	.000793 .000740 .000795	33 62 33



### Remarks on preceding Tables.

In the first table the whole of the observations and reductions are shown, but in the second the observations and only the important steps in the reduction.

The thickness given in the tables is that of the material under test plus the two layers of glycerine. The correction for the glycerine is too small to be taken into account, except in the case of glass and of the mixture of sulphur and French chalk.

The corrected values for glass are : at 35°, .00248, .00243, .00244 ; at 55°, .00258 ; at 68°, .00261, .00263 ; at 79°, .00272, and for the cement at 32°, .00182 ; at 61°, .00155.

The glass was ordinary window glass, known as 22 oz. In the second series of experiments with it, the glycerine contact layers had been replaced by shellac (see p. 420).

The naphthaline disc was cast between glass plates and ground down to the required thickness.

The two naphthol discs were obtained by sawing slices from two large blocks of the material and grinding the surfaces.

The sulphur disc was made by pouring melted sulphur on to a glass plate and grinding the upper surface down till it was parallel to the lower, and the disc of the



requisite thickness. The surface which had been in contact with the glass had a crystalline appearance, the other not.

The disc of sulphur and French chalk was made in the same way. It contained about equal quantities of the two constituents.

The shellac disc was cast between glass plates, and contained a few small air holes.

The fibre was the ordinary "white fibre" of electrical instrument makers.

As the result of these experiments, it may be stated that the tendency at about 40° C. of the thermal conductivities of solid substances not very good conductors of heat, is to diminish with increase of temperature, at a percentage rate which appears to vary with the nature of the substance. Amongst the materials tested, glass is the only exception to this rule.

Substance.	$k_{35^{\circ}\text{C.}}$	Mean percentage change per degree between 35° and 60° C.
Glass (22-oz. window) . . . . .	·00245	+ ·0025
Naphthaline . . . . .	·00095	— ·0052
$\alpha$ Naphthol . . . . .	·00076	— ·0105
$\beta$ " . . . . .	·00080	— ·0085
Sulphur . . . . .	·00067	— ·0036
Sulphur and French chalk . . . . .	·00180	— ·0052
Ebonite . . . . .	·00042	— ·0019
Shellac . . . . .	·00058	— ·0055
White Fibre . . . . .	·00079	— ·0020

The above values of the conductivities agree with those found by a different method (LEES, 'Phil. Trans.,' A, 1892, p. 506) for temperatures between 25° and 35°:—glass, ·00243; sulphur, ·00045; ebonite, ·00040; shellac, ·00060, except in the case of sulphur, where the large difference is probably due to the disc tested in the present experiment not being wholly crystalline.

## PART II. CONDUCTIVITIES AND TEMPERATURE COEFFICIENTS OF LIQUIDS.

### *Preliminary Experiments.*

The apparatus used originally in the experiments on the conductivities of liquids, consisted of two circular copper discs, *U* and *M*, cemented by rubber cement to the opposite sides of a circular sheet of rubber, *R*, fig. 10. To the free surface of the upper disc, a flat coil, *C*, of insulated platinoid wire was cemented, and served to supply the heat required in the experiment.

These discs were supported by means of a three-legged support, *T*, screwed to the upper disc, above the third copper disc, *L*, which was provided with a raised edge. The legs of the support were long enough to raise the lower surface of the disc, *M*, about 13 centim. above the upper surface of the disc, *L*. The liquid, the thermal

conductivity of which was to be determined, was poured on to the lower disc, where it was retained by the raised edge. The three discs were then placed on the horizontal top of a closed vessel through which a stream of cold water was kept flowing, and they could, if necessary, have a cover placed over them. On sending a current through the coil, *C*, part of the heat generated flowed through the discs to the top of the cold water vessel, the rest was lost by conduction and convection in the air above the coil. After the distribution of temperature has become steady, the heat flowing through the rubber from *U* to *M* proceeds either through the layer of liquid to *L*, or is lost from the surfaces of *M* and of the liquid. As the temperature of the water flowing through the vessel underneath the discs was always a few degrees less than that of the air of the room, it was possible, by regulating the rate of flow of the water, to arrange that the temperature of the surface from which this heat was lost was nearly identical with that of the air, and the loss was thus reduced to so small an amount that it could be neglected. By means of thermo-junctions of copper and platinoid wires, soldered into holes about 3 millims. deep in the edges the discs, *U* and *M*, and in holes 2 centims. deep in the edge of the disc, *L*, the temperature of the upper disc, and the differences of temperature between the upper and middle, and between the upper and lower discs, were found by balancing the thermo-electric E.M.F. in each case against the requisite fractional part of the E.M.F. of a Leclanché cell, which had been compared with a standard Clark cell. It was found that for small temperature differences the E.M.F.s so determined, could be taken as proportional to the differences of temperature.

Fig. 10.

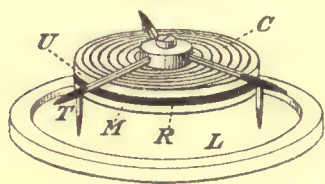


Fig. 11.

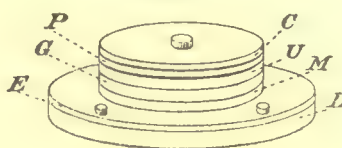
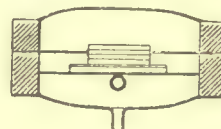


Fig. 12.



From the difference of temperature on the two sides of the rubber sheet, and the thickness area and conductivity of the rubber, the amount of heat entering the liquid could be calculated, and if the thickness of the liquid, the area of flow, and the difference of temperature are known, the thermal conductivity of the liquid can be found.

The curvature of the lines of flow in the liquid near the edge of the disc, *M*, will render the area of flow in the liquid greater than in the rubber, and a small correction would have to be applied if the thermal conductivity were made to depend on that of the rubber. It is better, however, to base the determination on the conductivity of water, which can be substituted for the liquid, and tested under the same conditions. In this case the areas of flow in the two liquids may be assumed to be equal, and the calculation simplified.



With the apparatus in this form, determinations of the thermal conductivities of water, ethyl alcohol, acetic acid, and glycerine were made.

The values found in different experiments for the relative conductivities of water and glycerine, and mixtures of them, were concordant, and are given in the following table along with H. F. WEBER'S\* values for water and glycerine between 9° and 15°, but experiments on ethyl alcohol and acetic acid showed discrepancies which were, after some time, traced to the evaporation of the liquid at points not covered by the middle disc. As most of the liquids to be tested were more volatile than water and glycerine, it was decided to get rid of this evaporation by entirely enclosing the liquid, and the apparatus used throughout the greater part of the investigation was then constructed.

Liquid.	$U - M.$	$M - L.$	$k.$	Temp.	H. F. WEBER'S values.
Water . . . . .	ohms. ·97	ohms. ·28	·00140†	20° C.	·00136
+25 per cent. } glycerine in water	·93	·32	·00119	"	
20·8 " }				"	
50 " }	·953	·387	·00101	"	
44·2 " }				"	
75 " }	·91	·46	·00081	"	
70·4 " }				"	
Glycerine . . . . .	·89	·52	·00070	"	·00067

### *Description of Apparatus.*

Between two nickel-plated copper discs,  $U$  and  $M$ , fig. 11, p. 419, 4 centims. in diameter and ·3 centim. thick, a disc of glass,  $C$ , ·28 centim. thick, was cemented by means of thin layers of shellac. In order to prevent bubbles of air occurring in the layer of shellac, the surfaces of the copper and glass discs to be cemented together were each covered with a thin layer of shellac, which was allowed to dry. Both surfaces were then smeared with glycerine, and slid carefully together, air being prevented from getting between the surfaces by the presence of abundance of glycerine. Pressure and heat were then applied, the pressure forcing out the glycerine and bringing the layers of shellac into contact, and the heat joining them. Both contacts were made in this way, and withstood a large amount of usage without coming apart.

A flat spiral coil of platinoid wire,  $P$ , was placed on the top of the upper disc,  $U$ , and was held down by a thin copper disc,  $C$ , ·10 centim. thick, screwed to the disc,  $U$ . The coil was insulated from both discs by means of mica, and the surfaces of the discs exposed to the air were varnished to give them the same emissivity. This combination rested on the inner edge of a ring of ebonite,  $E$ , of 7 centims. external,

\* H. F. WEBER, 'Berliner Ber.', 1885, p. 809.

† The upper numbers are percentages by weight; the lower percentages by volume.

‡ Taken from the table, p. 425.



and 3.71 centims. internal diameter, which in turn was supported by a disc of nickel-plated copper,  $L$ , 7 centims. diameter and .3 centim. thick, to which it was cemented by rubber cement.

The copper discs will be called in order downwards, the "cover," "upper," "middle," and "lower" copper discs, respectively.

The discs were placed with the flat surfaces horizontal,\* in a flat cylindrical air bath, fig. 12, p. 419, consisting of an upper and a lower half, each capable of being heated by means of an insulated coil of platinoid wire wound round it, and the lower half provided with a compartment through which a stream of cold water could be sent to cool the bath rapidly. During a test of thermal conductivity, this compartment contained air only.

A thermo-junction was soldered inside the top of the upper part of the enclosure, and it is assumed that the air to which the top and side surfaces of the discs lose heat, has the temperature given by this thermo-junction. The thermo-junctions in the upper discs were those described (p. 419), and were arranged vertically under each other. In order to bring the junction in the lower disc vertically under those in the upper, the holes in the sides of the lower disc were made correspondingly deeper, and for a depth of 1.5 centims. wider than those in the upper discs, and were lined with thin glass tubes to prevent the wires making contact with the disc at any other points than the junctions.

The liquid to be tested was placed in the ebonite ring, and had therefore the same thickness as the ebonite.

The heat flowing through the under surface of the middle copper disc was conducted away partly by the liquid under test, partly by the ebonite. The amount conducted away by the ebonite could be calculated if the shape of the stream lines and the thermal conductivity of the ebonite were known. It is, however, much more accurate to determine this quantity by a separate experiment, substituting air for the liquid and so arranging the amount of heat supplied, that the temperature difference between the middle and lower discs is the same as in the experiment on the liquid. The heat conducted through the air is then small, and can be calculated from the known thermal conductivity of air.† The remainder of the heat flowing through the under surface of the middle disc is conducted through the ebonite to the lower disc, and is the quantity required.

#### *Theory of Ebonite Ring Apparatus.*

If  $r$  = radius of copper and glass discs,

$t_g$  = thickness of glass disc,

\* The disc of liquid being thin, convection currents only come in to a slight extent when the discs are not quite horizontal. In the case of a disc of water, an inclination of  $15^\circ$  to the horizontal only increased the apparent conductivity 1 per cent.

† WINKELMANN'S value has been used, 'Wied. Ann.,' vol. 48, p. 186, 1895.

$t_M$  = thickness of copper disc below glass, plus correction for thermo-wires,

$r_l$  = radius of disc of liquid,

$t_l$  = thickness of disc of liquid,

$v_U$  = excess of temperature of upper disc over that of the air in the enclosure.

$v_M$  = excess of temperature of middle disc                   ,,                   ,,                   ,,

$v_L$  = excess of temperature of lower disc                   ,,                   ,,                   ,,

then the heat flowing through the glass disc per second

$$= \pi r^2 k_G \frac{v_U - v_M}{t_G},$$

the heat entering the middle disc

$$= \pi r^2 k_G \frac{v_U - v_M}{t_G} - 2\pi r h t_G \frac{v_U + v_M}{4},$$

and the heat leaving the middle disc by its under surface

$$= \pi r^2 k_G \frac{v_U - v_M}{t_G} - 2\pi r h \left( t_G \frac{v_U + v_M}{4} + t_M v_M \right).$$

The amount passing through the disc of liquid

$$= \pi r_l^2 k_l \frac{v_M - v_L}{t_l},$$

the amount entering the ebonite ring surrounding the liquid may be put\*

$$= A \frac{v_M - v_L}{t_l},$$

where A is an unknown constant depending on the breadth of the ring under the middle copper disc, and on the distribution of the lines of flow in it. Between these quantities we have the relation†

$$(\pi r_l^2 k_l + A) \frac{v_M - v_L}{t_l} = \pi r^2 k_G \frac{v_U - v_M}{t_G} - 2\pi r h \left\{ \frac{t_G}{4} \cdot v_U + \left( \frac{t_G}{4} + t_M \right) v_M \right\},$$

from which we can determine  $k_l$  if A,  $k_G$  and  $h$  are known, by observations of  $v_U$ ,  $v_M$ , and  $v_L$ .

\* Strictly, a term  $Bv_L$  ought to be introduced here, since some of the lines of flow from the middle disc may pass through the ebonite to its upper surface. On account of the smallness of B, this term has been neglected.

† See note, p. (403).

The value of  $k_g$  and  $h$  are known from the observations made in Part I.

The value of  $A$  is determined most accurately, by carrying out an experiment with air substituted for the liquid, so that the term involving  $k_l$  on the left-hand side of the equation has a known small value. From the observations of temperature the value of the right-hand side of the equation is calculated, and hence the value of  $A$ .

### *Constants of Apparatus.*

$$r = 2 \quad \text{centims., therefore } \pi r^2 = 12.5 \text{ sq. centims.}$$

$$r_l = 1.855 \text{ centims., therefore } \pi r_l^2 = 10.8 \text{ sq. centims.}$$

$$t_l = .127 \quad ,,$$

$$t_g = .281 \quad ,, \quad \text{therefore } \frac{t_g}{4} = .070 \text{ centims.}$$

$$t_M = .320 + .075 = .395 \text{ centims.}$$

$$h = .00030, \text{ therefore } 2\pi r h \frac{t_g}{4} = .00026$$

$$\text{and } 2\pi r h t_M = .00148$$

$$\text{Sum} = .00174$$

### *Method of Experimenting with Ebonite Ring Apparatus.*

In carrying out a determination of the thermal conductivity of a liquid, the surfaces of the discs with which the liquid was to come into contact were cleaned, and the lower nickel-plated surface of the middle disc wet with the liquid. Liquid was then poured into the space inside the ebonite ring, till the upper surface of the liquid was higher than the upper surface of the ring. The middle disc was then placed on the ebonite ring with one edge touching the liquid, which was drawn into the space between disc and ring by capillarity. The disc was then slid slowly over the ring till the liquid was entirely enclosed, and any excess which flowed over the ring was removed by filter paper.

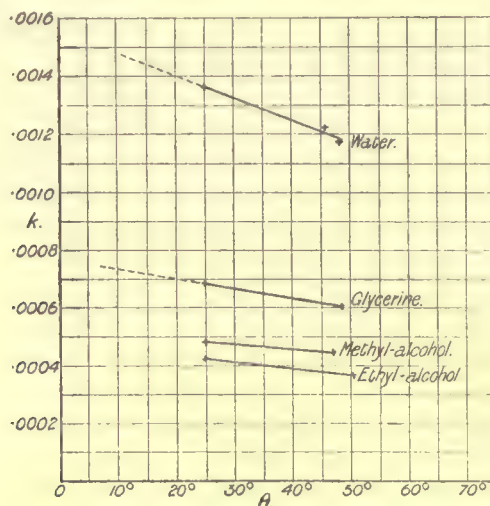
To prevent the upper discs moving during an experiment, they were secured to the lower, either by a silk thread passing from one side of the lower disc over the upper discs to the other side of the lower disc, or by a wire frame which could be attached to the lower disc, and was provided with a screw which pressed on a small disc of cork on the copper disc covering the heating coil.

The combination was then placed in the air bath and the process described (p. 413) carried out till the steady temperature state was attained, when the observations of power supplied, temperatures, and temperature differences were made.

Experiments were made on four liquids, with the results given in the following tables and curves.



Liquid.	$\theta_j$ .	U.	$\theta_v$ .	Reduced.			$\theta_N - \theta_L = v_N - v_L$ .	$\theta_N$ .	B.	$\theta_B$ .	Temperature slope in glass.	Heat transmitted through glass.	00026 cr.	0017 $v_M$ .	Heat transmitted through liquid and ebonite.	Heat transmitted per unit slope through liquid and ebonite.	Heat transmitted through liquid.	Heat transmitted per square cen- tim. = $k$ .	Temperature.
				ohms.	ohms.	ohms.													
Water . . . .	16.74	3.18	26.14	.600	0.480	ohms.	1.28	24.54	1.3	20.29	5.69	.168	.002	.007	.159	.0157	.0148	.00137	24
	16.91	12.02	48.38	.580	0.560	ohms.	1.41	46.93	10.32	44.01	5.18	.162	.001	.005	.156	.0141	.0132	.00122	46
	17.26	3.61	26.96	.590	0.475	ohms.	1.27	25.39	1.28	20.76	5.59	.166	.002	.008	.156	.0156	.0147	.00136	25
	16.80	12.68	49.91	.563	0.570	ohms.	1.43	48.50	11.16	46.69	5.01	.157	.000	.004	.153	.0135	.0126	.00117	48
Air . . . .	17.42	4.22	28.80	.340	2.56	ohms.	6.86	26.89	1.05	20.17	3.24	.096	.002	.012	.082	.0152	.0064	.000059	23
	17.27	4.04	28.17	.590	0.890	ohms.	2.37	26.60	1.77	22.08	5.59	.166	.002	.008	.156	.0325	.0737	.000682	25
Glycerine . .	17.50	12.62	50.45	.550	0.970	ohms.	2.44	49.08	10.55	45.20	4.87	.153	.002	.007	.144	.0750	.0662	.000613	48
	17.95	3.79	28.16	.570	1.28	ohms.	3.40	26.65	0.99	20.65	5.37	.159	.002	.010	.147	.0548	.0460	.000426	25
Ethyl alcohol .	17.38	14.09	53.98	.530	1.44	ohms.	3.58	52.67	11.77	48.19	4.66	.147	.002	.008	.137	.0486	.0398	.000369	51
	18.30	12.13	49.97	.520	1.21	ohms.	3.04	48.67	9.95	44.42	4.63	.145	.002	.007	.136	.0569	.0498	.000445	47
Methyl alcohol .	18.21	3.84	28.56	.575	1.16	ohms.	3.08	27.04	1.11	21.23	5.40	.160	.002	.010	.148	.0612	.0524	.000485	25



Liquid.	$k_{25}$ .	Mean percentage change per degree between 25° and 45°.	$k_{9-15}$ , H. F. W.
Water . . . . .	·00136	—·0055	·00136
Glycerine . . . . .	·00068	—·0044	·00067
Methyl alcohol . . . . .	·00048	—·0031	·000495
Ethyl alcohol . . . . .	·00043	—·0058	·000423

On account of the rapid way in which the methyl alcohol evaporated, it was necessary to work quickly, and the numbers given for it are not quite as reliable as those given for the other liquids.

The last column contains the values obtained by H. F. WEBER,\* for the mean thermal conductivities between 9° and 15°.

As a result of the experiments, we may state that the thermal conductivities of liquids decrease with increase of temperature in the neighbourhood of 30° C., at a percentage rate which appears to be roughly the same for a number of liquids.

### PART III. CHANGE OF CONDUCTIVITY ON MELTING.

Combining the last result with the one obtained for solids, Part I., and with the fact that solids in general conduct heat better than liquids, and liquids better than gases, we are led to conjecture that a given substance will, as its temperature is increased, decrease in conductivity, and that the decrease will continue during any change of state which the substance may undergo, owing to the increase in temperature. The

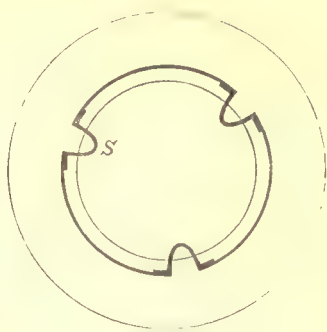
\* H. F. WEBER, 'Berliner Ber.,' 1885, p. 809.

question then arises: Is there any sudden change in the conductivity as the substance passes from one physical state to another, or is this change continuous?

In order to test what change occurs in the thermal conductivity of a substance as it passes from the solid to the liquid state, the apparatus used for liquids was modified slightly, so as to allow space between the lower and middle discs for change of volume of the substance tested on melting.

The ebonite ring was replaced by one of "white fibre," pierced by a hole 4.16 centims. in diameter, *i.e.*, greater in diameter than the middle copper disc (fig. 13),

Fig. 13.



with three strips, *S*, of fibre projecting inwards, in order to provide support for the middle disc, and keep the layer of substance tested of constant thickness. By means of a wire frame attached to the lower disc, and a screw, the upper discs were pressed down on to the strips of fibre, and the lifting of the upper discs, owing to expansion of the substance tested on melting, prevented.

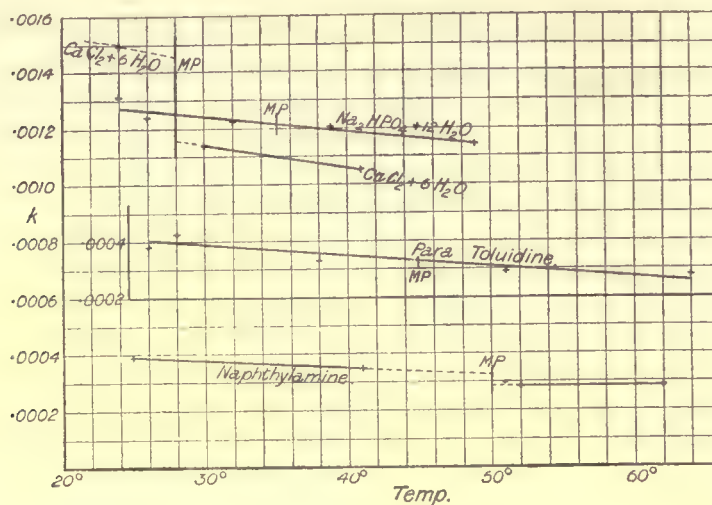
Some of the substance to be heated was placed within the ring, and the lower disc heated till the substance melted. The upper discs were then placed in position and the apparatus allowed to cool. The experiment was then conducted as in a test of a liquid.

The area of the projecting lugs of the fibre disc, through which heat could flow, was estimated as approximately .4 centim., hence the heat transmitted per second per unit slope of temperature through the fibre, and not through the substance under test, would be = .0003 gram degrees approximately.\* The nett area of flow through the substance has been taken = 13.3 sq. centims., but as this number is, to some extent, uncertain, owing to the difficulty of obtaining discs which, in the solid state, completely filled the space within the fibre ring, the results given in the following table for different substances are not comparable with each other to within 3 or 4 per cent., although those for the same substance, at different temperatures, are.

\* The conductivity of the fibre = .0008 (pp. 417, 418).



Substance.	Thickness, centims.	$\theta_1$ .	U.	$\theta_c$ .	Ohms reduced.		$\theta_M - \theta_L$ .	$\theta_M$ .	B.	$\theta_B$ .	Temperature slope in glass.	Heat transmitted through glass.	$\cdot 00026 v$ .	$\cdot 0017 v_M$ .	Heat transmitted through substance and fibre.	Temperature slope in substance.	Total heat/slope.	Heat through substance/slope.	Conductivity.	Temperature.
					U-M.	M-L.														
$\text{Na}_2\text{HPO}_4 + 12\text{H}_2\text{O}$ m.p. = $35^\circ\text{C}$ .	.125	15.82	3.72	25.92	.61	0.425	1.14	24.29	1.30	19.37	5.81	.171	.002	.008	.161	9.12	.0177	.0174	.00131	24
		16.26	6.76	34.29	.597	0.453	1.19	32.73	4.52	28.44	5.55	.167	.002	.007	.158	9.52	.0166	.0163	.00123	32
		16.68	9.16	40.91	.586	0.47	1.21	39.40	7.17	35.75	5.375	.165	.002	.006	.157	9.62	.0163	.0160	.00120	39
		16.87	12.91	50.58	.585	0.505	1.27	49.12	11.45	46.92	5.20	.163	.001	.004	.158	10.2	.0155	.0152	.00114	49
		16.67	4.33	28.34	.615	0.455	1.21	26.71	1.98	22.06	5.80	.173	.002	.008	.163	9.68	.0168	.0165	.00124	26
$\text{CaCl}_2 + 6\text{H}_2\text{O}$ m.p. = $29^\circ\text{C}$ .	.125	16.46	3.60	26.24	.63	0.385	1.03	24.54	1.24	19.9	5.98	.177	.002	.008	.167	8.24	.0203	.0200	.00150	24
		16.81	5.79	32.29	.62	0.50	1.32	30.66	3.4	26.00	5.80	.174	.002	.008	.164	10.6	.0155	.0152	.00114	30
		17.15	9.71	42.75	.60	0.535	1.37	41.22	7.3	36.5	5.44	.168	.002	.008	.158	11.0	.0144	.0141	.00106	41
Air . . . . .	..	18.20	2.59	25.22	.19	1.85	4.99	24.71	0.58	19.78	1.81	.053	.002	.008	.043	39.9	.0011	.00078	.000059	22
Naphthylamine m.p. = $50^\circ\text{C}$ .	.107	17.82	3.81	28.09	.58	1.10	2.93	26.55	1.08	20.76	5.48	.162	.002	.010	.150	27.4	.00547	.0052	.00039	25
		18.16	9.75	43.82	.565	1.27	3.25	42.38	7.37	37.73	5.12	.158	.002	.008	.148	30.4	.00487	.0046	.00035	41
		18.24	13.60	53.56	.533	1.467	3.66	52.24	11.15	47.51	4.70	.148	.002	.008	.138	34.2	.00404	.0037	.00028	52
		18.50	18.20	64.93	.520	1.495	3.63	63.67	15.76	59.12	4.48	.145	.002	.008	.135	33.9	.00398	.0037	.00028	62
Para-Toluidine m.p. = $45^\circ\text{C}$ .	.107	17.82	4.04	28.68	.555	1.065	2.84	27.21	1.15	21.13	5.23	.155	.002	.010	.143	26.5	.00540	.0051	.00038	26
		18.55	8.34	40.55	.563	1.204	3.103	39.10	(6.8)	29.35	5.16	.158	.003	.017	.138	29.0	.00473	.0044	.00033	38
		18.85	13.32	53.56	.52	1.355	3.38	52.27	10.81	47.22	4.59	.145	.002	.009	.134	31.6	.00424	.0039	.00029	51
		18.76	19.30	67.74	.50	1.455	3.48	66.55	16.89	61.96	4.23	.138	.002	.008	.128	32.5	.00394	.0036	.00027	64
		18.50	4.80	31.37	.65	1.22	3.23	29.65	3.95	29.12	6.11	.183	.001	.001	.181	30.2	.00600	.0057	.00043	28



The experiments show no apparent break in the regularity of the change of thermal conductivity at the melting point in the case of three of the solids, and a decrease of about 20 per cent. in the case of  $\text{CaCl}_2 + 6\text{H}_2\text{O}$ .\*

This salt was extremely difficult to work with, on account of the rate at which it absorbed moisture, and the results obtained cannot be relied on to the same extent as those for the other substances.

It seems, then, reasonable to conclude that, for salts at least, change of state on melting is not invariably accompanied by an abrupt change of thermal conductivity.

#### PART IV. THERMAL CONDUCTIVITIES OF MIXTURES.

After coming to the above conclusions with respect to pure, or approximately pure, substances, one is led naturally to experiment on mixtures of substances, and, since mixtures of liquids are most easily made, they were the first on which observations were carried out.

It is obviously advantageous to investigate mixtures of pairs of liquids, the conductivities of which differ as widely as possible. The number of miscible liquids which satisfy this condition is small, owing to the great number of organic liquids which have conductivities nearly alike. Water and glycerine are much better conductors than most liquids, and have, therefore, been used in several mixtures.

The following mixtures of water, ethyl alcohol, methyl alcohol, acetic acid, glycerine, and sugar were made, and their thermal conductivities investigated:—

\* BARUS ('Sill. Journal' (3), vol. 44, p. 1, 1892), has found a decrease of about 15 per cent. in the case of thymol, which melts at about  $12.5^\circ \text{C}$ .

Glycerine in water . . . . .	$\left\{ \begin{array}{l} 25, 50, 75 \text{ per cent., by weight.} \\ 20\cdot8, 44\cdot2, 70\cdot4 \text{ per cent., by volume.} \end{array} \right.$
Ethyl alcohol in water* . . . . .	$\left\{ \begin{array}{l} 25, 50, 75 \text{ per cent., by weight.} \\ 30, 58, 81 \text{ per cent., by volume.} \end{array} \right.$
Methyl alcohol in water . . . . .	$\left\{ \begin{array}{l} 23\cdot9, 50, 74\cdot7 \text{ per cent., by weight.} \\ 28\cdot8, 58, 80\cdot7 \text{ per cent., by volume.} \end{array} \right.$
Acetic acid in water . . . . .	$\left\{ \begin{array}{l} 25, 50, 75 \text{ per cent., by weight.} \\ 24, 48\cdot7, 74 \text{ per cent., by volume.} \end{array} \right.$
Glycerine in ethyl alcohol . . . . .	$\left\{ \begin{array}{l} 25, 49\cdot2, 74\cdot8 \text{ per cent., by weight.} \\ 17\cdot4, 38\cdot7, 65\cdot3 \text{ per cent., by volume.} \end{array} \right.$
Methyl alcohol in ethyl alcohol . . . . .	25, 48\cdot5, 74 per cent., by weight and by volume.
Sugar in water . . . . .	$\left\{ \begin{array}{l} 25, 50, 67 \text{ per cent., by weight.} \\ 17\cdot2, 38\cdot8, 62\cdot6 \text{ per cent., by volume.} \end{array} \right.$

Some of the experiments were carried out with the ebonite ring apparatus, fig. 11, p. 419, some with the earlier form of apparatus, fig. 10.

The following table gives the experiments made with the former apparatus:—

*Constants of Apparatus.*

$r = 2$ centims.	$\pi r^2 = 12\cdot5$ sq. centims.
$r_l = 1\cdot855$ „	$\pi r_l^2 = 10\cdot8$ „
$t_l = 0\cdot127$ „	
$t_G = 0\cdot281$ „	$\frac{t_G}{4} = \cdot070$ centims.
$t_M = \cdot320 + \cdot075 = \cdot395$ centims.	
$h = \cdot00030,$	$2\pi r h \frac{t_G}{4} = \cdot00026$
	$2\pi r h t_M = \cdot00148$
	<hr/> Sum = $\cdot00174$ <hr/>

\* HENNEBERG, 'Wied. Ann.,' vol. 36, p. 146, 1889, obtained results for mixtures of water and ethyl alcohol which agree closely with those that follow.



Liquid.	$\theta_j$	U.	$\theta_v$	Reduced.		$\theta_m - \theta_v$	$\theta_m$	B.	$\theta_n$	Temperature slope in glass.	Heat transmitted through glass.	$\cdot 00026v_m$ .	$\cdot 0017m$ .	Heat transmitted through liquid and ebonite.	Temperature slope.	Total heat/slope.	Heat through liquid/slope.	Conductivity.	Temperature.
		ohms.	°	ohms.	ohms.	°	°	ohms.	°	°	°								°
Water . . . . .	12.84	.57	14.4	.687	0.528	1.46	12.5	+ .02	13.0	6.76	.194	.0004	—	.194	11.5	.0169	.0160	.00149	11
25% } Ethyl alcohol in H <sub>2</sub> O	13.22	.07	13.4	.715	0.735	2.14	11.4	— .24	12.5	7.05	.202	.0003	—	.204	16.8	.0121	.0112	.00104	10
30 } do.																			
50% } do.	13.76	.23	14.4	.702	0.957	2.65	12.5	— .22	13.2	6.91	.198	.0004	—	.199	20.9	.0095	.0086	.00079	11
58 } do.																			
75% } do.	13.96	.48	15.3	.681	1.202	3.33	13.4	— .21	13.4	6.73	.193	.0006	.0000	.192	26.2	.0073	.0064	.00059	12
81 } Ethyl alcohol	13.84	.22	14.4	.692	1.492	4.13	12.5	— .32	12.9	6.83	.196	.0004	.0007	.196	32.5	.0060	.0051	.00047	10
25% } Acetic acid in H <sub>2</sub> O	13.92	.23	13.3	.715	0.631	1.91	11.3	— .33	12.9	7.05	.202	.0001	.0027	.205	15.0	.0136	.0127	.00118	11
24 } do.																			
50% } do.	14.12	.09	13.9	.704	0.879	2.44	12.0	— .32	13.2	6.94	.199	.0002	.0020	.201	19.2	.0105	.0096	.00088	11
48.7 } do.																			
75% } do.	12.86	+ .76	15.0	.680	1.110	3.08	13.1	— .04	12.8	6.69	.191	.0007	.0005	.191	24.2	.0079	.0070	.00064	11
74 } Acetic acid	13.28	+ .64	15.1	.670	1.551	4.30	13.2	— .20	12.7	6.62	.190	.0007	.0009	.190	33.9	.0056	.0047	.00043	11
25% } ethyl alcohol in H <sub>2</sub> O	13.80	.36	12.8	.715	0.753	2.09	10.8	— .34	12.8	7.05	.202	.0000	.0034	.205	16.4	.0125	.0116	.00107	9

■ The upper numbers are per cents. by weight, the lower by volume.

The conductivities given in the preceding table depend on that of the disc of glass between the upper and middle copper discs, which was determined in Part I.

Previous to the apparatus with the glass disc being constructed, experiments had been carried out with the apparatus in which rubber replaced the glass. As the conductivity of this rubber was unknown, the results obtained were all expressed in terms of the conductivity of water. By means of the value for water found in Part II., these relative values have been converted into absolute values, and are given in the table which follows.

The discs were not in this case surrounded by a vessel of known temperature, but were exposed to the air, the temperature of which was observed. The lower disc was placed in the top of a vessel through which cold water circulated, and the observations were taken in the manner previously described.

The theory of the method is identical with that given on p. 422, except that the constant A is zero.

*Constants of Apparatus.*

$r = 2$	centims.	$\pi r^2 = 12.5$	sq. centims.
$r_l = 1.855$	„	$\pi r_l^2 = 10.8$	„
$t_l = .132$			
$t_R = .110$ ,	therefore $t_R/4 = .027$	centims.	
$t_M = .320 + .075$	$= .395$	„	
$h = .00030$ ,	therefore $2\pi r h t_R/4 = .00010$		
	$2\pi r h t_M = .00148$		
	Sum . . = .0016		

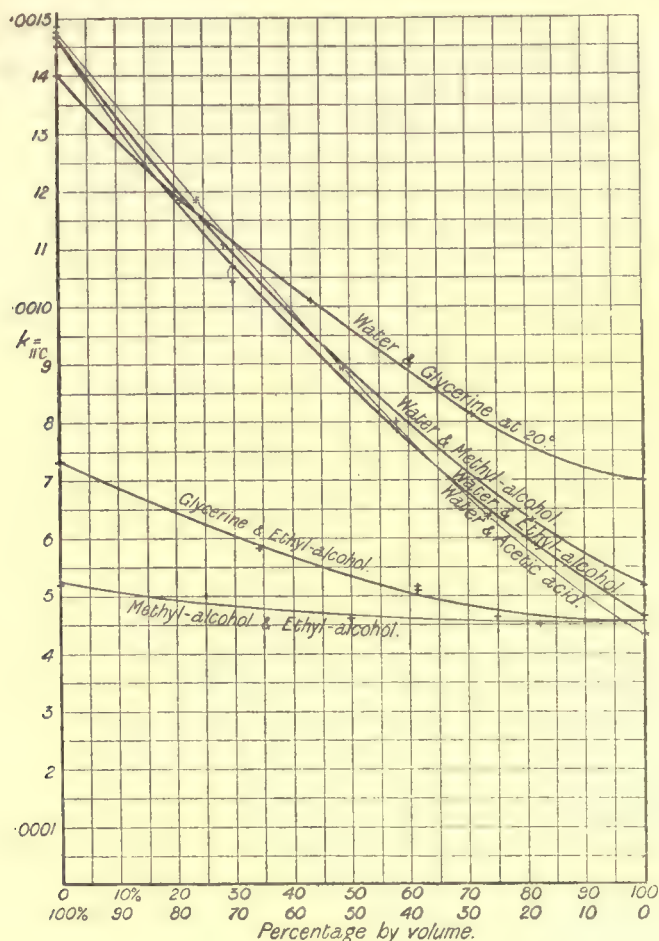
	$\theta$	$t$	$U - M$	$M - L$	$v$	$v - v_M$	$v_M - v_L$	$v_M$	Heat through rubber.	$\cdot 001 v$	$\cdot 0016 v_M$	Heat transmitted through liquid and ebonite.	Temperature slope in liquid.	Heat through liquid per unit slope.	Conductivity.	Temperature of liquid.	$k_{liq.}$
Water . . . . .	20	0.70	1.17	.31	1.9	3.17	0.84	-1.3	.101	.0002	-.002	.103	6.37	.0152	.00141†	18	.00147
" . . . . .	20	0.70	1.215	.325	1.9	3.29	0.88	-1.4	.105	.0002	-.002	.107	6.67	.0151	.00139	18	.00145
Ethyl alcohol . . . . .	20.5	1.00	1.19	.89	2.7	3.23	2.41	-0.5	.103	.0003	-.001	.104	18.3	.0048	.00044	19	.00046
" . . . . .	(20	1.26	1.17	.89	3.4	3.17	2.41	+0.2	.101	.0003	.000	.101	18.3	.0046	.00043	19	.00045
per cent. glycerine	{ 25 *	1.13	1.153	.877	3.1	3.12	2.38	0.0	.099	.0003	.000	.099	18.0	.0046	.00043	19	.00045
" . . . . .	{ 17.4 *	1.0	1.165	.795	2.7	3.16	2.15	-0.5	.101	.0003	-.001	.102	16.3	.0054	.00050	19	.00052
" . . . . .	{ 50	0.90	1.175	.718	2.4	3.18	1.95	-0.8	.101	.0002	-.001	.102	14.8	.0060	.00056	19	.00058
" . . . . .	{ 38.7	0.73	1.18	.315	2.0	3.20	0.854	-1.2	.102	.0002	-.002	.104	6.47	.0152	.00070†	19	.00148
Glycerine . . . . .	20.1	0.61	1.183	.404	1.7	3.21	1.095	-1.5	.102	.0002	-.002	.104	8.30	.0116	.00107	19	.00111
Water . . . . .	20.6	0.69	1.12	.52	1.9	3.04	1.41	-1.1	.097	.0002	-.002	.099	10.7	.0084	.00078	19	.00080
" . . . . .	20.7	0.69	1.10	.63	1.9	2.98	1.71	-1.1	.095	.0002	-.002	.097	13.0	.0066	.00061	19	.00063
" . . . . .	21	0.77	1.13	.77	2.1	3.06	2.09	-1.0	.098	.0002	-.002	.100	15.8	.0054	.00050	19	.00052
Methyl alcohol . . . . .	20.9	0.90	1.17	.82	2.4	3.17	2.22	-0.8	.101	.0002	-.001	.102	16.8	.0052	.00048	20	.00050
" . . . . .	25 per cent. ethyl alcohol	1.46	1.08	.76	4.0	2.93	2.06	+1.1	.093	.0004	+.002	.091	15.6	.0049	.00045	18	.00046
" . . . . .	50	1.54	1.095	.815	4.2	2.97	2.21	+1.2	.095	.0004	+.002	.093	16.7	.0047	.00044	18	.00046
" . . . . .	75	0.82	1.15	.35	2.2	3.12	0.95	-0.9	.100	.0002	-.001	.101	7.2	.0131	.00122	17	
Water . . . . .	18	0.9	1.14	.41	2.4	3.09	1.11	-0.7	.099	.0002	-.001	.100	8.4	.0110	.00102	17	
" . . . . .	19	0.92	1.16	.48	2.5	3.14	1.30	-0.6	.100	.0003	-.001	.101	9.85	.0094	.00087	18	
" . . . . .	19	1.00	1.14	.385	2.7	3.09	1.04	-0.4	.099	.0003	-.001	.100	7.9	.0118	.00109	18	
Ammonia $\rho = .907 = \left\{ \begin{smallmatrix} 26 \\ 36 \end{smallmatrix} \right\}$ per cent. of liquid ammonia	19																

\* Upper numbers by weight, lower by volume, throughout.

† Taken from table, p. 424.

‡ Taken from table, p. 420.





When the results given in these two tables are expressed by curves, with composition by weight as abscissæ and conductivities as ordinates, the curves for mixtures of methyl and ethyl alcohol and glycerine with water differ widely from each other, and no general law connecting the conductivities of mixtures directly with the conductivities and relative masses of the constituents can be traced. If, however, proportions by volume are taken as abscissæ, instead of proportions by weight, the curves for mixtures of methyl and ethyl alcohol and acetic acid with water become nearly identical, and we are led to the conclusion that in further work proportion by volume should be made the basis of comparison.

A further conclusion from an inspection of these curves is that the conductivity of a mixture of two liquids is less than the value calculated by the linear formula

$$k = \frac{k_1 v_1 + k_2 v_2}{v_1 + v_2},$$

when  $k_1$  and  $k_2$  are the conductivities,  $v_1$  and  $v_2$  the volumes, of the constituents present, and that the difference is greater the greater the difference in the conductivities of the constituents.

It is evident, therefore, that to investigate the law of variation of conductivity of a mixture with its constitution more closely, it is advisable to work with mixtures of constituents which differ in conductivity to a greater extent. If we confine ourselves to liquids, we are limited to ratios of conductivities of about one to three, and we must turn to mixtures of solids, or of solids and liquids to get greater ratios. In the case of mixture of solids with liquids which dissolve them, a difficulty arises as to which conductivity of the material entering into solution ought to be used in making the comparison—that of the material when in the solid, or when in the liquid state.

[The results given in the previous table for solutions of sugar in water are almost identical with those of mixtures of glycerine and water, so that it may be said that a solution of sugar behaves as if it were a mixture of water and a liquid having the same density as sugar and a thermal conductivity of '0007. I have, however, found, by the method described by me in 'Manchester Memoirs,' vol. 42, No. 5 (1898), that the thermal conductivity of the solid sugar was about '0012, so that from a knowledge of the thermal conductivity of a solid we can infer little or nothing as to how it will behave when it enters into solution. Thus, JÄGER, 'Wien. Ber.,' vol. 99, p. 245 (1890), found that solutions of sodium chloride, potassium chloride, and zinc sulphate of about the same strengths had conductivities nearly alike and slightly less than that of water, whereas I have found ('Manchester Memoirs' as above) that the first two salts in the solid state conduct eight or nine times as well as water, and the latter a little better than water. The result for ammonia solution given in the above table points in the same direction, so that it seems advisable, for the present at least, to confine our attention to mixtures in which neither constituent has changed its physical state. The further question, whether such mixtures should be treated as physical mixtures or as chemical compounds, must be held over till more information has accumulated.—13th June, 1898.]

If, on the other hand, solids are mixed with liquids in which they are not soluble, there is great difficulty in keeping the mixture experimented on uniform, and in determining its exact constitution.

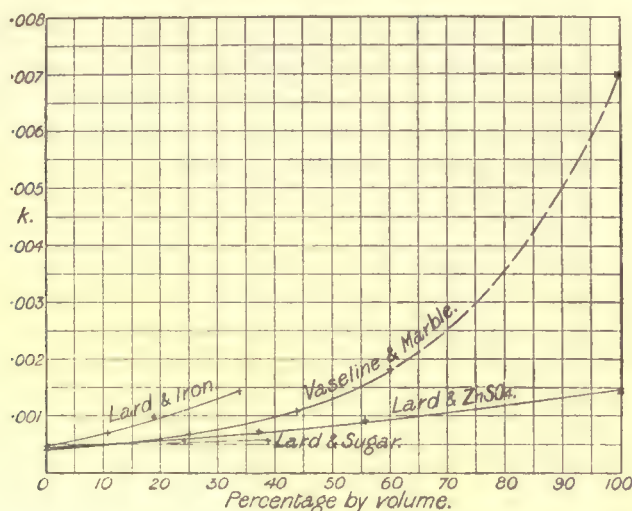
These considerations point to the conclusion that mixtures of solids would be most useful in leading to the discovery of any law. They have, however, the disadvantage of not being as readily made as mixtures of liquids, a disadvantage which may be removed by using as one constituent a semi-solid like lard or vaseline.

A series of experiments have, therefore, been made with the apparatus, fig. 11, p. 419, on mixtures of one of these substances with various amounts of reduced iron, marble, zinc sulphate, and sugar, in the form of powder, and the results are given in the following table and curves :—

Materials, per cent. by volume.	$\theta_j$	U.	U-M.	M-L.	$v_v$	$v_v - v_m$	$v_m - v_l$	$v_m$	Heat transmitted through glass.	$\cdot 00026 v_v$	$\cdot 0017 v_m$	Heat transmitted through mixture and ebullite.	Temperature slope.	Heat transmitted per unit slope.	Heat per unit slope through liquid.	k.
Lard . . . . .	16.7	.73	.671	1.456	2.0	1.85	4.03	+0.2	.192	.0006	.0003	.191	31.7	.00602	.0051	.00047
" . . . . .	14.8	.92	.669	1.413	2.5	1.85	3.90	0.7	.192	.0008	.0012	.190	30.7	.00618	.0053	.00049
" + 10.8 per cent. reduced iron	..	.61	.674	1.130	1.7	1.86	3.12	-0.2	.193	.0005	—	.193	24.6	.00784	.0069	.00064
" + 19.4 " "	..	.37	.693	0.851	1.0	1.91	2.36	-0.9	.198	.0003	—	.199	18.6	.0106	.0097	.00090
" + 32.3 " "	..	—	.717	0.600	-0.5	1.98	1.66	-2.5	.206	.0002	—	.210	13.1	.0160	.0151	.00140
" + 23.8 per cent. $ZnSO_4$	..	+	.661	1.193	+1.8	1.83	3.29	0.0	.189	.0005	.0000	.189	25.9	.00730	.0064	.00059
" + 37.5 " "	..	.43	.691	1.086	1.2	1.91	2.86	-0.7	.198	.0004	—	.199	22.5	.00883	.0080	.00074
" + 55.5 " "	..	.79	.681	0.890	2.2	1.88	2.46	+0.3	.195	.0007	.0005	.194	19.4	.0105	.0091	.00084
" + 13.0 per cent. sugar	..	.50	.693	1.407	1.4	1.91	3.89	-0.5	.198	.0004	—	.198	30.6	.00647	.0056	.00052
" + 38.0 " "	..	.70	.691	1.224	1.9	1.91	3.38	0.0	.198	.0006	.0000	.197	26.6	.00739	.0065	.00058
Lard . . . . .	..	.61	.663	1.446	1.7	1.83	4.00	-0.1	.199	.0005	—	.189	31.5	.00600	.0051	.00047
Vaseline . . . . .	..	.88	.690	1.617	2.4	1.91	4.46	+0.5	.198	.0007	+	.196	35.1	.00558	.0047	.00044
" + 25 per cent. marble	..	.70	.691	1.172	1.9	1.91	3.24	0.0	.198	.0006	.0000	.197	25.5	.00771	.0068	.00063
" + 43 " "	..	—	.731	0.779	-0.6	2.02	2.15	-2.6	.209	—	.0044	.213	16.9	.0126	.0117	.00108
" + 60 " "	..	—	.727	0.483	-0.8	2.01	1.33	-2.8	.208	—	.0048	.213	10.5	.0203	.0193	.00179
Marble . . . . .	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	.0071*

\* From LEES, 'Phil. Trans. Roy. Soc., A, vol. 90, p. 506 (1892).





These results, taken in conjunction with those found for liquids and solutions, show conclusively that the thermal conductivity of a mixture is not a linear function of its composition. The observed conductivity is always less than that calculated from the linear law, and, on the other hand, greater than that calculated on the assumption that the resistivity is a linear function of the composition. The second assumption is, however, a closer approximation to the observed facts than the former.

Since neither of these simple assumptions seems capable of representing the facts, which, by their uniformity, appear, notwithstanding, to point to some general law of mixtures, it seemed advisable to calculate the conductivity of a model of a mixture built up in some simple way so as to lend itself readily to the process.

Suppose a cubic centimetre of some substance, having a conductivity  $\rho_0$ , to be divided by equidistant planes parallel to its faces, into 1000 small cubes of 1 millim. edge; and let  $n$  small cubes of the same size but of a material of conductivity  $\rho$  be substituted for  $n$  of the cubes of the cubic centimetre chosen at random. The large cube is then a mixture of two materials, and its conductivity may be readily calculated if the lines of flow are assumed to be parallel to one edge of the cube.

If  $\rho_0$  is the thermal resistivity of the original,  $\rho$  that of substituted small cubes, the probability that, when  $n$  are substituted,  $p$  of them will be found in any column chosen at random,

$$= \frac{n! (1000 - n)!}{p! (n - p)! (10 - p)! (1000 - n - 10 - p)!} \bigg/ \frac{1000!}{10! 990!}$$

$$= \frac{n! (1000 - n)! 10! 990!}{p! (n - p)! (10 - p)! (990 - n + p)! 1000!}$$

and the heat conducted through such a column, when its ends differ in temperature  $1^\circ \text{C.}$ ,

$$= \frac{n! (1000 - n)! 10! 990!}{p! (n - p)! (10 - p)! (990 - n + p)! 1000!} \cdot \frac{10}{(10 - p) \rho_0 + p\rho}.$$

Hence the conductivity of the mixture on the assumption that cross-transmission from column to column may be neglected,

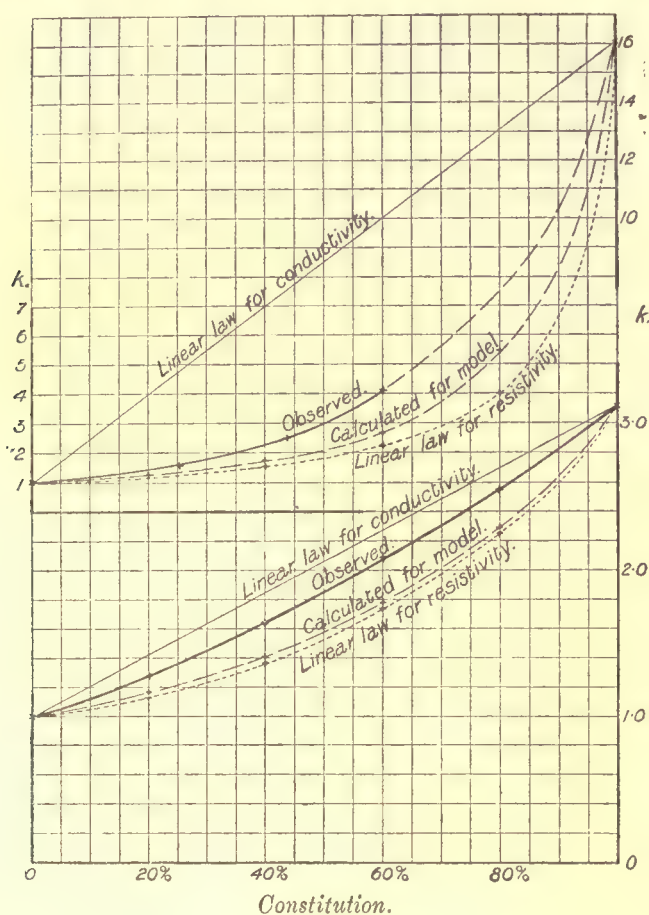
$$= \sum_{p=0}^{p=10} \frac{n! (1000 - n)! 10! 990!}{p! (n - p)! (10p)! (990 - n + p)! 1000!} \cdot \frac{10}{(10 - p) \rho_0 + p\rho}.$$

The values of the factorial expressions have been calculated for  $n = 200, 400, 600$ , and  $800$  with the aid of DE MORGAN'S tables of  $n!$ \* and are as follows :—

$p.$	$n = 200$	$n = 400$	$n = 600$	$n = 800$
0	·1062	·0054	·0001	·0
1	·2684	·0397	·0015	·0
2	·3036	·1204	·0103	·0001
3	·2021	·2156	·0420	·0007
4	·0878	·2521	·1112	·0053
5	·0260	·2013	·2013	·0260
6	·0053	·1112	·2521	·0878
7	·0007	·0420	·2156	·2021
8	·0001	·0103	·1204	·3036
9	·0	·0015	·0397	·2684
10	·0	·0001	·0054	·1062

The results of calculations of the conductivities of mixtures for which  $\rho_0 = 1$ , and  $\rho = \frac{1}{3.1}$  and  $\frac{1}{16}$  respectively, are given in the following curves, along with the observed curves for mixtures of water and alcohol, and of vaseline and marble, and the curves for conductivity and resistivity following the linear law, for comparison. The calculation gives conductivities always below the observed values, a result which is due in part, and it may be entirely, to the neglect of the transverse transmission of heat from column to column of the model. The agreement is, however, sufficiently near to justify the statements that the thermal conductivity of a substance is not greatly modified when it enters as one constituent in a physical mixture, and that the thermal conductivity of a mixture depends directly on the amounts and conductivities of its constituents.

\* 'Encyclopædia Metropolitana,' vol. 2, p. 486.



A reference to the values obtained for the thermal conductivities of solutions of a gas or a solid in water, will show that the above law does not apply to cases in which one constituent of the mixture changes its physical state on mixing. Further work is necessary before any law can be stated, but it seems probable that if the thermal conductivity of the gas or solid when liquefied were known and used as that of the material entering into solution, the above law would be found to cover solution as well as physical mixture.

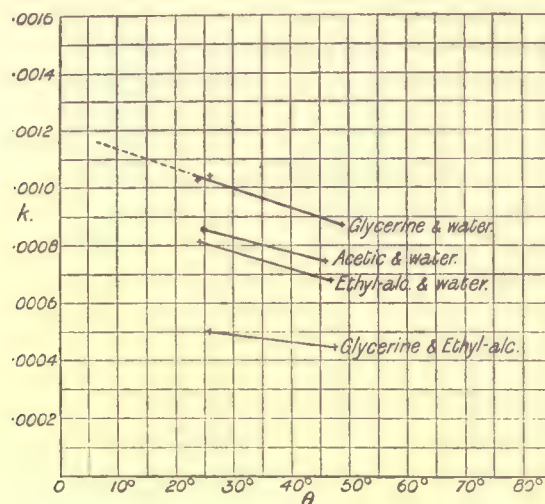
#### PART V. CHANGE OF CONDUCTIVITIES OF MIXTURES WITH TEMPERATURE.

In order to determine the effect of temperature on the thermal conductivities of mixtures, experiments were made on several of the 50 per cent. mixtures of liquids mentioned in the preceding section. They were carried out with the enclosed apparatus, fig. 11, p. 419. The observations and results are given in the following tables and curves :—



Mixture.	$\theta_j$ .	U.	$\theta_c$ .	Reduced.		$\theta_M - \theta_L$ .	$\theta_M$ .	B.	$\theta_B$ .	Temperature slope in glass.	Heat transmitted through glass.	·0006v.	·0017v.	Heat transmitted through liquid and ebonite.	Temperature slope.	Heat/slope.	Heat through liquid per unit slope.	k.	Temperature.
		ohms.	ohms.	ohms.	ohms.		ohms.	ohms.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.	°C.
50 per cent.* glycerine and water	17.44	3.45	26.74	59	.63	1.68	25.17	1.10	20.44	5.59	.166	.002	.007	.157	13.2	.0119	.0110	.00102	24
	18.92	12.12	50.48	.562	.718	1.80	49.08	10.16	45.59	4.98	.154	.001	.005	.148	14.2	.0104	.0095	.00088	48
	18.76	3.77	28.89	.62	.66	1.75	27.25	1.78	23.57	5.84	.174	.001	.006	.167	13.8	.0121	.0112	.00104	26
50 per cent. acetic acid and water	18.20	3.40	27.36	.582	.733	1.955	25.81	1.06	21.09	5.51	.163	.002	.007	.154	15.4	.0108	.0091	.00084	25
	18.40	11.44	48.27	.52	.79	1.98	46.97	9.57	45.58	4.63	.144	.001	.005	.138	15.6	.0085	.0080	.00074	46
	18.74	3.38	27.83	.582	.727	1.94	26.28	1.23	22.07	5.51	.163	.002	.006	.155	15.3	.0101	.0092	.00085	25
	14.12	— .09	..	.704	.879	..	..	— .32	..	..	..	..	..	..	..	..	..	.00088	11
50 per cent. ethyl alcohol and water.	17.50	3.46	26.83	.591	.78	2.08	25.26	.98	20.17	5.59	.166	.002	.007	.157	16.4	.00957	.0087	.00081	24
	17.53	12.21	49.41	.562	.888	2.225	48.01	10.09	44.15	4.98	.154	.002	.006	.146	17.5	.00834	.0074	.00068	47
	17.34	3.63	27.12	.60	.80	2.14	25.52	1.80	22.23	5.70	.169	.001	.005	.163	16.8	.00970	.0088	.00081	24
	13.76	.23	..	.702	.957	..	..	— .22	..	..	..	..	..	..	..	..	..	.00079	11
50 per cent. glycerine and ethyl alcohol	17.36	3.93	28.95	.565	1.105	2.93	27.46	1.15	20.49	5.30	.158	.002	.010	.146	23.1	.00632	.0054	.00050	26
	18.00	12.11	49.62	.523	1.197	3.00	48.31	9.86	43.88	4.65	.145	.002	.007	.136	23.6	.00576	.0049	.00045	47
	17.84	4.30	29.40	.563	1.127	2.99	27.91	1.63	22.27	5.30	.158	.002	.008	.148	23.5	.00630	.0054	.00050	26

\* By weight throughout.



Mixture 50 per cent. by weight of	$k_{25}$	Mean percentage change per degree between 25° and 45° C.
Glycerine and water . . . . .	·00103	— ·0063
Acetic acid and water . . . . .	·00085	— ·0058
Ethyl alcohol and water . . . . .	·00080	— ·0068
Glycerine and ethyl alcohol . . . . .	·00050	— ·0050

It will be seen from this that the thermal conductivities of mixtures decrease with increase of temperature at roughly the same rate per cent. as their constituents.

#### STATEMENT OF RESULTS.

The preceding results may be summarised as follows :—

1. Solids which are not very good conductors of heat, in general decrease in conductivity with increase of temperature in the neighbourhood of 40° C. Glass is an exception to this rule.
2. Liquids in the neighbourhood of 30° C. follow the same law.
3. The conductivity of a substance does not invariably change abruptly at the melting point.
4. The thermal conductivity of a mixture lies between the conductivities of its constituents, and seems connected with the constitution and conductivities by a simple law.
5. Mixtures of liquids decrease in conductivity with increase of temperature in the neighbourhood of 30° C. at about the same rate as their constituents.

# XI. *Experiments on Aneroid Barometers at Kew Observatory, and their Discussion.*

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*Communicated by the Author at the Request of the Kew Observatory Committee.*

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## *Preliminary.*

§ 1. THE ordinary aneroid barometer is an instrument whose chief recommendations are its portability and the ease with which it can be used by travellers. It is essentially a field instrument, and it would be largely waste of time to treat it as if intended for the laboratory.

The first object of the present investigation is to acquire knowledge likely to increase the usefulness of the aneroid under the conditions in which it is actually employed.

Aneroids have for many years been tested at Kew Observatory, and records exist of the performance of many hundreds of various sizes. The test has hitherto been applied as follows: The aneroid has been put in the receiver of an air pump, and,



pressure being reduced at the rate of 1 inch in 3 or 4 minutes, readings have been taken of the aneroid and a mercury gauge attached to the pump. The readings are taken with the pressure temporarily stationary, usually for each inch, but sometimes for each half-inch of pressure. After the reading has been taken at the lowest point for which verification is desired the pressure is reduced a very little further, and then maintained constant for some minutes. It is then allowed to rise by the readmission of air, stoppages being made and readings taken during the ascent of pressure precisely as during the descent.

From the observed differences between the aneroid and gauge—the readings of the latter being corrected of course for temperature—corrections are calculated separately for the descending and ascending readings of the aneroid, and these are given on the certificate issued.

If the aneroid possesses, as is most usual, a scale of altitudes in feet or metres, that is read as well as the pressure scale, and corrections are given to it based on Airy's table connecting altitude and barometric pressure. Whether the unrestricted application of Airy's table is the best means of interpreting observations taken during mountain ascents is an interesting question, but I do not propose to discuss it here, confining my attention to the aneroid as a measurer of pressure.

Aneroids intended for the Meteorological Office are usually constructed only for the narrow range, 31 to 26 inches. In their case there are exact regulations as to the size of error permissible and other points, and the issue of a certificate implies the attainment of a certain standard of excellence. With this exception, however, there exist no rules for rejection, and the certificate merely represents the report of an expert on an instrument submitted to him. The owner is supposed to draw his own conclusions from the figures submitted. This view of a certificate was apparently that originally dominant at Kew Observatory, but of late years there has been an increasing tendency on the part of the public to substitute the idea of a "certificate of excellence" for that of a "certificate of examination." What is sought is protection against inferior instruments rather than the means of applying corrections to observed readings. It has thus become desirable to ascertain how the excellence of an aneroid may be judged of, and one of the chief objects of the present investigation was to obtain data suitable for the purpose.

### *General Character of Phenomena.*

§ 2. An aneroid is usually graduated by direct reference to a mercury gauge during a reduction of pressure. An examination, made under the same conditions as the graduation, tests the accuracy of the workman, but not necessarily the quality of the instrument. That something more is desirable becomes obvious, when, after reducing the pressure several inches, one keeps it constant and continues comparative readings of the gauge and aneroid for some time. A gradual fall in the readings of the latter

soon becomes apparent. Again, on allowing the pressure to rise to its original value, one finds the aneroid read lower than at the start, this depression gradually disappearing. The instrument, in fact, behaves as if an imperfectly elastic body. The strain, under a uniform stress, tends to increase, and as a natural concomitant, there is elastic after-effect (*elastische Nachwirkung*). That the whole of the difference between readings with pressure descending and ascending (or, as we shall call them for brevity, descending and ascending readings) represents, in all cases, a true after-effect, I am not prepared to say. If we regard a rise of temperature as analogous to a fall of pressure, a thermometer presents a somewhat parallel case; and we know that there the difference between ascending and descending readings is due to at least two causes, viz., change in the mercury meniscus, with consequent alteration of internal pressure, and temporary change of zero following exposure to the higher temperature. Of these two causes, the latter only is a true after-effect; but the former may also conceivably have its counterpart in the mechanism of the aneroid.

An aneroid showing large after-effect is not a suitable instrument for travellers. Readings taken with it, for instance, during a mountain descent, are largely influenced by the elevation of the summit and the time spent there. Thus, from the outset, I have regarded the after-effect phenomena as specially requiring investigation.

§ 3. As a first step, I examined the records of about 300 aneroids, tested over the ranges 30-15, 30-18, 30-21, 30-23, 30-24, and 30-26 inches. The differences between the descending and ascending readings were noted for the several ranges, and the general character of the phenomena ascertained. There existed, however, no data for connecting the results from the several ranges, it being customary to test each aneroid over one range only. Special experiments were thus requisite, in which the same aneroids should be taken over a variety of ranges. For such a purpose it was impossible to employ the ordinary working apparatus, with a due regard to the regular work of the Observatory. Application was accordingly made by the Kew Observatory Committee to the Government Grant Fund, and £30 was obtained for a new air-pump and receiver, and for a set of aneroids to be experimented on.

The only special feature of the apparatus\* is a second or auxiliary receiver, between the pump and the main receiver which contains the aneroids. On the tube connecting the two receivers are two stop-cocks, a third cock being placed between the auxiliary receiver and the pump. One can thus exhaust the auxiliary receiver separately, and then by manipulating the cocks lower the pressure in the main receiver at any desired rate. This arrangement avoids the sudden changes of pressure to which aneroids are exposed when in a receiver connected directly to the pump. The pump, the receivers, and the mercury gauge—which shared the pressure of the main receiver—were rigidly attached to a stout board which rested on a table. The

\* The apparatus was obtained from Mr. J. J. HICKS, 8, Hatton Garden, London. It was solidly constructed and proved very satisfactory. The experimental aneroids were likewise purchased from Mr. HICKS.



board was tapped before each reading, this being the invariable practice in the ordinary Observatory test. Though not free from objections, this seems to reproduce most closely natural conditions.

*Differences of Descending and Ascending Readings, from Kew Verifications.*

§ 4. It being most convenient to extract data from books not in current use at the Observatory, I have employed, in the following discussion, data from aneroids tested between 1885 and 1891. I have, however, examined a sufficient number of the more recent data to assure myself that the ordinary aneroid has since then undergone no important modification. On the whole, there is, perhaps, a slight reduction in the average size of the after-effect phenomena.

Table I. shows the mean excess, in inches, of the descending over the ascending reading at each inch of pressure for the several ranges specified, also the sum of the differences between the descending and ascending readings, and the mean difference for each range. It also mentions the number of instruments on which each set of results is based. The results, with the exception of those from the second group of 30 aneroids taken over the range 30–24 inches, are shown graphically in the several curves of fig. 1. Abscissæ represent air pressures, ordinates the excess of descending over ascending readings.

Fig. 1.

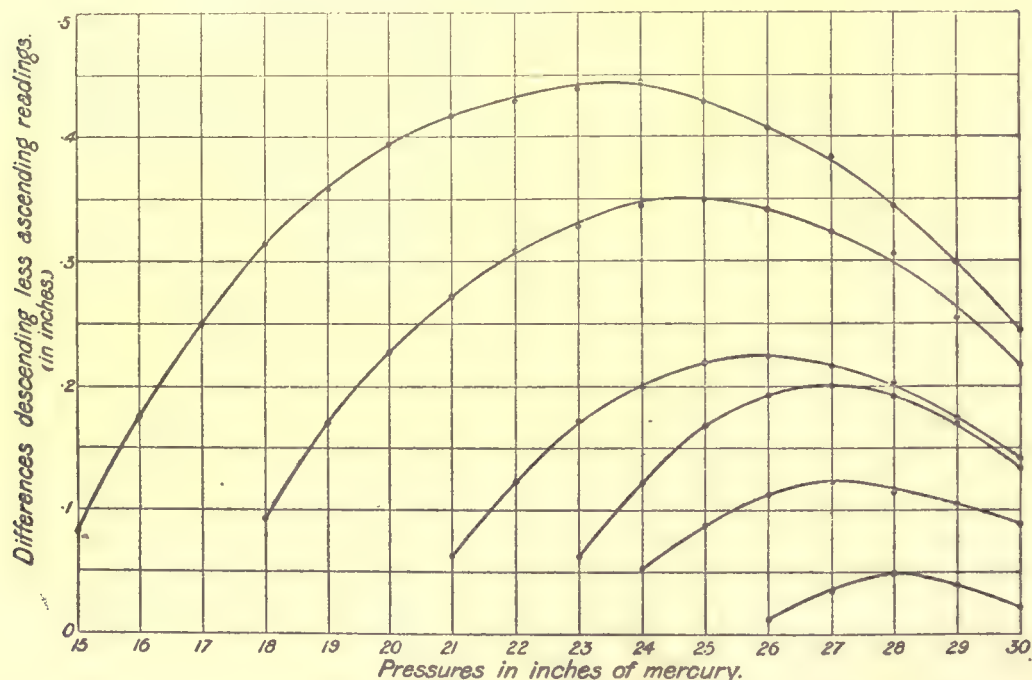




TABLE I.—Descending less Ascending Readings in inches (old records).

Range.	Number of instru- ments utilised.	Pressure in inches.											Sum of differ- ences.	Mean differ- ence.					
		15	16	17	18	19	20	21	22	23	24	25			26	27	28	29	30
inches. 30-26	101	..	..	..	..	..	..	..	..	..	..	..	·014	·035	·050	·037	·020	·156	·031
„ 24	13	..	..	..	..	..	..	..	..	·053	·086	·115	·124	·124	·114	·105	·087	·684	·098
„ 24	30	..	..	..	..	..	..	..	..	·051	·086	·138	·135	·135	·126	·084	·093	·102	·015
„ 23	33	..	..	..	..	..	..	..	..	·064	·120	·168	·193	·199	·191	·172	·133	1·240	·155
„ 21	44	..	..	..	..	..	..	·064	·120	·174	·199	·218	·223	·215	·204	·174	·141	1·732	·173
„ 18	35	..	..	..	·093	·171	·228	·271	·310	·328	·346	·350	·341	·323	·306	·254	·219	3·540	·272
„ 15	50	·083	·176	·249	·315	·357	·395	·417	·429	·438	·446	·428	·408	·385	·346	·299	·245	5·416	·338

The second group of aneroids tested over the range 30–24 inches were all large and of a special make, and so best treated apart. They showed such exceptionally small after-effect that I multiplied the actually-observed mean differences by seven, except in the last two columns, to make the figures similar in size to those given by the first group of 13 aneroids tested over the same range. In this way one sees more clearly that the general laws obeyed by the after-effect are to all appearance independent of its absolute amount.

In no case were the aneroids read to nearer than 0·01 of an inch, the third significant figure in Table I. being introduced merely in taking arithmetical means. Thus the law of variation throughout the range of the quantity tabulated is difficult to ascertain with great exactness over the shortest ranges, especially when the aneroids are of such exceptional quality as the second group tested between 30 and 24 inches.

When results are derived from a number of aneroids, such disturbing factors as local irregularities of graduation lose their influence; whereas with one or a small number of aneroids they may affect the smoothness of the results, however numerous be the observations.

§ 5. A glance at fig. 1 shows a similarity of type in the curves; but this is somewhat obscured by the variety in the pressure ranges and in the lengths of the maximum ordinates. The method of bringing this similarity into clearer relief will be most easily followed by taking a particular case, say, that of the range 30–21 inches. Here there are 10 differences of descending and ascending readings. Call these, starting from 21 inches and proceeding upwards,  $d_1, d_2 \dots d_{10}$ . The mean difference is

$$\bar{d} \equiv (d_1 + d_2 + \dots + d_{10})/10,$$

and  $d_1/\bar{d}, d_2/\bar{d}$ , &c., are the ratios of the several differences to the mean. Now draw a curve whose ordinates represent the size of these ratios on any convenient scale, while the abscissæ represent the corresponding fractions, 0, ·1, ·2, &c., of the range measured from the lowest pressure. The curves obtained in this way for the four longest ranges of Table I. are shown in fig. 2. In a short range the number of points on the curve determined by the data is so small as to leave the shape a little uncertain. Thus to have given together curves for all the data in Table I. would have sacrificed the clearness of detail without securing adequate compensation. It will be observed that the curves in fig. 2 are almost coincident near the central part of the range. At the lower end of the range there is a slight but appreciable difference, the ordinate tending to decrease as the range becomes longer. The closeness of the curves, striking as it is, would, I think, be improved by altering the cycle and the method of treating the results in the following three respects:—

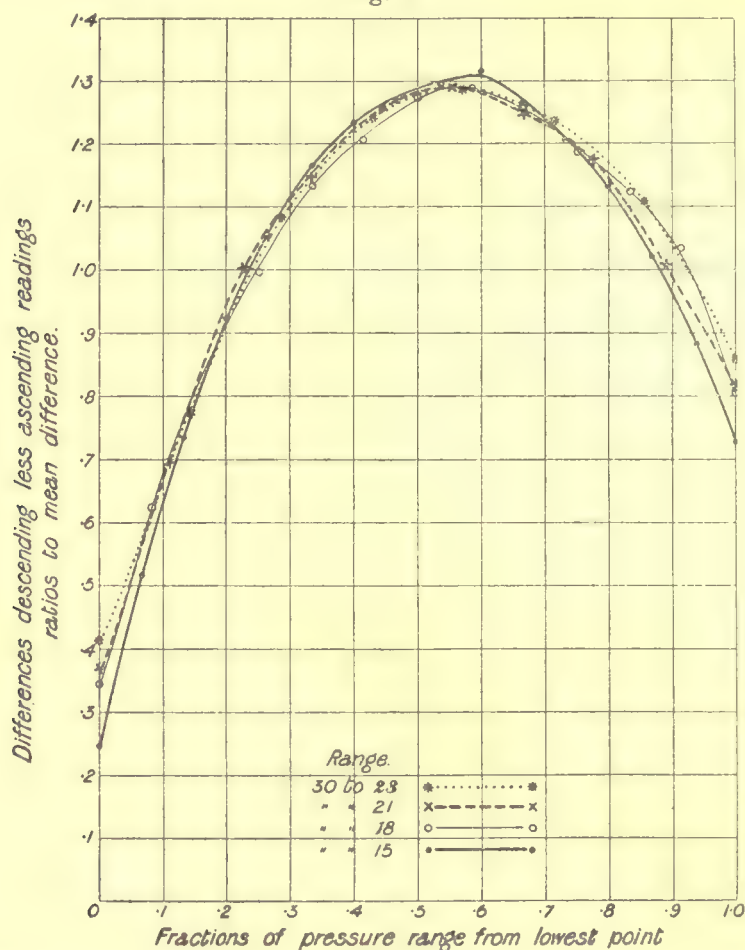
- (i.) Making the duration of the stoppage at the lowest point proportional to the length of the range.

(ii.) Making the number of points of observation the same in all ranges by suitably varying the size of the pressure steps.

(iii.) Allowing only half-weight to the two terminal differences; for instance in the range 30–21 inches taking  $\bar{d}$  as the mean of nine quantities, of which  $(d_1 + d_{10})/2$  is one.

The last of these changes was, of course, within my power, but as its advantages after all could not be large, and might be nil, it did not seem worth while to undertake the large amount of arithmetic that would have been entailed by its substitution for the method originally adopted.

Fig. 2.



§ 6. In actual practice, the change (ii.) specified above would be inconvenient, entailing readings at fractions of inches difficult to determine exactly on an ordinary gauge. In ranges where the points of observation are sufficiently numerous the same object can be obtained pretty satisfactorily by interpolation. For instance, in the range 30–21 inches, the middle of the range is 25.5 inches, and  $(d_5 + d_6)/2$  may be regarded as a tolerably good first approximation to the corresponding difference.



I have calculated in this way from Table I., in the case of all but the shortest range, the differences of the descending and ascending readings answering to the fractions 0, .1, .2, .3, &c., of the range, measured from the lowest point. For the range 30–24 inches the first group of aneroids was used. The ratios borne by these calculated differences to the mean differences in Table I. are given in Table II.

It should be noticed that the mean of the 11 differences found for the fractions 0, .1, &c., of the range is not generally identical with the difference  $\bar{d}$  given in Table I.; consequently the sum of the 11 entries in each row of Table II. is not identically 11, though nearly so. In deciding to take the  $\bar{d}$  of Table I. as the consequent of the ratios, I avoided the confusion entailed by having two slightly different *mean differences* for each range. There are, however, disadvantages, which seemed to me to prevail in the case of the mean results for the 5 ranges of Table II. In their case, accordingly, I made the small alterations required—an increment of less than  $\frac{1}{7}$  per cent.—to make the sum of the eleven ratios exactly 11.

TABLE II. (Old Observations).—Ratios of Differences Descending less Ascending Readings to Mean Difference.

Range.	Fraction of range—										
	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
30 inches to 24	.541	.743	.936	1.115	1.210	1.265	1.204	1.145	1.089	.998	.888
" " 23	.413	.666	.898	1.100	1.213	1.265	1.274	1.238	1.159	1.034	.858
" " 21	.370	.661	.943	1.106	1.215	1.274	1.270	1.223	1.143	.986	.814
" " 18	.342	.670	.900	1.081	1.192	1.271	1.278	1.226	1.149	.971	.804
" " 15	.245	.628	.931	1.111	1.232	1.281	1.318	1.235	1.137	.953	.724
Mean for 5 ranges	.382	.674	.923	1.104	1.214	1.273	1.270	1.215	1.137	.989	.819

Before treating further the results shown in Tables I. and II. it is convenient to notice some of the special experiments.

*Differences of Descending and Ascending Readings. Special Experiments.*

§ 7. The first 24 special experiments were intended to serve for comparison of the phenomena over different pressure ranges. They took place as follows, the numbers showing the chronological order :—

Range.	Order of experiments.	Number over range.
30 inches to 26	Nos. 1, 2, 12, 13, 22, 23, 24	7
" " 24	3, 4, 5, 20, 21	5
" " 21	6, 7, 18, 19	4
" " 18	8, 9, 16, 17	4
" " 15	10, 11, 14, 15	4

Two aneroids, No. 1 and No. 4, were subjected to all these experiments; two others, No. 2 and No. 3, divided only to 20 inches, were not taken below 21 inches of pressure.

The order of the experiments was adopted with a view to the possible permanent influence of the lower pressures on the aneroids.

The procedure in these experiments was uniform. The change of pressure during both the descent and the ascent was at the rate of 1 inch in 5 minutes. At the lowest point there was a stoppage of 10 minutes, preceded however by no slight lowering of pressure below the point at which the last descending reading was taken.

After each experiment the aneroids were left for some days at atmospheric pressure, so as to be fully rested at the beginning of each pressure cycle. The experiments were made in 1895, and extended over a period of five months. As will be seen later, there would not appear to have been any serious change in the after-effect phenomena during this interval.

§ 8 The first thing investigated was the law of variation throughout each range of the differences of the descending and ascending readings, it being advisable to make sure that the aneroids were fair specimens, and gave results sufficiently similar in type to those derived from the old verification books. I have thought it unnecessary to reproduce the results analogous to Table I., but pass at once to those analogous to Table II. The procedure followed was exactly the same as that already described, so that the results of Tables III. and II. are immediately comparable.

TABLE III. (Special Experiments).—Ratios of Differences Descending less Ascending Readings to Mean Difference.

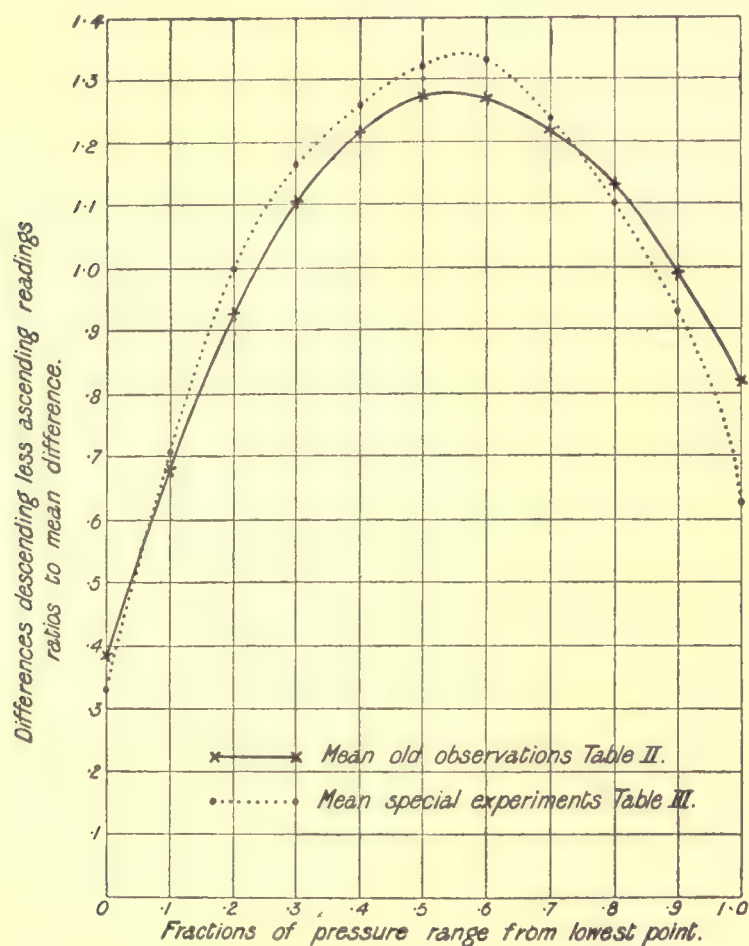
Range.	Fraction of range.										
	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
30 inches to 26	·430	·707	0·984	1·168	1·260	1·352	1·292	1·231	1·140	1·017	·895
" " " 24	·437	·771	1·029	1·137	1·195	1·228	1·293	1·309	1·230	0·974	·630
" " " 21	·294	·708	0·992	1·163	1·306	1·334	1·250	1·169	1·116	0·975	·693
" " " 18	·262	·715	0·940	1·142	1·276	1·416	1·425	1·227	1·049	0·887	·495
" " " 15	·197	·650	1·084	1·228	1·255	1·323	1·442	1·274	0·989	0·823	·433
Mean for 5 ranges	·323	·707	1·002	1·163	1·252	1·325	1·334	1·237	1·100	0·931	·626

The alteration required to make the sum of the ratios in the last line exactly 11, was slightly greater than in the case of Table II., but still was less than one half per cent.

There is a tendency in the values at the extremities of all the ranges to be lower in Table III. than in Table II., entailing of course a slight difference in the opposite direction towards the centre of the range. This is probably due mainly to the difference between the procedure followed in the old and new observations. The

difference in the results, though worthy of notice, is for practical purposes of little consequence. As we shall see later in Table VI., unity in Table III. represents at the very utmost 0.25 of an inch, so that a difference of 4 or 5 per cent. in one of the ratios answers to a quantity which is in general considerably less than the probable error of an observation.

Fig. 3.



To facilitate comparison of the old and new observations, I show side by side in fig. 3, curves of the same type as in fig. 2, the one representing the mean results of Table II., the other the mean results of Table III. The former curve is exceptionally smooth, and the latter even is very fairly regular; but curves for the individual ranges of Table III.—which I do not reproduce—show very appreciable irregularities. This of course is only what one would expect from the limited number of observations. Whilst the data in Table III. thus possess defects which might have been much reduced by an increase in the number of experiments, they appeared quite sufficiently consistent for the purpose I had in view.



*Formulae for the Variation of the Differences of the Descending and Ascending Readings throughout the Range.*

§ 9. The absolute size of the differences between the descending and ascending readings varies largely from aneroid to aneroid; thus formulæ reproducing the data of Table I. would not be immediately useful. On the other hand, the law of variation, as shown by the ratios of these differences, at different points of the range, to the mean difference, appears to be nearly the same in all ordinary aneroids. I have thus tried to represent the variations shown in Tables II. and III. by simple algebraic formulæ of the type

$$y = a_0 + a_1x + a_2x^2 + \dots \dots \dots (1),$$

where  $y$  is the ratio of the difference of the descending and ascending readings to the mean difference,  $x$  the fraction of the range measured from the lowest point.

One can secure absolute agreement between the observed and calculated values in such a case by taking enough terms; but, for practical purposes, the question is whether with a comparatively small number of terms one can secure a sufficiently close agreement. In the present instance, four terms are enough for practical purposes. To get the best general agreement we ought, of course, to determine  $a_0$ ,  $a_1$ ,  $a_2$ ,  $a_3$  by the method of least squares, taking account of the whole eleven values in each range. In dealing, however, with material such as that here available, the slightly increased accuracy thus attainable would not, in my opinion, be an adequate return for the labour expended. I thus simply determined the constants so that the observed and calculated values were identical for the values .1, .3, .7, and .9 of  $x$ . The values thus found for the constants are given in Table IV., the first six sets of values relating to the old observations of Table II., the last set to the mean data of Table III.

TABLE IV.—Values of Constants in  $y = a_0 + a_1x + a_2x^2 + a_3x^3$ .

Range.	$a_0$ .	$a_1$ .	$a_2$ .	$a_3$ .
30 inches to 24	+·425	+3·680	−5·209	+2·031
„ „ „ 23	+·332	+3·677	−4·079	+0·958
„ „ „ 21	+·322	+3·808	−4·264	+0·948
„ „ „ 18	+·377	+3·219	−2·947	+0·115
„ „ „ 15	+·264	+4·067	−4·391	+0·802
Mean old observations	+·345	+3·699	−4·186	+0·969
„ new „	+·358	+3·922	−4·362	+0·792

The value of  $a_1$  shows little variation. In the shortest range the values of  $a_2$  and  $a_3$  are undoubtedly unduly large numerically, and in the range 30–18 inches there is

a similar defect in the opposite direction. In these cases the method of least squares would no doubt have given values closer to the mean.

The closeness with which the data of Tables II. and III. are reproduced by the formulæ based on the above values of the constants is shown by Table V., which gives the algebraical excess of the observed over the calculated values. There is, it will be remembered, absolute agreement when  $x = \cdot 1, \cdot 3, \cdot 7$ , and  $\cdot 9$ .

TABLE V.—Observed less Calculated Values. (Ratios of Differences to Mean Difference.)

Range.	$x =$							Mean of differences.
	0	$\cdot 2$	$\cdot 4$	$\cdot 5$	$\cdot 6$	$\cdot 8$	1.0	
30 inches to 24	+·116	−·033	+·016	+·048	+·007	+·014	−·039	±·039
" " " 23	+·081	−·014	+·002	−·006	−·003	+·006	−·020	±·019
" " " 21	+·048	+·021	−·009	−·004	−·007	+·019	·000	±·015
" " " 18	−·035	−·004	−·009	+·007	+·005	+·023	+·040	±·018
" " " 15	−·019	+·024	−·006	−·018	+·022	+·018	−·018	±·018
Mean old observations	+·037	−·003	−·003	+·003	−·004	+·015	−·008	±·010
" new "	−·035	+·028	−·024	−·002	+·022	−·010	−·083	±·029

There is not a single instance here where the difference between the observed and calculated values represents as much as 0·01 of an inch in absolute pressure.

For ordinary purposes—unless special importance is attached to the phenomena at the two ends of the range—the formula

$$y = \cdot 35 + 3\cdot 7x - 4\cdot 2x^2 + \cdot 97x^3 \quad \dots \dots \dots (2)$$

is quite good enough for experiments made on the old plan over any range not shorter than 6 nor longer than 15 inches.

*Position of Maximum Difference, Descending less Ascending Reading.*

§ 10. As illustrating the practical use of (2), let us find where it makes the difference between the descending and ascending readings largest. For this, putting

$$\frac{dy}{dx} = 0 \text{ in (2),}$$

we find

$$2\cdot 91x^2 - 8\cdot 4x + 3\cdot 7 = 0 \quad \dots \dots \dots (3),$$

whence

$$x = \cdot 54.$$

Here  $x$ , it will be remembered, represents the fraction of the range measured from the lowest point. To test the accuracy of this result, I have calculated the position of the maxima for the several ranges from Table I., which contains, it should be noted, the direct results of observation. The results are as follows :—

	Range 30 inches to—						Mean.
	26	24	23	21	18	15	
	Position of maximum, $x =$	·56	·50	·54	·56	·58	·58

*Relations between Phenomena for different Ranges.*

§ 11. Our previous work may be regarded as determining the differences between the descending and ascending readings at every point of any range in terms of the mean difference for that range. The next step is to express the mean difference for a range of  $n$  inches as a product of  $f(n)$ —a function whose sole variable is  $n$ —and a factor which is constant for a given aneroid.

The mean differences between the descending and ascending readings (defined as in § 5) given by the first 24 experiments with the 4 aneroids are recorded in Table VI.

TABLE VI.—Mean Differences (in inches) from the first 24 Special Experiments.

Range 30 inches to—							
			26	24	21	18	15
Aneroid No.	1	. . .	·043	·084	·146	·217	·315
"	2	. . .	·033	·063	·123	..	..
"	3	. . .	·046	·089	·179	..	..
"	4	. . .	·027	·043	·077	·122	·190

The most convenient way of dealing with these results is to find the ratios borne by the several mean differences to one of their number for each aneroid separately. For this purpose, I have taken the differences in the range 30–21 inches as standards. The results are given in Table VII.



TABLE VII.—Ratios of Mean Differences to Mean Difference in Range 30–21.

	Range 30 inches to—				
	26.	24.	21.	18.	15.
Aneroid No. 1 . . . . .	·295	·575	1	1·486	2·158
„ „ 2 . . . . .	·268	·512	1		
„ „ 3 . . . . .	·257	·497	1		
„ „ 4 . . . . .	·351	·558	1	1·584	2·468
Mean for all 4 aneroids . . .	·293	·535	1		
„ „ Nos. 1 and 4 . . .	·323	·566	1	1·535	2·313

Assume now that the ratio of the mean difference  $\bar{d}_n$  for a range of  $n$  inches to the mean difference  $\bar{d}_9$  for a range of 9 inches is given by

$$\bar{d}_n/\bar{d}_9 = \phi(n) = b_0 + b_1n + b_2n^2 \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Determining  $b_0$ ,  $b_1$ ,  $b_2$  from the three ranges in Table VII. common to all four aneroids, we find

$$\phi(n) = -\cdot028 + \cdot053n + \cdot0068n^2 \quad . \quad . \quad . \quad . \quad . \quad (5),$$

with of course

$$\phi(9) = 1.$$

In arriving at this expression, we utilised only the data from the three shortest ranges in Table VII. Thus a check on the general suitability of the formula is supplied by the good agreement between the mean results for the two longest ranges in the table and the corresponding values

$$\phi(12) = 1\cdot587, \quad \phi(15) = 2\cdot297$$

calculated from (5).

§ 12. As illustrating the practical application of (4) and (5), suppose we find that in a particular aneroid the mean difference between the descending and ascending readings for the range 30–15 inches is 0·338 inch. Then we should conclude that the mean difference for a range of  $m$  inches,  $m$  being any specified number, is given by

$$\bar{d}_m = \cdot338 \times \phi(m) \div \phi(15).$$

For instance, we should find for this aneroid

$$\bar{d}_4 = \cdot043, \quad \bar{d}_6 = \cdot078, \quad \bar{d}_7 = \cdot099, \quad \bar{d}_9 = \cdot147, \quad \bar{d}_{12} = \cdot234.$$

The value assumed in the above illustration is that found in Table I. for the range 30–15 inches. Hence the values just deduced for  $\bar{d}_i$ , &c., are what the mean differences for the other ranges would have been in the case of the average aneroid sent for trial over the range 30–15 inches. Comparing these calculated values with the mean differences actually recorded in Table I., we should draw the conclusion that the average aneroid sent for testing over the longest range is less subject to after-effect than the average aneroid sent for testing over any other range except the shortest. This is, I believe, really the case. The exception in favour of the shortest range is due to the exceptional size of a number of the aneroids intended to cover the range 30–26 inches.

§ 13. The sum of the differences of the descending and ascending readings is really the source of our knowledge of the mean differences, and for some purposes it may well be the more convenient quantity of the two. The formula for it is deducible at once from (5), for

$$\begin{aligned} \frac{\text{Sum of differences when range } n \text{ inches}}{\text{Sum of differences when range 9 inches}} &= \frac{n+1}{10} \phi(n), \\ &= -.0028 + .0025n + .0060n^2 + .00068n^3 \dots (6), \\ &= \psi(n), \text{ say.} \end{aligned}$$

The following results are easily verified :—

$$\psi(4) = .147, \quad \psi(6) = .375, \quad \psi(9) = 1, \quad \psi(12) = 2.066, \quad \psi(15) = 3.680.$$

Using  $S_n$  to denote the sum of the differences for a range of  $n$  inches, then, according to the formula,

$$S_n \div \psi(n) = S_m \div \psi(m)$$

for all values of  $n$  and  $m$ .

To illustrate the degree of accuracy attained by the use of such formulæ, I have taken the case of the first 24 special experiments, and determined the arbitrary constant for each aneroid so as to make the total sum of the differences over all the ranges for which the instrument was tried the same for the calculated as the observed values. As it may be well to show how this was done, take the case of aneroid No. 2. Here the sums of the differences observed in the three shortest ranges were respectively .163, .443 and 1.228 inches, amounting in all to 1.834; while by the formula,

$$\psi(4) + \psi(6) + \psi(9) = 1.522.$$

Thus, for a range of  $n$  inches with this aneroid, we may assume

$$\text{Sum of differences} = \psi(n) \times 1.834 \div 1.522.$$

In Table VIII. C denotes the values calculated in this way; O, those actually observed.

TABLE VIII.—Sums of Differences Descending less Ascending Readings.

	Range 30 inches to—									
	26		24		21		18		15	
	O.	C.	O.	C.	O.	C.	O.	C.	O.	C.
Aneroid No. 1	·214	·204	·589	·523	1·463	1·393	2·820	2·878	5·040	5·126
" " 2	·163	·177	·443	·452	1·228	1·205	..	..	..	..
" " 3	·231	·254	·622	·651	1·790	1·737	..	..	..	..
" " 4	·134	·118	·304	·302	0·770	0·804	1·588	1·661	3·048	2·959

The differences between the observed and calculated values in Table VIII. are of the same order as those met with in different experiments over the same range under the most favourable conditions.

§ 14. The preceding results go a long way to secure one of the principal objects of the investigation—the more complete utilisation of the ordinary Kew test. Suppose, for instance, an aneroid has been taken over any convenient range in the usual way, and a summation made of the differences of the observed descending and ascending readings. This sum, being based on a considerable number of independent readings, is but little affected by errors of observation, if ordinary care is used. It is also but little affected by any ordinary irregularities in the sub-divisions, and thus affords a satisfactory basis for a general diagnosis of the quality of the instrument so far as after-effect is concerned.

By means of §§ 11, 12, and 13 we can deduce from the observed sum, or from the mean difference, the values of the corresponding quantities for any other range. In this way a common standard of excellence could be utilised for all aneroids of given type, irrespective of the range covered by their graduation.

§ 15. In the laboratory one can change the pressure at a uniform rate and keep the temperature within narrow limits. In the field, however, the conditions are very variable. It was thus obviously desirable to aim at a fuller knowledge of the properties of aneroids, which might be of service to a wider circle than that concerned with the mere testing of these instruments.

Some valuable work in this direction was done many years ago by Dr. BALFOUR STEWART, and more recently by Mr. EDWARD WHYMPER. Neither writer, however, made any very serious attempt to ascertain the exact connection between the several phenomena. I thus postpone consideration of their work to a later stage, when its significance will be more easily understood.



Whilst I regard the present experiments as throwing much fresh light on the subject, it will, I hope, be clearly understood that I do not profess to have established a complete physical theory, by the aid of which one can foretell the exact behaviour of an aneroid under any arbitrary set of conditions.

*Fall of Reading at the Lowest Point.*

§ 16. In the 24 experiments described in § 7, readings were taken at 2-minute intervals during the 10 minutes' stoppage at the lowest pressure. It was soon obvious that the fall in a given time was at least approximately proportional to the range, *i.e.*, to  $n$ , where  $30-n$  represents the lowest pressure in inches.

To show this in a simple way, I give in Tables IX. and X. the mean results obtained by multiplying the observed falls by  $(18,000 \div \text{range})$ . The constant was chosen so as to avoid decimals. The actual fall in inches is of course obtainable by multiplying the figures in the table by  $(\text{range} \div 18,000)$ .

TABLE IX.—Fall at Lowest Point  $\times (18,000 \div \text{range})$ .

Range.	Aneroid No. 1.					Aneroid No. 4.				
	Time.					Time.				
	2	4	6	8	10	2	4	6	8	10
30 inches to 26	39	45	71	77	90	26	39	45	45	51
" " 24	24	54	54	78	102	6	30	36	42	60
" " 21	30	65	80	85	95	10	30	30	30	40
" " 18	34	53	68	83	98	11	15	23	30	41
" " 15	27	45	63	78	87	12	15	21	30	36
Means . .	31	52	67	80	94	13	26	31	35	46

TABLE X.—Same quantity as in Table IX.

Range.	Aneroid No. 2.					Aneroid No. 3.				
	Time.					Time.				
	2	4	6	8	10	2	4	6	8	10
30 inches to 26	32	39	58	64	71	32	45	64	71	77
" " 24	18	54	54	66	90	24	54	66	84	108
" " 21	20	55	60	65	80	30	60	65	80	95
Means . .	23	49	57	65	80	29	53	65	78	93

Considering the limited number of experiments, the irregularities in the tables are by no means surprising, when we reflect that even at 15 inches pressure the average fall of reading in 10 minutes was only about .07 of an inch for aneroid No. 1, and .03 of an inch for aneroid No. 4. In the case of the readings after 2 or 4 minutes' exposure to pressures such as 26 or 24 inches, the uncertainty is of course much greater. It will be noticed that the figures for aneroids Nos. 1 and 3 are very similar and are approximately double the figures for No. 4.

§ 17. Subsequently three experiments, Nos. 33, 34, and 35, were devoted to elucidating the law connecting the fall of reading with the time elapsed since the stationary pressure was reached. Pressure was reduced at the normal rate, 1 inch in 5 minutes, but kept for  $2\frac{1}{2}$  hours at the lowest point of the range, readings being taken at intervals. The lowest pressures in these three experiments were respectively 24, 21, and 18 inches. It would occupy too much space to give full details, but Tables XI. and XII. show the falls observed at 30-minute intervals. To avoid decimals the observed falls are multiplied by  $10^4/\text{range}$ ; so, to deduce the falls in inches, the figures in the table must be multiplied by  $n \times 10^{-4}$ , where  $n$  is the number of inches in the range.

TABLE XI.—Fall at Lowest Pressure  $\times (10^4/\text{range})$ .

Range.	Aneroid No. 1.					Aneroid No. 4.				
	Time, in minutes.					Time, in minutes.				
	30	60	90	120	150	30	60	90	120	150
30 inches to 24	83	133	150	155	172	50	78	83	83	100
" " 21	89	111	133	156	156	44	44	66	78	89
" " 18	92	108	142	158	175	50	67	83	83	92

TABLE XII.—Same quantity as in Table XI.

Range.	Aneroid No. 2.					Aneroid No. 3.				
	Time, in minutes.					Time, in minutes.				
	30	60	90	120	150	30	60	90	120	150
30 inches to 24	100	128	155	167	172	117	161	172	183	205
" " 21	111	122	144	156	178	111	133	156	178	189

In these tables each entry depends on only one experiment, so the agreement with the law, "fall proportional to range," could hardly be better.

In the interval between these experiments and those dealt with in Tables IX. and X., aneroid No. 1 had broken down and been altered. The figures obtained from the other three aneroids stand to one another in almost exactly the same proportion as before.

§ 18. To show more clearly that the law of variation of the fall of reading with the time was nearly if not exactly the same for all four aneroids, I have in the following Table XIII. multiplied the figures actually found for Nos. 1, 2, and 4 by  $6/5$ ,  $8/7$  and 2 respectively. The results are the means of the several ranges included in Tables XI. and XII., but the observations at all the time intervals are included.

TABLE XIII.—Quantity for Aneroid No. 3, same as in Table XII. ; for the others, multipliers are applied, as explained above.

Aneroid.	Time, in minutes.																	
	2	4	6	8	10	15	20	25	30	45	60	75	90	100	105	120	135	150
No. 1 . . .	7	34	63	86	83	101	89	96	106	128	140	158	170	173	174	187	196	202
" 2 . . .	10	31	51	70	71	91	102	109	120	134	143	155	170	183	184	185	191	200
" 3 . . .	9	33	47	69	75	89	98	103	114	133	147	156	164	173	175	180	189	197
" 4 . . .	16	40	46	58	58	78	94	100	96	124	126	142	154	160	162	162	192	198

After trying some logarithmic functions, I found that the results in Table XIII. were more in accordance with a formula of the type

$$\text{fall, under stationary pressure} = Ct^q \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where  $t$  denotes time elapsed since the pressure became stationary, while  $C$  and  $q$  are constants *for a given previous rate of fall*.  $C$  varies, of course, from aneroid to aneroid, but to all appearance there was at least a close approach to equality in the values of  $q$  for the above four aneroids. Accordingly, I took the mean of the results from all the aneroids in Table XIII., and determined  $q$  by trial. Table XIV. compares the mean thus found for the observed values with results calculated from the formula

$$\text{constant} \times \text{fall} = (31.3) t^{.369} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The corresponding mean results from the first 24 experiments, as given in Tables IX. and X., are added.



TABLE XIV.—Law of Fall of Reading under Constant Pressure.

	Time, in minutes.																	
	2	4	6	8	10	15	20	25	30	45	60	75	90	100	105	120	135	150
Calculated. . . .	40	52	61	67	73	85	95	103	110	127	142	154	165	171	174	183	191	199
Observed from—																		
Expts. 33 to 35 .	11	34	53	71	72	90	96	10.	109	130	139	153	164	172	174	179	192	199
„ 1 to 24 .	22	42	51	60	73													

After the first 6 minutes the agreement between the calculated and observed values could hardly be better. The observed values at 2 and 4 minutes are conspicuously lower than the calculated, but this discrepancy is least in the case of the 24 experiments, which, on account of their much greater number, should give a more reliable mean than the 3 special experiments. Taking either set of figures, however, one would most naturally draw the conclusion that the formula (8) does not apply exactly when  $t$  is small. Another explanation is, however, possible. On an average, an aneroid's reading at two successive intervals would be taken as unchanged if the true alteration were less than '005 of an inch, and this unquestionably was likely to happen frequently in the case of the readings at  $t = 0$  and  $t = 2$ . That the falls observed, during at least this interval, suffered from this or some similar cause, can hardly be doubted, for it is most improbable, as would appear from the 3 special experiments, that the fall in the first 2 minutes should be less than the fall in the second 2. Again, as all the operations, such as the reading of the aneroids and gauge, occupied some time, it is possible that the method of procedure adopted really tended to curtail the first 2-minute interval. It is certainly a little suggestive that the values calculated for  $t = 2, 4, 6$ , are so nearly identical with those observed in the first 24 experiments at  $t = 4, 6, 8$ , respectively.

§ 19. In connection with §§ 16 to 18 several considerations should be borne in mind. The change of pressure prior to the stationary stage is gradual. The circumstances are not analogous to that of a wire suddenly loaded. Thus, the fact that the fall of reading in a given time is proportional to the pressure range can hardly be regarded as merely a special case of the ordinary after-effect law. As we shall see very clearly later, the phenomenon is largely dependent on the way in which the pressure is reduced. A preliminary stoppage on the way down, or a change in the rate of reduction of pressure, each exert an important influence; the one affecting the magnitude of the fall, the other its law of variation with the time.

*Recovery after Pressure Cycle.*

§ 20. As already stated, it is customary for an aneroid to read lower on completion of a pressure cycle than at its commencement. Let the departure from the original reading be called the *deficiency*, and let  $D_t$  represent the deficiency when  $t$  minutes have elapsed after return to the original pressure. Thus  $D_0$ —which we may call the *original deficiency*—is the difference between the descending and ascending readings, answering to the pressure of 30 inches.

According to the first 24 experiments,  $D_0$  is, at least approximately, proportional to the pressure range. The evidence for this conclusion is summarised in Table XV.; the figures are deduced from the mean values calculated from the several experiments over the same range.

TABLE XV.—Value of  $D_0 \times (1800/\text{range})$  in inches.

Range.	Aneroid No.			
	1.	2.	3.	4.
30 inches to 26	15	13	16	13
" " 24	19	12	17	9
" " 21	19	15	29	11
" " 18	17	..	..	9
" " 15	16	..	..	10

The abnormal entry 29 for aneroid No. 3 is really due to the occurrence of appreciable permanent changes of zero in the course of 2 experiments over the range 30–21 inches. On one of these occasions, after a full day's rest, the aneroid read no less than .08 inch lower than before the experiment.

The irregularity in the other results is due partly to the comparative fewness of the experiments, and partly to the fact that the pressure at the end of the experiment was in reality to some extent different from that at the start. It was usual, in fact, to treat the atmospheric pressure at the time being as 30 inches, unless it exceeded this value by several tenths of an inch.

§ 21. In the 24 experiments the aneroids were read at intervals of 5, 10, 15, 20, 60, 120, and 1440 minutes after the return to atmospheric pressure. The similarity of the phenomena for the several aneroids and the several ranges is brought out more clearly by tabulating  $D_t/D_0$  than  $D_t$  itself. This is done in Table XVI. Initially, of course, *i.e.*, when  $t = 0$ ,  $D_t/D_0$  is unity.

TABLE XVI.—Recovery after Conclusion of Pressure Cycle, Values of  $D_t/D_0$ .

Range.	Aneroid No.	$t =$						
		5.	10.	15.	20.	60.	120.	1440.
30 inches to 26	1	·74	·65	·52	·39	·22	·13	·17
	2	·65	·65	·50	·40	·25	·20	·25
	3	·72	·64	·48	·44	·32	·16	·20
	4	·62	·57	·57	·52	·43	·52	·19
	Mean	·68	·63	·52	·44	·31	·25	·20
" " 24	1	·84	·76	·60	·56	·44	·48	·16
	2	·81	·69	·62	·44	·25	·31	·25
	3	·83	·70	·57	·48	·39	·39	·13
	4	·92	·75	·67	·50	·58	·75	·17
	Mean	·85	·73	·62	·50	·42	·48	·18
" " 21	1	·74	·68	·66	·61	·42	·32	·03
	2	·77	·77	·68	·58	·45	·35	·16
	3	·86	·81	·74	·69	·53	·48	·29
	4	·81	·71	·67	·57	·43	·48	·14
	Mean	·80	·74	·69	·61	·46	·41	·16
" " 18	1	·84	·73	·76	·67	·49	·40	·24
	4	·87	·74	·61	·57	·26	·22	—·09
	Mean	·86	·74	·69	·62	·38	·31	·08
" " 15	1	·79	·70	·66	·58	·32	·23	·04
	4	·82	·79	·70	·64	·39	·36	·33
	Mean	·81	·75	·68	·61	·36	·30	·19

In dealing with Table XVI. one must bear in mind the smallness of  $D_0$  in the shorter ranges, especially in the case of aneroid No. 4. The law of variation of a quantity of the size ·03 or ·04 of an inch is difficult to ascertain with great exactitude unless one can read to much less than ·01. The figures, as they stand, certainly suggest that in its early stage the recovery was distinctly more rapid in the case of the shortest range than in any other. Since, however, there is no trace of a like phenomenon in the next shortest range, its reality is at least doubtful, and I should be disposed to attribute it to experimental errors.

Over the other ranges the law of recovery is unquestionably almost, if not exactly, identical; and if, instead of grouping the results under pressure ranges, one had grouped them under the several aneroids, one would have found that the mean results were almost identical for the four. Thus, in deducing the law of variation, I combined the results from all the aneroids, utilising all the ranges except the shortest. The mean values thus obtained are compared in Table XVII. with values calculated from the formula

$$D_t/D_0 = (1 + t \times \cdot 1357)^{-\cdot 369} \dots \dots \dots (9),$$



where  $t$  represents as before the time elapsed in minutes since the return to atmospheric pressure.

TABLE XVII.—Law of Recovery, Observed and Calculated Values of  $D_t/D_0$ .

	$t =$							
	0.	5.	10.	15.	20.	60.	120.	1440.
$D_t/D_0$ observed . .	1	·83	·74	·67	·59	·41	·37	·15
„ calculated . .	1	·826	·729	·664	·616	·442	·349	·143

The agreement is certainly closer than I anticipated from the use of any formula with only two arbitrary constants. Of course I was led to try  $-.369$  in the index of (9) from having already found  $+.369$  in the index of (8); but this does not affect the significance of the fact that formulæ with the same numerical index should so closely reproduce the depression and recovery phenomena.

The expectation that (9) would apply to all aneroids treated as the 4 were, could, of course, only be justified by very wide experiment. As will be shown later, the law of recovery certainly depends on the rate and type of the pressure changes.

### *Effects of Temperature.*

§ 22. Aneroids are usually “corrected for temperature,” i.e., there is a compensating arrangement to prevent the reading altering when the instrument is exposed to varying temperature at ordinary atmospheric pressure. The experimental aneroids, tried as usual at three temperatures at atmospheric pressure, appeared correctly compensated. It would appear, however, from the special experiments, that this is not incompatible with imperfect compensation at lower pressures.

Temperature is likely to influence several parts of the mechanism, and that possibly in more than one way. It presumably alters the elasticity of the vacuum box and iron spring, and causes slight changes in the dimensions of the apparatus. The compensation works by producing slight curvature in a lever, but as the position of the lever varies with the pressure, a curvature that suffices at one pressure may be insufficient at another. Again, temperature might influence the after-effect phenomena, and so modify the reading in a variety of ways.

I thus investigated the influence of temperature on the readings with pressure descending, on the fall at the lowest pressure, and on the differences of the descending and ascending readings. The experiments bearing most directly on the question consisted of a group, Nos. 51 to 55, made on five consecutive days, May 10 to 14, 1897. No. 51 was preliminary, and was not utilised in the calculations. Of the

others, Nos. 52 and 55 were made at a temperature of 50° F., Nos. 53 and 54 at a temperature of 81° F. Use was made of the room employed for testing chronometers, the temperature of which is readily controlled. Information as to temperature effects is also obtainable from the other special experiments, because, being taken at all seasons of the year in an ordinary room, they answered to considerably varied temperatures.

The investigation of the influence of temperature on the descending readings is complicated by the fact that the index error of an aneroid is a somewhat variable quantity. To get rid, so far as possible, of this uncertainty, I subtracted from all the readings a constant equal to the error observed at 30 inches. This is equivalent to making the instrument correct at the start of each experiment. The excesses of the readings so modified over the true pressures at lower points of the range I shall call the *corrected errors*.

Table XVIII. shows the algebraic excess of the corrected errors from the mean of experiments 52 and 55 over the corrected errors from the mean of experiments 53 and 54.

TABLE XVIII.—Corrected Errors temperature 50° F., less Corrected Errors temperature 81° F.

Aneroid.	Pressure in inches.								
	29	28	27	26	25	24	23	22	21
No. 2 . .	·005	·00	·04	·05	·055	·095	·09	·11	·13
„ 3 . .	·01	·02	·055	·045	·065	·085	·085	·105	·12
„ 4 . .	·015	·02	·03	·045	·065	·095	·085	·09	·11
„ 8 . .	·02	·01	·035	·055	·055	·085	·075	·105	·115
Mean for 4 aneroids	·01	·01	·04	·05	·06	·09	·08	·10	·12
Mean corrected error 30 — pressure	·012	·006	·013	·012	·012	·015	·012	·013	·013

There is here a very decided fall of reading accompanying rise of temperature, and the fall appears to increase directly as the pressure interval measured from 30 inches.

§ 23. Similar conclusions follow from the experiments as a whole. For the two aneroids Nos. 2 and 3 there were 22 experiments over the range 30–21 inches at the normal rate. These I divided into two groups, the 11 colder in the one, the 11 hotter in the other, and found the mean corrected errors for the two groups separately. Aneroid No. 4, for some reason, changed its behaviour during a long rest between experiments Nos. 46 and 47. Prior to the change, however, there were 25 experiments in which the pressure was lowered to or below 21 inches. Twelve of these with temperatures from 77° to 71° formed the hotter group. The colder group

included the rest, but half-weight only was attached to each of two experiments in which the temperature was 70°. The treatment applied to these groups was the same as for aneroids Nos. 2 and 3. Aneroid No. 1 was somewhat erratic almost from the start, and the results from it are not reproduced: they showed clearly enough the same general phenomena as in the others. The results for aneroids Nos. 2, 3, and 4 are given in Table XIX. The temperatures quoted are the means for the colder and hotter groups.

TABLE XIX.—Corrected Errors, influence of Temperature.

Pressure in inches.	Aneroid No. 2.				Aneroid No. 3.				Aneroid No. 4.			
	56° F.	75° F.	Difference for 19° F.	Difference 30—pressure.	56° F.	75° F.	Difference for 19° F.	Difference 30—pressure.	61° F.	74° F.	Difference for 13° F.	Difference 30—pressure.
29	−005	−021	+016	+016	−012	−028	+016	+016	+040	+041	−001	−001
28	−019	−037	+018	+009	−029	−062	+033	+017	−020	−029	+009	+004
27	−049	−087	+038	+013	−029	−083	+054	+018	−037	−047	+010	+003
26	−026	−069	+043	+011	−047	−092	+045	+011	+004	−011	+015	+004
25	−045	−096	+051	+010	−059	−111	+052	+010	+022	−008	+030	+006
24	−053	−127	+074	+012	−089	−159	+070	+012	−030	−060	+030	+005
23	−077	−163	+086	+012	−130	−205	+075	+011	−004	−038	+034	+005
22	−136	−232	+096	+012	−186	−276	+090	+011	−053	−095	+042	+005
21	−198	−305	+107	+012	−214	−304	+090	+010	−074	−117	+043	+005

The experiments being numbered in chronological order, the arithmetic means of the experiment numbers in the several groups were, for aneroids Nos. 2 and 3, colder 38, hotter 37; for aneroid No. 4, colder 24, hotter 27. This alone would suffice to show that the phenomena cannot be ascribed to any gradual change in the aneroids.

§ 24. I next consider the possible influence of temperature on the fall of reading at the lowest pressure. Taking the 16 earliest experiments in which pressure was reduced at the normal rate to 21 inches, and then maintained steady for 10 minutes, I divided them into four groups, as follows:—

Group	I.,	Experiments	Nos.	6,	7,	18,	19.
„	II.,	„	„	29,	30,	31,	32.
„	III.,	„	„	34,	36,	39,	40.
„	IV.,	„	„	41,	42,	43,	44.

Temperature did not vary much between individual experiments of the same group.



Table XX. gives the mean temperature for each group, and the mean falls of reading at the intervals 2, 4, 6, 8, and 10 minutes after the lowest pressure, 21 inches, was reached. To avoid decimals, the unit chosen is the  $\frac{1}{400}$  of an inch.

TABLE XX.—Fall at Lowest Pressure (21 inches), unity =  $\cdot 0025$  inch.

Group.	Temperature, Fahrenheit.	Aneroid No. 1.					Aneroid No. 2.					Aneroid No. 3.					Aneroid No. 4.				
		Time.					Time.					Time.					Time.				
		2	4	6	8	10	2	4	6	8	10	2	4	6	8	10	2	4	6	8	10
I.	66 $\frac{1}{2}$	6	13	16	17	19	4	11	12	13	16	6	12	13	16	19	2	6	6	6	8
II.	50 $\frac{1}{2}$	3	7	11	18	18	4	7	10	16	17	4	9	10	15	18	0	2	4	6	6
III.	75	..	..	..	..	..	2	6	12	13	15	4	12	16	21	24	1	2	4	5	6
IV.	72 $\frac{1}{2}$	..	..	..	..	..	3	8	12	16	18	4	8	11	16	20	1	3	7	11	12

The results in the last two groups for aneroid No. 1 are omitted as not comparable—owing to repair of the instrument—with the earlier ones.

In Table XX. there is no certain indication of any temperature effect.

In the special temperature experiments, Nos. 52 to 55, the observed mean falls in the colder and hotter groups were as follows:—

TABLE XXI.—Fall in 10 minutes at Lowest Pressure (21 inches) in inches (Experiments 52 to 55).

Experiments.	Temperature.	Aneroid.			
		No. 2.	No. 3.	No. 4.	No. 8.
52 and 55	50° F.	$\cdot 035$	$\cdot 065$	$\cdot 03$	$\cdot 04$
53 and 54	81° F.	$\cdot 035$	$\cdot 05$	$\cdot 025$	$\cdot 025$

There being only two experiments in each group, it would be unsafe to base on this any more definite conclusion than that if there is any temperature effect it is certainly not large.

§ 25. The influence of temperature on the differences of the descending and ascending readings next calls for consideration. The experiments bearing most directly on this point are the special ones, Nos. 52 to 55. The results derived from them are given in Table XXII., unity representing 1 inch.

TABLE XXII.—Mean Sums of Differences Descending less Ascending Readings (from Experiments 52, 53, 54, 55).

Experiments.	Temperature.	Aneroid.				Sum for 4 aneroids.
		No. 2.	No. 3.	No. 4.	No. 8.	
52 and 55	50° F.	0·96	1·35	1·19	0·98	4·48
53 and 54	81° F.	0·98	1·54	1·08	0·88	4·48

Here there is certainly no suggestion of any temperature influence.

The great majority of the later experiments had some peculiarity, such as a stoppage, which prevents their use for investigating the influence of temperature on the sum of the differences. I thus confine my attention to the earliest 24 experiments, subdividing those over each range into a colder and a hotter half.\* The mean sums of the differences, in inches, are given in Table XXIII.

TABLE XXIII.—Mean Sums of Differences Descending less Ascending Readings (Experiments 1 to 24).

Aneroid.	Range 30 inches to 26.		Range 30 inches to 24.		Range 30 inches to 21.		Range 30 inches to 18.		Range 30 inches to 15.	
	Temperature.		Temperature.		Temperature.		Temperature.		Temperature.	
	57° F.	67° F.	66½° F.	72° F.	62½° F.	70½° F.	68½° F.	72° F.	69° F.	74½° F.
No. 1	·207	·217	·530	·625	1·375	1·550	2·750	2·890		
„ 2	·170	·177	·400	·480	1·230	1·225				
„ 3	·217	·210	·500	·740	1·745	1·825				
„ 4	·143	·143	·290	·315	·785	·755	1·555	1·620	2·920	3·175

Some of the data, more especially those for the range 30 to 24 inches, suggest an increase in the sum of the differences as temperature rises. In the range 30 to 26 inches, however, where the temperature difference was greatest, there is no trace of temperature effect. In the two longest ranges the temperature difference was small and the experiments very few.

In the four experiments dealt with in Table XXII. the deficiency in the reading

\* In the case of the range 30 to 26 inches the two groups consist of the three hotter and three colder experiments, the odd experiment being left out of account. The hotter were also the earlier experiments in this instance.

on return to atmospheric pressure was exceptionally small in the two high temperature experiments ; there was, however, no corresponding phenomenon in the case of the experiments dealt with in Table XXIII.

*Secular Change in Aneroids.*

§ 26. Variations in the reading of an aneroid at a standard pressure, say 30 inches, may arise from two causes—true secular change of zero, and elastic after-effect. In practice it is by no means easy to separate the two causes with absolute accuracy ; it is desirable, however, that the theoretical distinction should be clearly grasped. When a low pressure is maintained for a long time, the interval required for the after-effect to disappear on return to the standard pressure is, as Mr. WHYMPER has found, correspondingly great. This increases, of course, the difficulty of tracing the true secular change. In all but the latest experiments at Kew Observatory there was little trace of after-effect when 24 hours had elapsed since the return to atmospheric pressure. It is thus comparatively easy to trace the true secular change in the experimental aneroids, with at least a close approach to accuracy. This is done in Table XXIV. The quantity tabulated for each aneroid is the error or algebraic excess of its reading over the true pressure, as registered by the mercury gauge, at the commencement of experiments preceded by at least 40 hours' exposure to atmospheric pressure. Experiments during the same month were grouped together, and the mean of the errors found.

All the aneroids except No. 4 had a zero error to commence with.



TABLE XXIV.—True Secular Change of Zero, error in inches.

Date.	Aneroid.					
	No. 1.	No. 2.	No. 3.	No. 4.	No. 7.	No. 8.
1895.						
April . . . . .	—·09	—·05	—·23			
May . . . . .	—·05	+·13	—·23	+·00		
June . . . . .	—·02	+·16	—·39	+·05		
July . . . . .	—·02	+·16	—·54	+·09		
August . . . . .	—·11	+·17	—·54	+·10		
September . . . .	—·19	+·19	—·51	+·11		
October . . . . .	—·25	+·13	—·59	+·11		
November . . . .	—·01	..	..	+·07		
December . . . .	—·14	+·12	—·59	+·05		
1896.						
January . . . . .	—·34	+·10	—·62	+·04		
February . . . . .	—·39	+·09	—·61	+·06		
March . . . . .	—·34	+·09	—·62	+·06		
April . . . . .	—·29	+·12	—·59	+·08		
May . . . . .	—·42	+·13	—·57	+·08		
June . . . . .	—·56	+·15	—·56	+·10		
July . . . . .	—·67	+·16	—·57	+·13		
August . . . . .	—·71	+·14	—·58	+·14		
December . . . .	—·69	+·14	—·56	+·16		
1897.						
April . . . . .	..	..	..	+·09	—·05	—·04
May . . . . .	..	+·15	—·55	+·12	+·06	+·01
August . . . . .	..	..	..	+·15	+·07	+·03
October . . . . .	..	..	..	+·02	—·08	—·03
November . . . .	..	..	..	+·00	—·21	—·06
December . . . .	..	..	..	—·04	—·23	—·08
1898.						
March . . . . .	..	+·08	—·60	—·04	—·30	—·11

A horizontal line indicates the break down and repair of the instrument; readings above and below it are not comparable.

The changes were certainly in a good many instances discontinuous. Thus aneroid No. 1 experienced at least 9 sudden changes of reading of ·1 inch or more, 3 of these being rises and 6 falls. No. 2 had only one sudden change so large as this. It occurred in May, 1895, and amounted to +·18 inch. No. 3 had two sudden falls of appreciable magnitude, both in June, 1895. No. 7 had two sudden changes exceeding ·1 inch, one a rise, the other a fall. In the case of Nos. 4 and 8, except on the occasion of the former breaking down, there was no sudden change as large as ·1 inch. The constancy of zero in the former aneroid was especially notable, because it experienced more variations of pressure than any of the others. The cause of the

discontinuities is not known. In some instances they may have been brought about by a somewhat too vigorous tapping preparatory to taking a reading. Sometimes they occurred during exposure of the aneroids to atmospheric pressure. Unless a discontinuous change were of a size considerably larger than  $\cdot 01$  of an inch it would be difficult to demonstrate its existence. It is quite possible, in fact, that all the changes in the experimental aneroids were discontinuous. There is certainly no clear indication of a general tendency in the zeros of these aneroids to alter in a definite direction with age. The fact that discontinuous changes may not unlikely arise from time to time in the zero of an aneroid, shows the absolute necessity of periodic comparisons with a mercury barometer.

§ 27. Apart from shift of zero, it is conceivable that secular change might occur in at least two ways. There might exist something equivalent to gradual alteration of elastic moduli, whereby changes of reading answering to definite pressure changes would either increase or diminish. Quite independently, there might be change in the qualities whereon depend the elastic after-effect, manifesting itself in the alteration of such quantities as the sum of the differences of the descending and ascending readings in a given pressure cycle.

To examine into the first possibility I took note of the size of the corrected errors (see § 22) in the descending readings of a number of the experiments. The earliest 24 experiments were dealt with first. Aneroids Nos. 2 and 3 were used in 16 of these experiments. In the earlier 8 the pressure 24 inches was reached 5 times, the pressure 21 inches only twice; the later 8 experiments differed only in that the pressure 24 inches was reached no more than 4 times. Subtracting the mean corrected error at each inch of pressure in the earlier group from the corresponding mean error in the later group, we obtain the algebraical increase in the error (or rise in the reading) during the interval.

Aneroid No. 4 was exposed to the whole 24 experiments. In the earlier 12 the pressure 24 inches was reached 9 times, 21 inches 6 times, 18 inches 4 times, and 15 inches twice; the later 12 experiments differed only in that the pressure 24 inches was reached but 8 times. The mean corrected errors in the two groups of experiments were dealt with exactly like the corresponding errors in aneroids Nos. 2 and 3. Aneroid No. 1, being so much more erratic than the others, is not discussed. The data obtained from the other three aneroids are recorded in Table XXV.

TABLE XXV.—Corrected Errors from later less corrected Errors from earlier of first 24 Experiments.

Aneroid.	Pressure in inches.														
	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15
No. 2	-.001	-.018	-.011	-.010	-.028	-.074	-.095	-.085	-.07						
" 3	-.002	-.017	-.018	-.024	-.022	-.029	-.025	-.05	-.04						
" 4	-.011	-.021	-.020	-.014	-.010	-.012	-.011	-.01	-.008	-.02	-.01	-.012	+.02	+.03	+.035

The table shows pretty decisive evidence of a slight fall in the readings of Nos. 2 and 3 as time advanced. In their case the differences in the table increase on the whole fairly regularly as pressure falls, a phenomenon at least consistent with a diminution of true elastic moduli. With No. 4 the figures suggest a slight fall in reading down to the pressure of 18 inches, but a rise at lower pressures. Both conclusions, however, are very doubtful. In only 4 of the experiments was the pressure carried below 18 inches, and in the two earlier temperature averaged 5° F. higher than in the two later. As we have seen in §§ 22 and 23, the difference in temperature would tend to relatively depress the readings in the earlier experiments. Again, while the entries in the last line of the table are persistently negative between 29 and 18 inches, they show no tendency to increase numerically as pressure falls. In aneroid No. 4, as might be conjectured from Table XIX., the scale was not very regular near 30 inches, so that the apparent index error, answering nominally to 30 inches, but in reality to a pressure which might be slightly greater, or slightly less, was exposed to a fictitious variation, liable to influence the corrected errors at lower points of the range.

§ 28. To throw more light on the general question, I calculated the mean corrected errors shown by aneroids Nos. 2 and 3 in the years 1895, 1896, and 1897 separately, confining my attention to the part of the scale most used, and including only those experiments in which pressure was lowered at the normal rate. Particulars are given in Table XXVI.

TABLE XXVI.—Corrected Errors, Yearly Means.

Year.	Number of Experiments.	Mean Temperature.	Aneroid No. 2.				Aneroid No. 3.			
			29 inches.	28 inches.	27 inches.	26 inches.	29 inches.	28 inches.	27 inches.	26 inches.
1895	18	64° F.	-.008	-.022	-.060	-.038	-.012	-.031	-.037	-.045
1896	13	68° F.	-.016	-.031	-.072	-.049	-.020	-.048	-.063	-.074
1897	7	64° F.	-.009	-.027	-.067	-.054	-.020	-.044	-.051	-.074



In accordance with § 22 a trifling + correction ought to be applied to the figures for 1896 to allow for its 4° excess of temperature over 1895 and 1897. On the whole, there appears to be satisfactory evidence of a slight secular lowering of reading in the two aneroids, such as would accompany alteration of a true elastic modulus; but it certainly did not increase to any sensible extent after the end of the first year.

In pushing the inquiry further in the case of aneroid No. 4, I took the 25 experiments already dealt with in Table XIX., but divided them into an earlier and a later half, treating experiments Nos. 16 and 17 as but one. The results appear in Table XXVII.

TABLE XXVII.—Mean Corrected Errors in Aneroid No. 4.

	Temperature.	Pressure in inches.								
		29.	28.	27.	26.	25.	24.	23.	22.	21.
Earlier half . .	68° F.	+·042	—·023	—·043	—·006	+·001	—·052	—·031	—·079	—·103
Later half . .	67° F.	+·039	—·025	—·041	—·001	+·012	—·038	—·012	—·068	—·088
Later less earlier	..	—·003	—·002	+·002	+·005	+·011	+·014	+·019	+·009	+·015

At the higher pressures the apparent difference between the earlier and later series is microscopic; at the lower pressures the figures would point to a secular *rise* in the corrected errors; but, as there is no visible tendency in this to increase at the very lowest pressures, its true character is somewhat doubtful. Probably the only safe conclusion to be drawn is that if any real change occurred in aneroid No. 4 it certainly was very small.

§ 29. For investigating possible secular change in the differences of the descending and ascending readings, the data are somewhat limited, because few of the later experiments were altogether of the normal type. The most suitable data are utilised in Table XXVIII., which gives the sums of the differences of the descending and ascending readings in experiments of the normal type separated by appreciable intervals of time. When two experiments are combined, the mean sum is taken to the nearest ·01 of an inch.

TABLE XXVIII.—Sums of Differences, Descending less Ascending Readings.

Range 30 inches to 26.					Range 30 inches to 24.					Range 30 inches to 21.					Range 30 inches to 18.		
Experiments.	Aneroid No.				Experiments.	Aneroid No.				Experiments.	Aneroid No.				Experiments.	Aneroid No.	
	1.	2.	3.	4.		1.	2.	3.	4.		1.	2.	3.	4.		1.	4.
1, 2	·22	·14	..	·13	3, 4	·63	·47	·68	·33	6, 7	1·48	1·18	1·83	·74	8, 9	2·85	1·52
12, 13	·21	·18	·24	·14	20, 21	·61	·44	·63	·30	18, 19	1·45	1·28	1·76	·81	16, 17	2·80	1·65
23, 24	·21	·16	·21	·14						36, 39	..	1·15	1·67	·72			
										56	..	1·09	1·67	·73			

The blanks are due to some uncertainty in the readings of No. 3 on one occasion, and to the later experiments with No. 1 not being comparable with the earlier. Between experiments No. 6 and No. 36 there intervened fully a year, and between No. 36 and No. 56 nearly a second year.

The table does not, I think, afford any clear proof of a change in the sum of the differences. If any change occurred, the evidence is in favour of its being a decrease.

In all the experiments utilised in Table XXVIII., the aneroids had been exposed to atmospheric pressure for at least two days previously. This introduces a difference between these experiments and five of the ordinary type, Nos. 52, 53, 54, 55, and 59, which were preceded by experiments over the same range, 30–21 inches, on the previous day. The difference in this respect is probably the chief reason for the relative smallness of the mean sums of the differences of the descending and ascending readings given by these five experiments, viz., ·96 for No. 2, 1·43 for No. 3 and ·66 for No. 4.

#### *Influence of Rate of Change of Pressure.*

§ 30. In ordinary mountain ascents the rate of change of pressure is much slower than 1 inch in 5 minutes; it is thus important to know how the various phenomena are modified when the rate of change of pressure is reduced. A variety of experiments were devoted to this investigation. In eight of these, Nos. 25, 26, 27, 28, 37, 38, 57, and 58, pressure was altered at the rate of 1 inch in 10 minutes, or half the normal rate, with a stoppage of the normal length, 10 minutes, at the lowest point. In experiments 27 and 28 the pressure range was 30–15 inches, in the other six instances it was 30–21 inches.

The two pairs of experiments, 37, 38 and 57, 58, were each preceded and followed by an experiment of the normal type over the same range. The experiments 36, 37, 38, 39 were taken on consecutive days, July 6–9, 1896; the experiments 56, 57,

58, 59 were taken on May 19, 21, 25, and 26, 1897. During either of these two groups of experiments the condition of the aneroids may reasonably be regarded as constant, except in so far as earlier experiments of a group may have affected the later. The temperature was practically the same for all the four experiments of either group; so that the corresponding observations are peculiarly well fitted for investigating the influence of the rate of reduction of pressure on the descending readings. The general nature of the phenomena will be found, I think, sufficiently illustrated by Tables XXIX. and XXX., which give the corrected errors at the lowest point of the range.

TABLE XXIX.—Corrected Error at Lowest Point.

Experi- ment.	Tempera- ture, Fahren- heit.	Aneroid No. 1.		Aneroid No. 2.		Aneroid No. 3.		Aneroid No. 4.	
		Time to fall an inch.		Time to fall an inch.		Time to fall an inch.		Time to fall an inch.	
		5 mins.	10 mins.	5 mins.	10 mins.	5 mins.	10 mins.	5 mins.	10 mins.
36	73	—·23	..	—·29	..	—·30	..	—·11	..
37	74	..	—·27	..	—·32	..	—·32	..	—·16
38	75	..	—·19	..	—·32	..	—·30	..	—·16
39	76	—·22	..	—·34	..	—·29	..	—·14	..
Means . . .		—·225	—·23	—·315	—·32	—·295	—·31	—·125	—·16

TABLE XXX.—Corrected Error at Lowest Point.

Experiment.	Temperature, Fahrenheit.	Aneroid No. 2.		Aneroid No. 3.		Aneroid No. 4.		Aneroid No. 7.		Aneroid No. 8.	
		Time to fall an inch.		Time to fall an inch.		Time to fall an inch.		Time to fall an inch.		Time to fall an inch.	
		5 mins.	10 mins.	5 mins.	10 mins.	5 mins.	10 mins.	5 mins.	10 mins.	5 mins.	10 mins.
56	66	—·27	..	—·30	..	+·06	..	—·07	..	—·03	..
57	66	..	—·29	..	—·31	..	+·04	..	—·05	..	—·04
58	64	..	—·27	..	—·28	..	+·05	..	—·06	..	—·02
59	63	—·22	..	—·20	..	+·07	..	—·03	..	—·00	..
Means . .		—·245	—·28	—·25	—·295	+·065	+·045	—·05	—·055	—·015	—·03

In both tables the mean corrected error is more negative, or the aneroid reads lower, when the rate of change of pressure is reduced. The difference between the



two rates appears larger in Table XXX. than in Table XXIX.; this is mainly due, however, to experiment No. 59, and very probably arises from the fact that the time interval between this and the previous experiment was shorter than the average.

Taking the figures as they stand, we should conclude that the mean reading at 21 inches was lowered by only  $\cdot 02$  of an inch when the time of reducing the pressure was increased from 45 to 90 minutes. At first sight, the smallness of this difference seems truly remarkable, because after lowering the pressure to 21 inches at the normal rate we should, in accordance with Tables IX. and X., have observed a mean fall of  $\cdot 02$  of an inch in some 4 minutes.

The influence of a still slower rate of pressure was tried in a very recent group of experiments, Nos. 71, 72, 73, 74, and 75, carried out on March 16, 18, 22-23, 29-30, and April 1, 1898. Of these, Nos. 71, 72, and 75 may be regarded as of the normal type, except that the stoppage at the lowest pressure, 21 inches, lasted 2 hours. Nos. 73 and 74 differed from these only in that all the pressure intervals were nine times longer. These last experiments lasted of course nearly two days, the length of stoppage at the lowest point being chosen so as to make this feasible without night work. In reality, it was found convenient to reduce the stoppage in experiment No. 74 by 45 minutes, and the corresponding curtailment, 5 minutes, was made in the case of No. 75. I have disregarded this trifling difference, treating Nos. 71, 72, and 75 as identical, and Nos. 73 and 74 as identical. From these experiments I have calculated, and give in Table XXXI., the differences between the mean corrected errors throughout the range for the slower rate (1 inch in 45 minutes) and the faster rate (1 inch in 5 minutes). The results are given to the nearest  $\cdot 01$  inch.

TABLE XXXI.—Mean Corrected Error Slower Rate, less Mean Corrected Error Faster Rate.

Aneroid.	Pressure in inches.								
	29.	28.	27.	26.	25.	24.	23.	22.	21.
No. 2 . .	+ $\cdot 01$	$\cdot 00$	— $\cdot 01$	— $\cdot 02$	— $\cdot 02$	— $\cdot 03$	— $\cdot 04$	— $\cdot 04$	— $\cdot 06$
„ 3 . .	+ $\cdot 01$	— $\cdot 01$	— $\cdot 02$	— $\cdot 03$	— $\cdot 02$	— $\cdot 04$	— $\cdot 05$	— $\cdot 07$	— $\cdot 08$
„ 4 . .	+ $\cdot 02$	+ $\cdot 01$	+ $\cdot 01$	$\cdot 00$	+ $\cdot 01$	$\cdot 00$	— $\cdot 01$	— $\cdot 01$	— $\cdot 02$
„ 7 . .	$\cdot 00$	— $\cdot 01$	$\cdot 00$	+ $\cdot 01$	— $\cdot 01$	— $\cdot 01$	— $\cdot 03$	— $\cdot 04$	— $\cdot 04$
„ 8 . .	$\cdot 00$	— $\cdot 01$	$\cdot 00$	— $\cdot 01$	— $\cdot 01$	— $\cdot 02$	— $\cdot 03$	— $\cdot 03$	— $\cdot 04$
Means .	+ $\cdot 01$	$\cdot 00$	$\cdot 00$	— $\cdot 01$	— $\cdot 01$	— $\cdot 02$	— $\cdot 03$	— $\cdot 04$	— $\cdot 05$

Here, as in Tables XXIX. and XXX., there is, at least at the lower parts of the range, a lowering of the reading accompanying reduction in the rate of fall of pressure. Also the mean lowering of the reading at the lowest point,  $\cdot 05$  of an inch, is notably

greater than in the two previous tables. As temperature was about  $4^{\circ}$  lower in the experiments at the slower rate, its influence must have tended if anything to reduce the apparent difference.

As we should have anticipated *a priori*, the differences in Table XXXI. are greatest in the case of the two aneroids Nos. 2 and 3, which showed the largest after-effect phenomena.

Taking both the Tables XIX. and XXXI. into consideration, we see that the fall of temperature naturally experienced in mountain ascents, and the reduction in the rate of change of pressure relative to that of the ordinary test, must occasionally neutralise one another's effects; but the consequences, however satisfactory, can hardly, I think, be justly ascribed to design. Though the differences exhibited in Table XXXI. are certainly quite appreciable, they are very much less than would naturally be anticipated from experiments in which pressure is reduced rapidly and then maintained constant for a lengthened period.

§ 31. The influence of the previous rate of reduction of the pressure on the fall of reading at the lowest point is hardly capable of being satisfactorily ascertained from the earlier groups of experiments, because with a stoppage of only 10 minutes we have to do with changes of reading so small relative to the probable error of observation that a large number of experiments are required to give results of much value. In the last group of experiments, Nos. 71 to 75, however, the stoppages were sufficiently long to produce considerable falls of reading. To avoid decimals I have applied a common large multiplier, and give the results in Table XXXII.

TABLE XXXII.—Fall of Reading at Lowest Pressure (21 inches).

Aneroid.	Rate 1 inch in 5 minutes.								Rate 1 inch in 45 minutes.			
	Time in minutes.								Time in minutes.			
	2.	5.	10.	15.	20.	30.	60.	120.	18.	45.	90.	1080.
No. 2 . . .	6	12	24	30	30	44	54	68	9	12	18	69
„ 3 . . .	6	16	28	34	42	48	68	86	6	6	12	75
„ 4 . . .	0	4	8	12	15	24	32	36	3	3	6	33
„ 7 . . .	2	8	14	18	18	28	36	40	6	6	12	39
„ 8 . . .	0	0	4	10	12	20	32	36	6	6	12	42
Sums . .	14	40	78	104	117	164	222	266	30	33	60	258

The agreement between the falls in 120 minutes in the one set of experiments and in 1080 minutes ( $120 \times 9$ ) in the other is remarkably close. A comparison of the falls at the three shorter intervals, 2, 5, and 10 minutes in the one case, and 18, 45,

90 minutes in the other, though naturally showing conspicuous irregularities, points to the same general conclusion. It is clear that the *creep*, or fall of reading, at the lowest point is largely dependent on the rate of the previous change of pressure. A slow rate of change affords opportunity for simultaneous creep, but at the same time it reduces the tendency to creep at lower pressures.

If we may judge from Table XXXII., supposing the time taken to produce a given fall of pressure to be increased in a given proportion, then the time required for the subsequent creep to attain a certain value is likewise increased in the same or nearly the same proportion. If the proportion were *exactly* the same in the two cases one would hardly, I think, expect to meet with the differences in the corrected errors shown by Tables XXIX., XXX., and XXXI.

§ 32. In treating of the influence of the rate of change of pressure on the differences between the descending and ascending readings, I shall consider first the final group of experiments Nos. 71 to 75. The following Table XXXIII. shows the ratio of the mean difference of the descending and ascending readings at each point of the range from the two experiments at the slower rate (1 inch in 45 minutes) to the corresponding mean from the three experiments at the faster rate (1 inch in 5 minutes).

TABLE XXXIII.—Ratios Differences Descending less Ascending Readings,  
slower : faster.

Aneroid.	Pressure in inches.									
	21.	22.	23.	24.	25.	26.	27.	28.	29.	30.
No. 2 .	1·02	1·05	1·03	1·08	1·14	1·08	1·05	1·08	1·19	·94
„ 3 .	·87	1·01	1·04	·97	1·11	·98	1·00	·99	1·04	·87
„ 4 .	·92	·99	·99	1·03	1·14	·99	1·07	1·11	1·24	·81
„ 7 .	·98	·90	1·01	1·15	1·09	1·18	1·03	·94	1·11	1·33
„ 8 .	1·17	·92	·94	·96	1·08	1·10	1·14	1·07	1·06	·98
Means . .	·99	·97	1·00	1·04	1·11	1·07	1·06	1·04	1·13	·99

The mean of the means is 1·04.

We can hardly avoid drawing the conclusion that, provided all the time intervals in the cycle be altered in the same proportion, the sum of the differences between the descending and ascending readings, and the law of variation of the differences throughout the range, are either absolutely unaffected or very nearly so. The departure of the mean ratio in Table XXXIII. from unity is mainly due to experiment No. 72. The interval allowed for recovery after the immediately preceding experiment was possibly insufficient. The agreement between experiments 71 and 73 and between experiments 74 and 75 could hardly have been improved. In the



latter pair, for instance, the sums of the differences of the descending and ascending readings were as follows :—

	Aneroid.				
	No. 2.	No. 3.	No. 4.	No. 7.	No. 8.
Experiment No. 74 (rate 1 inch in 45 minutes)	1·36	1·93	·84	1·14	1·15
"      "  75      "      "  5      "	1·42	1·88	·85	1·12	1·24

In devising the earlier groups of experiments, for comparing the two rates 1 inch in 10 and 1 inch in 5 minutes, I did not at the time foresee the laws of the phenomena, and so allowed equal lengths to the stoppages at the lowest point in the two cases. The results are notwithstanding of considerable interest. Taking the experiments over the range 30–21 inches, in order to allow for alterations in the aneroids, I grouped them as follows :—

Group.	1 inch in 5 minutes.	1 inch in 10 minutes.
I.	Experiments Nos. 6, 7, 18, 19	25, 26
II.	"      "      36, 39	37, 38
III.	"      "      56, 59	57, 58

Taking Group III. for example, the differences of the descending and ascending readings at each inch were summed for the aneroids Nos. 2, 3, 4, 7, 8 in experiments 57, 58, and the ratios taken to the corresponding sums in experiments 56, 59. The preponderance thus allowed to aneroids Nos. 2 and 3, in which the after-effect was largest, is of little consequence, as all the aneroids showed the same general phenomena. The results appear in Table XXXIV.

TABLE XXXIV.—Ratios Differences Descending less Ascending Readings  
(1 inch in 10 : 1 inch in 5 minutes).

Group.	Pressure in inches.										Mean for group.
	21.	22.	23.	24.	25.	26.	27.	28.	29.	30.	
I.	·94	·84	·86	·90	·84	·87	·90	·91	·84	·81	·87
II.	·62	·79	·87	·90	·83	·84	·88	1·04	·77	·75	·83
III.	1·00	·91	·81	·96	·98	·98	1·07	·86	1·01	·97	·95
Means	·85	·85	·85	·92	·88	·90	·95	·94	·87	·84	·88

The ratios would seem to be somewhat above their average near the centre of the range, and below it at both extremities; but there are hardly sufficient experiments to justify any too positive conclusion. That the mean ratio is less than unity is absolutely clear; and there can be but little doubt that this is due to the fact that the stoppage at the lowest point in the case of the slower experiments was not doubled along with the other time intervals.

I likewise compared experiments Nos. 27 and 28 over the longest range, 30–15 inches, at the slower rate, 1 inch in 10 minutes, with earlier experiments, Nos. 10, 11, 14, 15, over the same range at the normal rate. Only one aneroid, No. 4, was available for calculation, owing to erratic behaviour in No. 1. It will thus suffice to mention that the results obtained were analogous to those in Table XXXIV., and that for the ratio of the sums of the differences of the descending and ascending readings at the two different rates I found

$$\frac{\text{case 1 inch in 10 minutes}}{\text{case 1 inch in 5 minutes}} = \cdot 93.$$

In both cases, as already stated, only 10 minutes stoppage was allowed at the lowest pressure.

§ 33. The influence of the rate of change of pressure on the rate of recovery at the end of a pressure cycle was less exactly ascertained. After experiments in which the pressure changes are slow, the variations observed in the reading during the first hour or two of the recovery are so small as to be largely affected by small errors of reading. After 12 or 24 hours recovery, readings are affected by various sources of uncertainty, *e.g.*, change of temperature or atmospheric pressure, and possible permanent shift of zero.

Table XXXV. compares the recovery following pressure cycles at the two rates 1 inch in 10 and 1 inch in 5 minutes.

The data for the normal rate are taken from Table XVII., and  $D_t/D_0$  has the same meaning as there. The data for the slower rate were obtained from the experiments 25, 26, 37, 38 over the range 30–21 inches.

TABLE XXXV.—Recovery after Pressure Cycle, Values of  $D_t/D_0$ .

Rate in cycle.	Interval in minutes.					
	0.	5.	10.	15.	20.	60.
1 inch in 5 minutes	1	·83	·74	·67	·59	·41
„ 10 „	1	·87	·83	·72	·74*	·49

\* If No. 4 omitted, only ·69. For some reason, reading recorded for No. 4 showed on two occasions apparent *fall* of ·01 inch in interval 15 to 20 minutes.

The number of experiments at the slower rate was insufficient to give very smooth results, and it would be waste of time to fit a formula to them. We should not be justified in regarding the coincidence between the data under 10 and 20 minutes in the second line with those under 5 and 10 minutes in the first as more than accidental. At the same time there seems conclusive evidence that the rate of recovery becomes slower when the rate of the previous pressure changes is reduced.

This is further illustrated in Table XXXVI., which compares the recovery in experiments Nos. 73 and 74 with that exhibited in experiments Nos. 71, 72, and 75.

TABLE XXXVI.—Recovery after Pressure Cycle, values of  $D_t/D_0$ .

Aneroid.	Pressure rate 1 inch in 5 minutes (stoppage at lowest point 2 hours).							Pressure rate 1 inch in 45 minutes (stoppage at lowest point 18 hours).					
	Time in minutes.							Time in minutes.					
	0.	5.	10.	15.	20.	120.	210.	0.	15.	30.	45.	1020.	2460.
No 2 . . .	1	.89	.85	.78	.70	.29	.20	1	.94	.88	.83	.65	.47
" 3 . . .	1	.91	.84	.81	.74	.38	.27	1	.96	.88	.80	.60	.44
" 4 . . .	1	1.00	.92	.85	.85	.67	.20	1	1.14	1.14	1.00	.14	.00
" 7 . . .	1	.89	.83	.89	.87	.67	.13	1	.87	.87	.86	.50	.38
" 8 . . .	1	.80	.65	.65	.60	.50	-.06	1	.85	.77	.80	.38	.31
Means .	1	.90	.82	.80	.75	.50	.15	1	.95	.91	.86	.45	.32

The uncertainties referred to above as unavoidable in a small number of experiments of this kind are conspicuous in Table XXXVI. ; but as to the recovery being very much slower in the experiments with the slow pressure change there is no room for doubt. That it is exactly or almost exactly nine times as slow in the one case as in the other is a conclusion for which the evidence is insufficient ; but there is at least nothing to demonstrate the contrary. The observations in the case of the experiments at the slower rate would naturally have been taken at the intervals 45, 90, 135, &c., minutes after return to atmospheric pressure, but daylight did not last long enough to admit of this. The extreme slowness of recovery after experiments such as Nos. 73 and 74 is a somewhat serious obstacle to research.

#### *Effects of Stoppage.*

§ 34. The influence of a stoppage on the readings at lower pressures is shown by three groups, each of three experiments, in all of which pressure was reduced at the rate of 1 inch in 5 minutes—in most cases with a subsidiary stoppage—to 15 inches, and maintained at that point for an hour.



In the first two groups there was in two out of the three experiments a subsidiary stoppage of 1 hour at a pressure above the lowest. In the third group there was a subsidiary stoppage of about 26 hours' duration in all three experiments. As the order of the experiments is of significance, I give particulars :—

Experi- ment.	Date.	Stoppage at—	Experi- ment.	Date.	Stoppage at—	Experi- ment.	Date.	Stoppage at—
		inches.			inches.			inches.
62	1897. Oct. 11	18 and 15	65	1897. Nov. 15	21 and 15	68	1897. Dec. 21	16 and 15
63	„ 13	15	66	„ 17	15	69	1898. Jan. 5	18 and 15
64	„ 15	21 and 15	67	„ 19	18 and 15	70	„ 20	21 and 15

As experiments Nos. 63 and 66 were of the normal type, so far as descending readings are concerned, I have used them as a standard to which to refer the others. To see, for instance, how an hour's stoppage at 18 inches influences the reading when the pressure 15 inches is reached, I compare the mean error at 15 less the mean error at 18, prior to the stoppage, in experiments 62 and 67, with the mean difference between the errors at 15 and 18 in experiments 63 and 66. For instance, in aneroid No. 4, I found :—

from experiments 62 and 67, mean error (before stoppage) at 18 = + '01,  
 „ „ 62 and 67, „ „ „ at 15 = + '015,  
 and so error at 15 less error at 18 = + '005 ;

from experiments 63 and 66, mean error „ „ at 18 = + '01,  
 „ „ 63 and 66, „ „ at 15 = + '04,  
 and so error at 15 less error at 18 = + '03.

Thus an hour's stoppage at 18 has lowered the reading at 15 by

$$.03 - .005, \text{ or } .025 \text{ inch.}$$

Again, to determine the influence of the above stoppage on the total creep which takes place as pressure is being lowered from 18 to 15, I compare the mean error at 15 less the mean error at 18, *subsequent* to the stoppage, in experiments 62 and 67 with the difference between the mean errors at 15 and 18 in experiments 63 and 66. For instance, in aneroid No. 4,

from experiments 62 and 67, mean error (after stoppage) at 18 = - '06,  
 „ „ 62 and 67, „ „ „ at 15 = + '015,  
 and so error at 15 less error at 18 = + '075 ;

and this is greater than in experiments 63 and 66 by

$$\cdot 075 - \cdot 03, \text{ or } \cdot 045 \text{ inch.}$$

Thus an hour's stoppage at 18 diminished the total creep between 18 and 15 by  $\cdot 045$  inches.

These examples should suffice to explain Table XXXVII.

TABLE XXXVII.

Stoppage.		Consequent lowering of reading, in inches, at 15 inches.				Consequent diminution of total creep, in inches, between stoppage and 15 inches.			
At	For	No. 4.	No. 7.	No. 8.	Mean.	No. 4.	No. 7.	No. 8.	Mean.
inches.	hours.								
21	1	$\cdot 015$	$\cdot 025$	$\cdot 005$	$\cdot 02$	$\cdot 04$	$\cdot 05$	$\cdot 05$	$\cdot 05$
18	1	$\cdot 025$	$\cdot 015$	$\cdot 05$	$\cdot 03$	$\cdot 045$	$\cdot 06$	$\cdot 04$	$\cdot 05$
21	26	$\cdot 04$	$\cdot 04$	$\cdot 07$	$\cdot 05$	$\cdot 08$	$\cdot 10$	$\cdot 06$	$\cdot 08$
18	26	$\cdot 12$	$\cdot 135$	$\cdot 14$	$\cdot 13$	$\cdot 06$	$\cdot 065$	$\cdot 06$	$\cdot 06$
16	26	$\cdot 185$	$\cdot 185$	$\cdot 19$	$\cdot 19$	$\cdot 035$	$\cdot 045$	$\cdot 04$	$\cdot 04$

§ 35. In arguing from a limited number of experiments allowance must be made for accidental coincidences. The general conclusions to be drawn from Table XXXVII. seem however pretty obvious.\* A stoppage has two effects, tending somewhat to neutralise one another. It lowers the reading, but while doing so it diminishes the tendency to further creep as pressure is further lowered. For instance, stoppage for 26 hours at 18 inches lowered the reading at 18 inches in aneroid No. 4 by  $\cdot 18$  inch, but lessened the subsequent creep between 18 and 15 inches by  $\cdot 06$  inch. Consequently on arrival at 15 inches the aneroid read only  $\cdot 12$  inch lower than if there had been no stoppage at 18 inches. When the stoppage occurred at 21 inches, and lasted only an hour, the diminution in the creep between 21 and 15 inches nearly obliterated all influence on the reading at 15 inches. With increased duration of stoppage the direct lowering of reading tended more distinctly to predominate.

The lower the pressure at which stoppage occurs, the greater influence does it exert in both directions. To realise this one should notice that stoppage at 21 inches influenced the creep over the 6 inches 21 to 15, whereas stoppage at 16 inches influenced the creep over a single inch only.

§ 36. The way in which the influence of a stoppage tends to disappear the further the pressure is lowered below the stationary point, is brought out most clearly by a consideration of the phenomena observed after the lowest pressure is attained.

In treating of this I have utilised, in addition to experiments Nos. 62 to 70, two

\* At the same time, so far as concerns phenomena occurring during actual change of pressure, it would be difficult to distinguish between a temporary rise in true elastic moduli and a diminished tendency to creep.

earlier experiments, Nos. 60 and 61, made respectively on August 17 and 20, 1897. Of these 60 agreed in type with 63 and 66, while 61 was identical with 62 and 67. Taking the means of the observed falls in reading during stated intervals of exposure to 15 inches pressure in experiments 60, 63 and 66 as standards, I have found the ratios borne to these by the corresponding mean falls in the other groups of experiments during both the subsidiary stoppage and the stoppage at 15 inches. The results appear in Table XXXVIII. The other data in the table are of the following character :—

“Mean ratio” signifies the mean deduced from the values at the seven intervals 5, 10...60 minutes. “Theoretical ratio” is what the value should have been according to the law that the creep in a given interval of time is proportional to the pressure interval measured from 30 inches. For instance, in experiments 64 and 65 the reading during the subsidiary stoppage at 21 inches should according to the law have fallen  $(30-21) \div (30-15)$ , or  $\cdot 60$ , times as much in any given interval as it did in the same interval during exposure to the pressure of 15 inches in experiments 60, 63 and 66. In reality the fall in the former case was on the average only  $\cdot 58$  of that in the latter. Accordingly the presumption is that the tendency to creep in experiments 64 and 65 was less than in the three standard experiments in the ratio 58 : 60. This result is utilised in calculating the true influence of the stoppage at 21 inches on the fall of reading during the subsequent stoppage at 15 inches. According to the actual observation figures the mean ratio of the falls at 15 inches in experiments 64 and 65 to the falls in the standard experiments was  $\cdot 93$ , but on the assumption that this suffered from the same cause as the creep at 21 inches, it should be multiplied by  $60/58$ , in order to eliminate the difference between the aneroids’ mean state during the two groups of experiments. The result of this multiplication is termed the “Corrected ratio.”

TABLE XXXVIII.—Ratios to Falls of Reading (in same intervals) at 15 inches in Standard Normal Experiments.

Experiments.	Fall at—  inches.	Time in minutes.							Mean ratio.	Theoretical ratio.	Corrected ratio.
		5.	10.	15.	20.	30.	45.	60.			
64, 65	21	$\cdot 57$	$\cdot 79$	$\cdot 55$	$\cdot 47$	$\cdot 55$	$\cdot 55$	$\cdot 60$	$\cdot 58$	$\cdot 60$	..
64, 65	15	$1\cdot 07$	$\cdot 96$	$\cdot 88$	$\cdot 97$	$\cdot 87$	$\cdot 85$	$\cdot 91$	$\cdot 93$	..	$\cdot 96$
61, 62, 67	18	$\cdot 62$	$\cdot 71$	$\cdot 65$	$\cdot 76$	$\cdot 70$	$\cdot 76$	$\cdot 77$	$\cdot 71$	$\cdot 80$	..
61, 62, 67	15	$\cdot 71$	$\cdot 85$	$\cdot 77$	$\cdot 85$	$\cdot 81$	$\cdot 79$	$\cdot 83$	$\cdot 80$	..	$\cdot 90$
68	16	not observed									
68	15	$\cdot 29$	$\cdot 00$	$\cdot 20$	$\cdot 17$	$\cdot 08$	$\cdot 07$	$\cdot 10$	$\cdot 13$	..	..
69	18	$\cdot 86$	$\cdot 75$	$\cdot 85$	$\cdot 78$	$\cdot 76$	$\cdot 71$	$\cdot 65$	$\cdot 77$	$\cdot 80$	..
69	15	$\cdot 43$	$\cdot 33$	$\cdot 33$	$\cdot 33$	$\cdot 42$	$\cdot 39$	$\cdot 39$	$\cdot 37$	..	$\cdot 38$
70	21	$\cdot 57$	$\cdot 50$	$\cdot 65$	$\cdot 66$	$\cdot 55$	$\cdot 60$	$\cdot 65$	$\cdot 60$	$\cdot 60$	..
70	15	$\cdot 71$	$\cdot 66$	$\cdot 85$	$\cdot 78$	$\cdot 72$	$\cdot 71$	$\cdot 72$	$\cdot 74$	..	$\cdot 74$



We should conclude from the table that an hour's stoppage at 21 inches lowered the creep during the subsequent stoppage at 15 inches by only 4 per cent. An hour's stoppage at 18 inches was distinctly more effective, but still not very serious. Stoppage for 26 hours had a much larger influence, reducing the creep at 15 inches from 100 to 74 when it occurred at 21 inches, from 100 to 38 when it occurred at 18 inches, and from 100 to 13 when it occurred at 16 inches.

According to the table :

$$\frac{\text{Influence of an hour's stoppage at 21}}{\text{Influence of an hour's stoppage at 18}} = \frac{4}{10} = \cdot 40,$$

$$\frac{\text{Influence of 26 hours' stoppage at 21}}{\text{Influence of 26 hours' stoppage at 18}} = \frac{26}{62} = \cdot 42.$$

This *suggests* that the influence of a stoppage on the creep at a lower stationary pressure may be expressible as a product of two factors, one a function of pressure only, the other having for sole variable the duration of the subsidiary pressure.

Again, taking the three experiments when the subsidiary stoppage lasted 26 hours, it will be noticed that the mean ratios,  $\cdot 13$ ,  $\cdot 38$ , and  $\cdot 74$ , are to one another nearly in the proportion 1 : 3 : 6, which corresponds to the pressure intervals 16 to 15, 18 to 15, and 21 to 15 inches.

Whether we have to do here with chance coincidences or with physical laws it would be impossible to say without further somewhat elaborate experiments.

§ 37. Before leaving Table XXXVIII. we may note its bearing on the laws established in §§ 16 to 19 for change of reading under steady pressure. The data deduced from the subsidiary stoppages at 21 and 18 inches, and the stoppages at 15 inches in the three standard experiments, are in good agreement with the laws that the fall of reading in a given time during a stoppage, preceded by a reduction of pressure at a uniform rate, is proportional to the pressure range, and that the mode of variation of the reading with the time is independent of the range. The first of these laws is supported by the closeness of the "mean" and "theoretical" ratios, the second by the absence of any clear tendency in the ratios of the falls at the specified intervals to either increase or decrease as the intervals become longer.

Further support of the first law is supplied by the fact that the mean observed falls during 26 hours' exposure to the pressures of 21, 18, and 16 inches were respectively  $\cdot 13$ ,  $\cdot 19\dot{3}$ , and  $\cdot 22\dot{6}$  inch, and we have

$$\cdot 13/9 = \cdot 014, \quad \cdot 19\dot{3}/12 = \cdot 0161, \quad \cdot 22\dot{6}/14 = \cdot 0162,$$

values nearly equal.

A very interesting point is that the ratios borne by the falls at 15 inches pressure, in the several experiments where a subsidiary stoppage existed, to the corresponding falls in the standard experiments are, to all appearance, constant throughout the hour's stoppage at 15. There are, of course, irregularities in the figures, but there

is no clear tendency in the ratios to increase or decrease as the time increases from 5 to 60 minutes. There would thus appear to be a very vital distinction between the influence of a stoppage and that of a reduction in the rate of fall of pressure. In both cases there is a reduction in the creep observed in a given time while the pressure is maintained steady at its lowest point, but the stoppage has apparently no appreciable influence on the law of variation of the creep with the time, while the reduction of the rate of fall of pressure has a notable influence in this direction.

§ 38. The influence of a stoppage is also brought out in an instructive way by comparing the differences of the descending and ascending readings at points below and at points above the stationary pressure with the corresponding data from experiments of the normal type. Taking as standards the normal experiments 63 and 66, I have instituted this comparison for experiments 62 and 67, 64 and 65, 68, 69, 70, and exhibit the results in Table XXXIX. The following example will show how the results were arrived at and what they signify.

The mean of the experiments 63 and 66 gave, in the case of aneroid No. 4, for the sum of the differences, descending less ascending readings, from 15 to 18 inches inclusive  $\cdot 605$  inch, and from 18 to 30 inches inclusive  $2\cdot 635$  inches. Now in experiment 69, with a prolonged subsidiary stoppage at 18 inches, the sum of the differences at 15, 16, 17 and 18 inches (after stoppage in descent) was  $\cdot 26$  inch, while the sum of the differences at 18 inches (before stoppage in descent) and 19 to 30 inches inclusive was  $3\cdot 19$  inches. Thus stoppage for 26 hours at 18 inches diminished the sum of the differences below this point by  $100 (\cdot 605 - \cdot 26) \div \cdot 605$ , or 57 per cent., and raised the sum of the differences above this point by  $100 (3\cdot 19 - 2\cdot 635) \div 2\cdot 635$ , or 21 per cent.

TABLE XXXIX.—Percentage Change in Sum of Differences, Descending less Ascending Readings, Below and Above Pressure of Stoppage (+ gain, — loss).

Aneroid.	Stoppage at 21 inches.				Stoppage at 18 inches.				Stoppage at 16 inches.	
	For 1 hour.		For 26 hours.		For 1 hour.		For 26 hours.		For 26 hours.	
	Below.	Above.	Below.	Above.	Below.	Above.	Below.	Above.	Below.	Above.
No. 4	—7	+3	—24	+12	—19	+10	—57	+21	—83	+57
„ 7	—6	+8	—22	+9	—23	+4	—48	+19	—68	+49
„ 8	—6	+4	—20	+13	—19	+11	—43	+17	—83	+51
Means .	—6	+5	—22	+11	—20	+8	—49	+19	—78	+52

The percentage loss in the sum of the differences below the stationary pressure is,



as a rule, considerably larger than the percentage gain throughout the higher part of the range. The latter, however, is a substantial quantity except in the case of the stoppage at 21 inches for an hour only.

§ 39. As pressure was raised above the point where it was stationary in the descent, the influence on the difference of the descending and ascending readings gradually diminished, and by the time atmospheric pressure was reached there was in every case a close approach to the normal deficiency in the reading. In some cases the deficiency was even below the normal. There is, however, ground for believing that even after return to the atmospheric pressure the stoppage exercised some influence. The evidence for this conclusion is contained in Table XL., which shows the rate of recovery in cases in which there was, and in other cases in which there was not, a subsidiary stoppage, in addition to the stoppage of an hour at the lowest point, 15 inches. The results are means for the three aneroids Nos. 4, 7, and 8.

TABLE XL.—Ratio of Deficiency to Original Deficiency,  $D_t/D_0$ .

Experiments Nos.	Additional stoppage.		Time interval in minutes—				
	At	For	0.	5.	10.	15.	20.
60 and 66	inches. none	hours. none	1	·88	·81	·61	·58
64 „ 65	21	1	1	·81	·75	·71	·67
62 „ 67	18	1	1	·92	·88	·86	·84
68	16	26	1	·89	·90	·87	·83

The experiments are, of course, not numerous enough to justify exact numerical deductions, but it seems quite clear that the recovery in the two last cases in the table was decidedly slower than in the first case.

§ 40. In addition to the experiments already referred to, there were two of earlier date, Nos. 45 and 46, in which there was a subsidiary stoppage, lasting 24 hours, during the descent. In the one the stoppage occurred at 26, in the other at 24 inches. In both, pressure was reduced to 21 inches, the stoppage there lasting only 10 minutes, and the ascent was of the normal type. At points below the stoppage the differences between the descending and ascending readings were very decidedly less than in the case of the normal experiments Nos. 36 and 39 treated as standards. The effect was considerably larger when the stoppage occurred at 24 than when it occurred at 26 inches. In neither case, however, was the recovery at the end of the experiment noticeably slow.

§ 41. After our discussion of the effects of a stoppage during the reduction of pressure, a brief reference will suffice to a variety of experiments, Nos. 29, 30, 31, 32, 40, 41, 42, 43, and 44, in which there was a prolonged subsidiary stoppage during the ascent of pressure. In all, the lowest pressure was 21 inches, and the stoppage there lasted 10 minutes.



Speaking generally, the reading of the aneroids tended to fall or to rise during the subsidiary stoppage according as the stationary pressure was below or above 26 inches; but unless it was below 25 or above 27 inches such change of reading was extremely small.

The influence of the stoppage in these experiments on the readings during the subsequent rise of pressure, can be traced with nicety, because the part of the cycle preceding the subsidiary stoppage was strictly of the normal type, and so could be utilised at once in conjunction with complete normal experiments to fix the standard. In this way I calculated the percentage increment produced by the subsidiary stoppage in the sum of the differences of the descending and ascending readings at points further up the scale. The results appearing in Table XLI. are the means for aneroids Nos. 1, 2, 3, and 4; + signifies an increase, — a diminution. The subsidiary stoppage lasted 24 hours when it occurred at 24 or 26 inches; in the other cases it lasted only one hour.

TABLE XLI.—Influence of Stoppage during Ascent of Pressure. Percentage Change of Sum of Differences (Descending less Ascending Readings) at Higher Pressures.

	Stoppage at						
	22.	23.	24.	26.	27.	28.	29.
Percentage change.	+10	+17	+27	+15	—11	—38	—40

A slight rise of reading during a prolonged stoppage is not incompatible with an increase in the sum of the differences of the descending and ascending readings at higher pressures. For, as we have already seen, there is reason to believe that a stoppage tends to make subsequent recovery slower.

#### *Theoretical Deductions.*

§ 42. Reasoning from the experimental data I have built up a theory, of a somewhat empirical kind it is true, which reproduces satisfactorily the phenomena presented by the normal type of test at Kew Observatory.

A lowering of pressure is supposed to produce two independent lowerings of an aneroid's reading. One is a perfectly reversible or wholly elastic phenomenon, the other is the source of the after-effect and is termed the *creep*. It is supposed that during the constant interval—called 5 minutes for brevity—occupied in the fall of pressure from  $30 - n$  to  $30 - \overline{n + 1}$  inches the magnitude of the creep is  $k(n + \overline{n + 1})/2$ . Here  $k$  is constant, for a particular aneroid, so long at least as the temperature is unchanged. During the fall of pressure to, say,  $30 - p$  inches, the accumulated creep thus amounts to

$$\frac{1}{2} k (1 + 3 + \dots + 2p - 1) = \frac{1}{2} kp^2.$$

During the stoppage for a constant interval—termed 10 minutes for brevity—at the lowest pressure, the creep augments by  $kmp$ , where  $m$  is independent of the aneroid.

During the rise of pressure, matters are more complicated. As pressure rises from  $30 - p + r - 1$  to  $30 - p + r$  inches, it is assumed that in virtue of the rise of pressure above the lowest point the creep will diminish to the extent  $kl(r - 1 + r)/2$ , or  $kl(2r - 1)/2$ , where  $l$  is constant in the same sense that  $m$  is. On the other hand, the creep tends to increase in virtue of the pressure being still below 30 inches. During the 5 minutes ascribe to the pressure its mean value  $30 - p + r - \frac{1}{2}$ . The length of time elapsed from the beginning of this 5 minutes to the beginning of the corresponding 5 minutes during the descent is  $5r + 10 + 5(r - 1)$ , or  $5(2r + 1)$  minutes. Then what I assume is that the creep downwards in this 5 minutes bears to the creep in the corresponding 5 minutes in the descent the same ratio that the creep in the interval from  $5(2r + 1)$  to  $5(2r + 2)$  minutes after pressure becomes steady bears to the creep in the first 5 minutes after pressure becomes steady. This does not absolutely assume the creep to be the same in *magnitude* when pressure is changing as when it is steady; but it is, of course, considerably speculative.

In the ordinary Kew test, the time occupied by the experiment is not in excess of the duration of the steady pressure during the special experiments Nos. 33, 34, 35, used in constructing Table XIV. In the following calculations I have employed, not the direct results of these experiments, but the values supplied by the formula based on them.

§ 43. An example of the calculations will serve, I hope, to illustrate the process. Take the cycle 30–24–30 inches, supposed of the normal type.

As pressure falls the creep augments by  $\cdot 5k$  between 30 and 29 inches, by  $(1 + 2)k/2$  or  $1\cdot 5k$  between 29 and 28 inches, and so on. Thus the accumulated creep amounts to  $\cdot 5k$  at 29,  $2k$  at 28,  $4\cdot 5k$  at 27, and so on.

During the stoppage at 24 inches, the creep augments by  $6mk$ . As pressure rises from 24 to 25 inches we regard it as 24·5, or 5·5 inches below 30. Also the 5 minutes occupied by the rise represents the interval 15 to 20 minutes elapsed since pressure in its descent entered the stage 25 to 24 inches.

Now, by Table XIV.,

$$\frac{\text{fall in interval 15–20 minutes}}{\text{fall in interval 0–5 minutes}} = \frac{10}{57} = \cdot 18.$$

Thus, for the increase of creep during this 5 minutes, we take  $\cdot 18 \times 5\cdot 5k$ , or  $1\cdot 0k$ , as against a decrease of  $\cdot 5k$ .

The following scheme shows the magnitude of the creep as the pressure attains the values specified in the first column.

The common factor  $k$  is to be understood.

Pressure in inches.	Accumulated creep.		Descending less ascending reading.
	Descent.	Ascent.	
30	0	$6m + 20.3 - 18.0l$	$6m + 20.3 - 18.0l$
29	0.5	$6m + 20.3 - 12.5l$	$6m + 19.8 - 12.5l$
28	2.0	$6m + 20.2 - 8.0l$	$6m + 18.2 - 8.0l$
27	4.5	$6m + 20.0 - 4.5l$	$6m + 15.5 - 4.5l$
26	8.0	$6m + 19.6 - 2.0l$	$6m + 11.6 - 2.0l$
25	12.5	$6m + 19.0 - 0.5l$	$6m + 6.5 - 0.5l$
24	18.0	$6m + 18.0$	$6m$

This gives sum of differences descending less ascending  $= k(42m + 91.9 - 45.5l)$ .

§ 44. As  $k$  is a common multiplier; all quantities such as ratios of differences of ascending and descending readings to the mean difference, or ratios of sums of such differences for different ranges, depend on only two constants,  $l$  and  $m$ . When these are known,  $k$  may be found for any given aneroid from the observed sum of the differences of its descending and ascending readings over any given range.

For the old Kew observations, summarised in Table I., I found by trial

$$l = 0.83,$$

$$m = 1.3,$$

employing these values for *all* the ranges.

To test the theory severely, it seemed expedient to compare with it the observed results for the law of variation of the ratios of the differences of the descending and ascending readings to the mean difference over the several ranges. For this comparison I have utilised the figures resulting directly from the observations at each inch of pressure, and not the data in Table II., as the latter arise from a combination of actual observations at adjacent points of the scale. This should obviate the suspicion that naturally arises when data compared with theory have been subjected to any kind of manipulation.

The observed, "O," and the calculated, "C," values appear side by side in Table XLII. To adequately appreciate the agreement, the reader should remember that unity in the table answers to the mean difference between the descending and ascending readings. It thus answers to only about .34 inch even in the range 30–15 inches. In the range 30–26 inches it represents only a tenth of this.



TABLE XLII.—Ratios of Differences of Descending and Ascending Readings to the Mean Difference. "O," observed results (1885-91); "C," calculated from  $\begin{cases} l = 0.83, \\ m = 1.3. \end{cases}$

Pressure in inches.	Range.											
	30-26 inches.		30-24 inches.		30-23 inches.		30-21 inches.		30-18 inches.		30-15 inches.	
	O.	C.	O.	C.	O.	C.	O.	C.	O.	C.	O.	C.
15	..	..	..	..	..	..	..	..	..	..	0.25	0.28
16	..	..	..	..	..	..	..	..	..	..	0.52	0.53
17	..	..	..	..	..	..	..	..	..	..	0.74	0.73
18	..	..	..	..	..	..	..	..	0.34	0.33	0.93	0.90
19	..	..	..	..	..	..	..	..	0.63	0.61	1.05	1.04
20	..	..	..	..	..	..	..	..	0.84	0.83	1.17	1.15
21	..	..	..	..	..	..	0.37	0.40	1.00	1.01	1.23	1.23
22	..	..	..	..	..	..	0.69	0.72	1.14	1.15	1.27	1.29
23	..	..	..	..	0.41	0.46	1.00	0.97	1.20	1.24	1.29	1.31
24	..	..	0.54	0.50	0.77	0.83	1.15	1.14	1.27	1.29	1.32	1.31
25	..	..	0.88	0.90	1.08	1.08	1.26	1.25	1.29	1.30	1.26	1.28
26	0.35	0.61	1.17	1.14	1.24	1.23	1.29	1.29	1.25	1.27	1.21	1.22
27	1.14	1.05	1.26	1.26	1.28	1.28	1.24	1.26	1.19	1.20	1.14	1.13
28	1.51	1.23	1.16	1.25	1.23	1.22	1.18	1.17	1.12	1.09	1.02	1.02
29	1.19	1.19	1.07	1.11	1.11	1.08	1.01	1.01	0.93	0.94	0.88	0.88
30	0.82	0.91	0.89	0.85	0.86	0.83	0.81	0.79	0.80	0.74	0.72	0.71
Mean difference observed — calculated . . . . .	± .14		± .037		± .024		± .016		± .020		± .013	
Representing in inches . . . . .	± .004		± .004		± .004		± .003		± .005		± .004	

By varying  $l$  and  $m$  one could of course have slightly improved the agreement in individual ranges.

For instance, the assumption

$$l = 0.78,$$

$$m = 1.07,$$

would reduce the mean difference of the observed and calculated values in the range 30-23 to  $\pm .011$ , representing  $\pm .002$  inch.

§ 45. As an example of the calculation of  $k$ , we have for the sum of the differences of the descending and ascending readings in the cycle 30-24-30 :—

from mean of aneroids in Table I. (first group) .684 inch,

from formula . . . . .  $k(42m + 91.9 - 45.5l)$ .

Putting  $l = \cdot 83$ , and  $m = 1\cdot 3$ , as in Table XLII., we find

$$k = \cdot 684 \div (54\cdot 6 + 91\cdot 9 - 37\cdot 8) = \cdot 0063.$$

This is a mean value for the 13 aneroids dealt with as a group in Table I.; individuals of the group differed, of course, amongst themselves.

§ 46. In the case of the experimental aneroids, it might appear that  $m$  and  $l$  are at once determined by the experiments.

For instance, it might be supposed that

$$m = \frac{\text{fall of reading in 10 minutes at lowest point}}{\text{fall of reading in 5 minutes at lowest point}},$$

$$l = \frac{\text{recovery in first 5 minutes at end of cycle}}{\text{fall in first 5 minutes at lowest pressure}}.$$

If this were so, we should have from the first 24 experiments for the mean of aneroids Nos. 1, 2, 3, and 4,

$$m = 73/47 = 1\cdot 55,$$

$$l = \cdot 65, \text{ roughly.}$$

In the case of  $m$ , if we preferred to use the formula (8), we should have

$$m = 73/57 = 1\cdot 28.$$

These deductions, however, assume the creep phenomena the same whether pressure is steady or changing, which may not be strictly true. Further, even if there were no theoretical objection, the method would not, in practice, be very satisfactory, owing to the large probable error involved in determining such small quantities as an increment or decrement of creep in 5 minutes.

Another method that naturally suggests itself is a comparison of the observed sums of the differences of the descending and ascending readings in three ranges. This supplies two simple equations involving  $l$  and  $m$  as unknowns. This method, however, is in practice no better than the other, as will be recognised on inspection of Table XLIII. The quantities tabulated there are the ratios borne by the sums of the differences of the descending and ascending readings over several ranges to the sum for the range 30–21 inches. The observed values are taken from the first 24 experiments.

TABLE XLIII.—Ratios of Sums of Differences to Sum for Range 30–21 inches.

	Range.				
	30–26 inches.	30–24 inches.	30–21 inches.	30–18 inches.	30–15 inches.
Observed ratios. . . . .	·146	·375	1	2·00	3·65
Ratios calculated from $l = 0·65, m = 1·5$ . . .	·145	·370	1	2·08	3·72
„ „ „ 0·83, „ 1·5 . . .	·148	·374	1	2·06	3·67
„ „ „ 0·83, „ 1·3 . . .	·144	·369	1	2·08	3·73
„ „ „ 1·0, „ 1·0 . . .	·138	·363	1	2·10	3·79

The difficulty here is that large variations in the values of  $l$  and  $m$  have but little influence on the calculated ratios. The differences between the observed and calculated values in Table XLIII. are in no case in excess of reasonable limits of experimental error. The agreement is certainly best for  $l = ·83, m = 1·5$  and worst for  $l = m = 1$ .

The variation of the ratio of the difference of the descending and ascending readings to the mean difference over a range affords a much more sensitive method of determining  $l$  and  $m$ . The application of this method shows that, for the special experiments,  $l = m = 1$  is in reality a better choice than  $l = ·83, m = 1·3$ . The evidence for this conclusion is supplied by Table XLIV.

TABLE XLIV.—Mean Differences between Observed and Calculated Values of Ratios of Differences of Descending and Ascending Readings to the Mean Difference (Experiments 1–24).

	Range.				
	30–26 inches.	30–24 inches.	30–21 inches.	30–18 inches.	30–15 inches.
Mean difference, using $l = 1, m = 1$ . . .	±·066	±·070	±·057	±·043	±·048
„ „ „ „ $l = 0·83, m = 1·3$ . . .	±·080	±·094	±·063	±·086	±·095

*Previous Work Bearing on the Subject.*

§ 47. Of investigations into elastic after-effect, more especially in metal wires, there is an excellent summary in vol. 1 of WINKELMANN'S 'Handbuch der Physik.' This I studied, having previously read a number of the papers it refers to, notably some by KOHLRAUSCH. Of quite recent papers on the subject I have not seen many,



and with one or two exceptions—*e.g.*, a paper by Mr. L. AUSTIN, in 'Wied. Ann.,' vol. 50, 1893—the theories they dealt with seemed of a hopelessly involved character for practical application. I did indeed try one or two of the formulæ proposed, but conditions so complicated as those of the Kew aneroid test soon led to expressions of a formidable character. The numerical results I actually reached did not show a promising accord with experiment, and having regard to the many claims on my time, I did not push the calculations further.

§ 48. The following are the only earlier investigations on aneroid barometers, having any direct bearing on the results of the present paper, that have come under my notice:—

1. "On the Errors of Aneroids at Various Pressures" (Dr. BALFOUR STEWART), 'B.A. Report,' 1867, pp. 26, 27, of 'Transactions of the Sections.'

2. "An Account of Certain Experiments on Aneroid Barometers made at the Kew Observatory at the Expense of the Meteorological Committee" (Dr. BALFOUR STEWART), 'Roy. Soc. Proc.,' vol. 16, 1869, pp. 472-480; 'Phil. Mag.,' vol. 37, 1869, pp. 65-74.

3. "How to Use the Aneroid Barometer," by EDWARD WHYMPER; London: JOHN MURRAY, 1891.

Of these, No. 1 is a preliminary report apparently of the work more fully dealt with in No. 2. It mentions the tendency in the reading to fall under continued exposure to low pressure, and also the general nature of the recovery after return to atmospheric pressure.

No. 2 commences with a slight reference to the secular variation of zero. It then describes some tests showing that at atmospheric pressure changes of temperature have little effect on "well-made compensated" instruments. The writer adds, "I am unable to say what effect a change of temperature would have at a diminished pressure."

The main part of the paper describes the phenomena observed in pressure cycles, 30-19-30 inches of the following kind. Pressure was reduced an inch and then kept steady for 10 minutes, when a reading was taken; it was then lowered a second inch, and so on. At the lowest pressure there was a stoppage of  $1\frac{1}{2}$  hours, and thereafter pressure was raised at the same rate as it had been lowered. The aneroids were always tapped before reading. The general superiority of large to small aneroids, and the increase in the tendency to creep when the range is lengthened are duly noted. The effects of stoppage on the top of a mountain are also remarked on.

A considerable number of aneroids were examined. The object was apparently, however, rather to ascertain the conditions under which they behaved fairly consistently than to investigate whether the variation in the readings obeyed any ascertainable laws.

One experiment, at least, was, however, made to ascertain whether the rate of reduction of pressure was of any consequence. Two aneroids were tried, a comparison being made of their readings when pressure was reduced at the two different rates, 1 inch in 10 and 1 inch in 30 minutes. In one of the aneroids the readings were sensibly lower all through the range at the slower rate; in the other the change of rate had no certain influence.

§ 49. No. 3, which has been already referred to, is divided into parts. Part 1, treating of "Comparisons in the Field," enumerates changes of zero and of reading observed in a number of aneroids used at lofty stations, but hardly bears on the present paper. Part 2, "Experiments in the Workshop," gives numerous details as to the creep during several weeks' exposure to low pressures. It also records a considerable number of observations showing the recovery of aneroids on return to atmospheric pressure after a week's exposure to a lower pressure.

Mr. WHYMPER was apparently unaware of Dr. BALFOUR STEWART's work, and regarded the creep at low pressures as a new phenomenon. He noticed that the rate of creep diminished as the time increased since the low pressure was reached, and that the recovery on return to atmospheric pressure was most rapid at first. He seems, however, to have made no serious attempt to study more exactly the laws of the phenomena. Judging from some remarks on his p. 25, he was discouraged from doing so by the belief that different aneroids varied very much in the laws they followed. The fall of reading during the first day's exposure to a low pressure might vary, he says, from about one-third\* to more than three-fourths of the fall in the first week.

For evidence of this he refers to a table on his p. 26, giving particulars of the creep in 29 aneroids during the first day and the first week. The evidence seems to me somewhat inconclusive. In the first place, there seems no information as to whether the original lowering of pressure was carried out at an invariable rate. Thus it is not clear whether an exact comparison is possible between the aneroids which were exposed to different pressures. Those exposed to the same pressure were presumably exposed to the same conditions, and in their case there are no variations at all approaching those mentioned by Mr. WHYMPER. In fact, I notice only two instances in the whole table in which the creep in the first day was less than the half of that in the first week; and in both these cases there was no companion aneroid tried at the same pressure. In the first instance a 3-inch Watkin aneroid, exposed to a pressure of 24 inches, is credited with a creep of '111 inch in a day and '273 inch in a week; in the second instance, that of a  $4\frac{1}{2}$ -inch Watkin aneroid, the creeps were '052 inch in a day and '144 inch in a week. The presumption is that the aneroids were only read to '01, or at most to '005, of an inch, the third decimal coming from the reading of the mercury barometer; in the second instance mentioned above, a trifling error in the reading would make a large difference in the result.

\* A footnote says "less than a fifth" in some exceptional cases not quoted.



In the other 27 aneroids the creep in the first day bore to that in the first week a ratio lying between '6 and '9. It is also not unreasonable to suppose that some at least of the extreme values were appreciably influenced by permanent changes of zero.

§ 50. Notwithstanding the absence of information as to the exact nature of the operations, Mr. WHYMPER's experiments are of much interest, inasmuch as they supply information of an almost unique kind relative to the effects of very prolonged exposure to low pressures. I have thus examined the results with considerable care, with a view to seeing what light they throw on the phenomena already described.

There are, in the first place, several tables whose contents afford the opportunity of determining how far it is possible to represent the creep over a series of weeks at a low pressure by a single algebraic term  $Ct^q$ , as in (7), § 18.

Thus, on his p. 17 Mr. WHYMPER gives the errors, relative to a mercury barometer, in six aneroids exposed together for six weeks to a pressure of  $22\frac{1}{2}$  inches. Readings were taken at the end of each week. How exactly equal the "weeks" were is not stated; but a few hours' variability in such a case would hardly matter. At the end of four weeks the readings had become, if not absolutely stationary, so nearly so that the probable error of reading began to be too serious.

In Table XLV. I give the ratios I find borne to the first week's creep by the total creeps up to the end of the second, third, and fourth weeks; also the values found for  $q$  in the algebraic term by taking separately the mean values of the above three ratios.

TABLE XLV.—Creep Ratios, calculated from data on Mr. WHYMPER's p. 17.

	Aneroid						Mean.	Corre- sponding $q$ .
	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.		
2 week's creep/1 week's	1·06	1·10	1·15	1·05	1·12	1·19	1·11	·151
3   "       "       "	1·13	1·20	1·26	1·21	1·15	1·33	1·20	·166
4   "       "       "	1·19	1·22	1·23	1·29	1·20	1·31	1·24	·155

The irregularities apparent in the data for some of the aneroids are not surprising, in view of the fact that the average creep recorded in the fourth week was only '013 of an inch.

The mean results agree well with the single term formula (7).

The next set of data I have utilised are given on Mr. WHYMPER's p. 19. They refer to three aneroids—distinct from those of Table XLV.—exposed for eight weeks to the pressure of 16 inches. In these the increase of creep can be fairly traced up to the end of the sixth week. The individual aneroids show somewhat irregular behaviour. I give the mean results, obtained in the same way as the corresponding results in our last table. "1w" stands for "one week's creep," and so on.



TABLE XLVI.—Creep Ratios, calculated from data on Mr. WHYMPER's p. 19.

	$2w/1w.$	$3w/1w.$	$4w/1w.$	$5w/1w.$	$6w/1w.$
Mean ratio . . . . .	1·21	1·29	1·38	1·41	1·48
Corresponding $q$ . . .	0·275	0·232	0·232	0·213	0·219

The mean value for  $q$ , viz., '234, is considerably higher than that given in Table XLV. There is also at least a suggestion that the first two weeks' creep was, relatively speaking, somewhat greater than according to the single term law.

§ 51. The last and much the most varied set of data which I have utilised are taken from tables on Mr. WHYMPER's pp. 26, 27, and 28. The table on p. 26 has been already described. That on p. 27 gives with some blanks the errors at the end of the 1st, 2nd, 4th, and 7th days' exposure in 21 of the 29 aneroids included on p. 26. The table on p. 28 gives the errors at the end of the first hour, and the end of the first day, in all the 29 aneroids. Though not apparently mentioned, it is clear, from the size of the errors specified at the end of the first day, that the three tables refer to the same experiments. I have thus dealt with them together. The results of my calculations are given in Table XLVII. for all the aneroids for which complete sets of readings are given, with the exception of one, of which a reading is queried by Mr. WHYMPER. In the case of the 5 lowest pressures in the table, the results are means for the 2 or 3 aneroids exposed to the same pressure.

TABLE XLVII.—Creep Ratios, calculated from data on Mr. WHYMPER's pp. 26, 27, 28.

Constant pressure in inches.	Number of aneroids.	1 Hour's 1 Day's	2 days' 1 day's	4 days' 1 day's	7 days' 1 day's
14	2	·62	1·11	1·27	1·50
15	2	·58	1·17	1·27	1·42
21	2	·42	1·14	1·32	1·41
22	3	·47	1·17	1·35	1·50
23	2	·63	1·13	1·41	1·64
24	1	·44	1·50	1·92	2·46
26	1	·62	1·14	1·23	1·33
Mean from 13 aneroids }		·53	1·17	1·36	1·55
Corresponding $q$ . . . }		·200	0·227	0·222	0·225
Mean from 12 aneroids }		·54	1·14	1·32	1·48
Corresponding $q$ . . . }		·194	0·189	0·200	0·201

The aneroid exposed to the pressure of 24 inches is one of the two exceptional ones already referred to. The mean values in the last two lines of the table are obtained by omitting it.

The agreement between the mean results from the 12 aneroids and those deduced from the single term

$$Ct^{.196}$$

is extremely close.

There are naturally very considerable irregularities amongst the individual results in the table, but, with the solitary exception already alluded to, the phenomena observed at the different pressures are very similar. The aneroids were made mainly by CASELLA and HICKS, but they varied in diameter from 2 to  $4\frac{1}{2}$  inches.

§ 52. Mr. WHYMPER's data as to recovery, though less varied, are also of interest. On his p. 30 he gives the original errors, the errors on return to the original pressure, and those observed 1 day and 1 week later for 36 aneroids, exposed for one week to pressures varying from 14 to 26 inches. In 12 instances the reading was higher after a week's recovery than it was prior to the experiment.

On his p. 31 Mr. WHYMPER repeats some of these data for 20 of the aneroids, adding also the errors observed after only 1 hour's recovery. I have omitted one of the 20 aneroids, No. 40, because after a week's recovery it read .29 inch higher than prior to the experiment, the figures suggesting the occurrence of a large permanent change of zero before the end of the first day's recovery. According to the figures on p. 31 there was also a permanent rise of zero in three other aneroids, Nos. 42, 22, and 23. In the last-mentioned case, however, this seems due to a misprint of .236 for .206 in one of the data, the latter figure appearing on p. 30. In the case of Nos. 22 and 42 the final reading exceeded the original by so small an amount it seemed best to retain the aneroids. This retention accounts for the minus sign in the second entry of the last column of Table XLVIII. When more than one aneroid was exposed to a given pressure, I give only the mean values of the ratios for the group. In many cases the aneroids are the same as were dealt with in Table XLVII.

TABLE XLVIII.—Ratios of Deficiencies to Original Deficiency (data from Mr. WHYMPER's pp. 30, 31).

Constant pressure in inches.	Number of aneroids.	Time.			
		0.	1 hour.	1 day.	1 week.
14	2	1	.52	.17	.04
15	2	1	.70	.11	-.06
18	2	1	.62	.30	.22
20	6	1	.63	.37	.09
22	3	1	.63	.35	.19
23	2	1	.57	.28	.04
24	1	1	.61	.56	.35
26	1	1	.70	.50	.17
Mean from	19	1	.62	.32	.11

Evidently, little weight attaches to individual results after a week's recovery. A small permanent change or a slight initial fatigue in an aneroid would have a large effect on the final data. Even after a day's rest the results are somewhat erratic.

So far as one can judge from the recovery during the first hour, the lowness of the pressure to which the aneroids were exposed was immaterial. There is certainly no indication of the recovery becoming slower as the stationary pressure is more remote from 30 inches. It must be remembered, of course, that the aneroids exposed to the different pressures were different, so that individual peculiarities were not eliminated.

Comparing Table XLVIII. with Tables XVII. and XXXVI., it will be seen that the rate of recovery in Mr. WHYMPER's experiments was slower than in the case of the normal Kew experiments, but faster than in those experiments where pressure was reduced at the rate of an inch in 45 minutes, and where the lowest pressure was maintained stationary for 18 hours.

§ 53. There are other data on Mr. WHYMPER's p. 33 relating to the recovery of 22 aneroids, which had been exposed for a week to a pressure of 21.692 inches. These data I have not considered, for the reason that the reading during recovery proved higher than the reading prior to the experiment in

1 aneroid on return to atmospheric pressure,						
5 aneroids 14 hours after return to atmospheric pressure,						
9	„	12 days	„	„	„	„
16	„	30	„	„	„	„

Such a wholesale tendency to a rise of zero, whatever its cause, would have introduced great uncertainty into any conclusions drawn, even as to the recovery during the first 14 hours, the shortest interval for which results are given.

§ 54. Part 3 of Mr. WHYMPER's pamphlet, dealing with the "Determination of Altitudes," discusses the best method of utilising the readings of aneroid barometers. I hardly think that Mr. WHYMPER makes due allowance for the fact that in mountain ascents the reduction of pressure is very gradual, even at the lower stages. This allows accommodation to take place, so that on arrival at the summit the instrument will not behave as it would have done if rapidly transported there. Again the difference between descending and ascending readings is very different in aneroids whose readings as pressure falls at a given rate may be equally correct, and the difference increases *ceteris paribus* with the length of time spent on the mountain summit. Thus the advantage to be derived from taking the mean of ascending and descending readings—a course which Mr. WHYMPER seems to suggest in some instances—is very problematical.

On his p. 51 Mr. WHYMPER gives an example of a Kew aneroid certificate and then devotes several pages to a criticism of it. On his p. 54 he says, "In the absence of directions to the contrary, it may be assumed by persons into whose



hands similar certificates come that the corrections stated in them are good under all conditions and for all time." In view of the footnote to the certificate, calling attention to the probable recovery from the residual error, exhibited on return to the original pressure, this standpoint could hardly, I think, be justified; and judging by a footnote to his p. 52 Mr. WHYMPER would presumably allow this himself. At the same time, he apparently considers it a point of view likely to present itself to travellers, a class as to whose scientific knowledge and general intelligence he is better qualified to speak than I am. If he is right in his conclusion, a change in the certificate is certainly desirable.

Before concluding my reference to Mr. WHYMPER's interesting pamphlet, I would take the opportunity of explaining that the rate of change of pressure in the ordinary Kew test is not, as he states on several occasions, about 1 inch in 2 minutes; the actual rate is only about half this.

§ 55. The 75 special experiments—some of a very tedious and exacting character—on which this paper is mainly based, were carried out with great care and discretion by Mr. W. HUGO, Senior Assistant at Kew Observatory. In addition to the observational work, Mr. HUGO reduced all the barometer readings and carried out some of the subsequent arithmetical operations. The bulk of these and the checking of the reductions were undertaken by myself.

It is, I allow, anomalous, and from various points of view undesirable, that a scientific man should have himself performed none of the experiments which he discusses. When, however, as in the present case, observation is being pushed to the utmost capabilities of the instruments employed, the absence of preconceived ideas in the actual observer is a valuable compensation.



XII. *On the Heat Dissipated by a Platinum Surface at High Temperatures.**By* J. E. PETAVEL, 1851 *Exhibition Scholar.**Communicated by* Lord RAYLEIGH, *F.R.S.*

Received May 19,—Read June 9, 1898.

[PLATES 18–23.]

FROM the beginning of the century, and more especially of late years, an almost uninterrupted series of papers have appeared on the above subject. Unfortunately few of the authors have reduced their measurements to absolute units, thus rendering comparison of the results obtained very difficult.

The first part of this paper is to some degree an extension to higher temperatures and pressures of the work so ably done by Dr. BOTTOMLEY,\* whilst the names of LANGLEY, PASCHEN,† WEBER,‡ and many others will occur to anyone glancing through the second and third parts.

With regard to temperature measurements, the present work is mainly founded on the researches of CALLENDAR and GRIFFITHS,§ which have been amply confirmed by the later investigations of HEYCOCK and NEVILLE.||

Full references to the above papers are given here once for all, as we shall have continually occasion to refer to one or other of them.

*On the Apparatus Employed.*

The preliminary experiments soon showed that to obtain reliable results a very thick wire must be used for the double purpose of radiator and thermometer, even if it were at the cost of a slight loss of sensitiveness, the reasons being the following :—

\* ‘Phil. Trans.,’ A, vol. 178, 1887, p. 429, and ‘Phil. Trans.,’ A, vol. 184, 1893, p. 593.

† ‘Wied. Ann.,’ vol. 49, p. 50, 1893; vol. 58, p. 455, 1896, and vol. 60, p. 663, 1897.

‡ ‘Physical Review,’ vol. 2, p. 112; ‘Berichte der Preussischen Akademie,’ 1888, p. 933, &c.

§ ‘Phil. Trans.,’ A, vol. 182, p. 119, 1891.

|| ‘Journ. of the Chem. Soc.,’ vol. 67, p. 160, 1895; same vol., p. 1024; ‘Phil. Trans.,’ A, vol. 189, p. 25, 1897. On this subject see also C. M. CLARKE, ‘The Electrician,’ vol. 38, pp. 175, 241, 372, and J. D. H. DICKSON, ‘Phil. Mag.,’ December, 1897, p. 445.



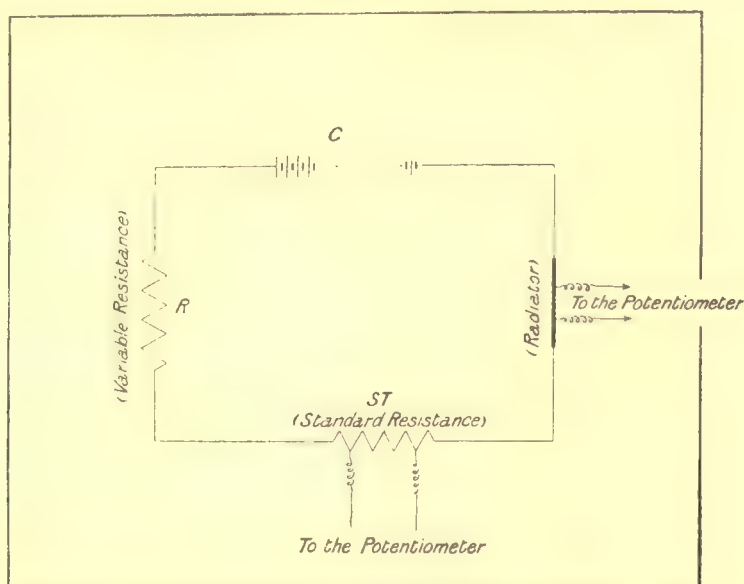
1. A thin wire maintained at any temperature above  $1200^{\circ}$  is subject to a rapid increase of resistance.

2. It is impossible to obtain consistent results with a thin wire at atmospheric pressure, owing to the constant fluctuation in temperature due to convection currents.

3. As pointed out by Professors AYRTON and KILGOUR, with fine wires the emissivity, referred to unit of surface, depends largely on the diameter, but the rate of change decreases rapidly as the diameter increases.

The diameter of the wire used was  $\cdot 112$  centim., or nearly 1 sq. millim. in cross-section. The resistance of the thermometers was about  $\cdot 01$  ohm at  $0^{\circ}$  C., or about one five-hundredth of the resistance usually employed.

Fig. 1.



The electrical connections can be briefly described as follows:—At C, in fig. 1, is a set of forty cells, which by the aid of a mercury commutator can be placed in any desired combination, either series or parallel. This commutator was used to make the first rough adjustment of the current. R is a variable resistance consisting of thirteen German silver tubes, which are kept at a constant temperature by a water circulation. The resistance is designed to carry up to 150 amperes. By two sliding contacts, one of which serves as a fine adjustment, the current can be regulated to within  $\cdot 01$  of its value. The current through the radiator is measured by aid of the standard resistance ST, the exact value of which, according to the Reichsanstalt certificate, is  $\cdot 0099965$  ohm at  $16\cdot 55^{\circ}$  C., its temperature coefficient being  $\cdot 0003$  per cent. A small motor was used to maintain a circulation of oil round the wires of this resistance, the oil being in its turn cooled by a water circulation. By this means the temperature of the standard was prevented from rising above  $25^{\circ}$  C.

All electrical measurements have been referred to this standard and to a Clarke's cell (N 5217 A), its value being taken as 1.4331 volts at 16° C.

Both the current through, and the electromotive force at the terminals of the platinum wire, were measured on a Wolff potentiometer. In none of the coils did the error exceed .02 per cent. The temperature coefficient was less than .001 per cent.

The radiators used were calibrated at frequent intervals, a slight increase in absolute resistance and in temperature coefficient being noticeable.

One of these sets of readings is given below :—

Radiator R <sub>2</sub> .	
R <sub>100</sub> = resistance at 100° C. . . . .	.012252
R <sub>0</sub> = resistance at 0° C. . . . .	.009907
$c = \frac{R_{100} - R_0}{100}$ . . . . .	.00002345
$\delta = 1.304.$	

The value of  $\delta$  is deduced from the resistance of the wire in sulphur vapour by CALLENDAR'S formula,

$$t - pt = \delta \left( \left[ \frac{t}{100} \right]^2 - \left[ \frac{t}{100} \right] \right),$$

where  $t$  is the temperature in degrees Centigrade,

$$pt = \text{temperature in platinum degrees} = \frac{R - R_0}{c}, \text{ and}$$

$\delta = \text{a constant.}$

The platinum thermometer has rarely been used to determine temperatures above 1100°. It is therefore necessary to obtain some confirmation of the above formula for temperatures up to 1500°. For this the melting point of palladium was chosen, and the value given by VIOLLE,\* 1500° C., taken as correct.

A length of about 3 millims. of palladium wire .01 centim. in diameter was held against the platinum radiator. The temperature being gradually increased, the palladium soon became viscous and adhered to the platinum, finally fusing at a very sharply-defined temperature.

To this method there is an obvious objection. With each fusion there is a certain weight (.0004 gram, or .03 per cent. in this case) of palladium added to the platinum. This would produce an appreciable change in the electrical constants. To obviate this difficulty it is necessary to fuse the palladium on a part of the platinum wire, not between the potential leads, but sufficiently near them to be substantially at the same temperature.

\* 'Comptes Rendus,' vol. 87, p. 983, 1878.

The melting point of palladium calculated in this way from the formulæ of CALLENDAR and GRIFFITHS was for the thermometer  $R_2$   $1489^\circ$ , as against  $1500^\circ$  as determined by VIOLLE.

A second confirmation of the temperature determinations will be found in the fact that in all the curves given below, the calculated temperatures agree with those determined directly by the fusion of platinum or palladium.

Finally, it may be well to state here that although every precaution was taken to ensure the accuracy of the results, the absolute resistance of the wires used was too small for me to lay claim to as great a degree of precision as that obtained by HEYCOCK and NEVILLE in their valuable researches on the melting point of metals. Up to about  $500^\circ$  I believe the results given to be correct to a fraction of a degree, but above  $1200^\circ$  it is difficult to prove that they are absolutely exact.

In Table I. will be found a determination of the volume specific resistance of platinum and palladium. The values obtained for platinum show that the temperatures above  $1500^\circ$  cannot be calculated by the ordinary equations; these, for instance, in the case of  $R_2$  would give for the melting point of platinum a temperature more than  $100^\circ$  too low.

#### Part I.—ON THE EMISSIVITY OF A BRIGHT PLATINUM SURFACE IN AIR AND OTHER GASES.

Throughout the experiments, the results of which are given in Tables II. to XIII., a cylindrical glass enclosure, 5.8 centims. in diameter and 24 centims. in height, was used. The straight platinum wire, which served at the same time as radiator and thermometer, was placed in the axis of the cylinder. A platinum wire, 2 millims. in diameter, bent into the form of a "J" and fused at its lower extremity on to the thermometer, formed the return lead. This wire, as well as the potential leads, was kept well to the side of the enclosure. The enclosure was always used with its axis vertical. This arrangement forms the simplest geometrical disposition of the surfaces of the radiator and enclosure that can be used with the method adopted, as for obvious reasons a sphere enclosed in a sphere is out of the question.

A sufficient length was left between the point of contact of the lower potential lead and the terminal of the radiating wire, for the wire to acquire a constant temperature.

As pointed out by Dr. J. T. BOTTOMLEY, the low thermal conductivity of glass is a very serious objection to its use as an enclosure in experiments on emissivity, it being possible for the inner surface to rise to a temperature very appreciably above that of the water circulation. Owing to this fact, it was not thought advisable to take any observations with the radiator at much less than  $100^\circ$  above the enclosure. At temperatures higher than this a slight error in the estimation of the temperature of the enclosure would not alter the results.



A large number of experiments were made, but only the most typical results have been recorded here. The observations are given in the tables according to the order in which they were taken. In Table IIA will be found an illustration of the way in which all the results were worked out.

The curves relating to air, hydrogen, carbon dioxide, oxygen, and steam are given in figs. 2, 3, 4, and 5. The pressure in centimetres of mercury is indicated on each curve. Observations for each of the first three gases are given at three distinct pressures, for steam and oxygen the emissivity is only determined at a pressure of 76 and 228 centims. respectively.

The degree of purity of the hydrogen used was 97 per cent., the remaining 3 per cent. being, in all probability, air. The carbon dioxide, which was prepared from calcium carbonate, contained less than 1 per cent. of impurities.

During the course of each experiment the exact height of the barometer and the temperature of the room, the standard resistance, and the potentiometer, were recorded. The temperature of the enclosure was also read at short intervals of time on a thermometer divided in tenths of a degree. These factors have too little effect on the main point at issue for each of the readings to be recorded here.

In the course of the preliminary experiments it was found that the emissivity, as far as these results are concerned, is practically independent of the condition of the glass surface (whether perfectly clean and freshly polished or covered with a thin layer of lamp-black), and of the exact position of the radiator in the enclosure.

The emissivity does not seem to bear any simple relation to the specific gravity of the gas. In hydrogen, and to a lesser degree both in carbon dioxide and in oxygen, the emissivity is greater than in air.

In fig. 2 the rise in emissivity caused by saturating the air with moisture will be seen. The difference in emissivity is nearly constant throughout the entire range of temperature, averaging .0001 therm. per second per square centimetre of surface, per degree above the enclosure. This is all the more remarkable as the average temperature of the air, and therefore the quantity of water vapour present, naturally increases with the temperature of the radiating wire.

In fig. 6 some curves will be found giving the relation between the temperature of the wire and the mean temperature of the gases in the enclosure. These curves were obtained by using the enclosure itself as a rough form of air thermometer, the measurements being made at atmospheric pressure. The three gases are here in the same order as when classed with regard to the experiments on emissivity. For all the gases included in this study, the rise of emissivity due to any given increase of pressure is not proportional to the initial value of the emissivity, but is more nearly constant at all temperatures.

For facility of reference some parts of the hydrogen curves given both in fig. 2 and fig. 3 (Plate 18), and the curve representing the emissivity in air at three atmospheres, will be found again in fig. 4 (Plate 19).

The emissivity in steam at atmospheric pressure is given in fig. 5. The slope of the curve is much sharper than for any of the gases previously studied. This fact, as we shall see from the results given below, cannot be entirely accounted for by the higher temperature of the enclosure.

I hope shortly to undertake a research on the emissivity of platinum in gases at a much higher pressure. For the present the results are too incomplete to allow any general theory to be formulated.

The values obtained for the emissivity are to some extent dependent on experimental conditions, such as the diameter of the radiating wire, its position in space (whether horizontal or vertical), the dimensions of the enclosure, its absolute temperature, and the material of which it is made. For the results to be of any general application, it is necessary to form an idea of the extent to which they are affected by these divers factors.

Curve I. in fig. 7 (Plate 20) is taken with the radiating wire in the square gun-metal box described in Part III. of this work. For Curves II. and III. the wire was placed in the axis of an iron cylinder 2.6 centims. in diameter, 27.5 centims. long. In all three cases the wire was horizontal.

Curve IV. has already been given in fig. 2; it refers, as do all the results given above, to a wire .112 centim. in diameter placed vertically in the axis of a glass cylinder 5.8 centims. in diameter.

The difference between the Curves I. and IV. is mainly due to the change of position. When the wire is vertical, the part on which the observations are taken is surrounded by a layer of gas which rises from below after being heated to substantially the same temperature as the wire itself. When the wire is horizontal, the hot gas rising from it is replaced by a fresh supply at nearly the same temperature as the enclosure. The variation between Curves I. and II. is about 10 per cent. It is a difference of this order that we may expect when the shape, size, and material of the enclosure are radically changed. In cases, however, where the ratio of the surface of the radiator to the surface of the enclosure is not kept very small, the total loss of heat depends very largely on the dimensions of the enclosure.

Curve III. is taken in the same iron cylinder as used for Curve II. In this case the iron was kept at a high temperature by surrounding it with a jacket containing boiling sulphur. The temperature of the enclosure, as measured by the platinum wire subsequently used as the radiator, was somewhat high. This temperature, namely,  $455^{\circ}\text{C}$ ., or ten degrees above the usual boiling point of sulphur, can easily be accounted for by the direct heating effect of the flame, for the prevention of which no precautions were taken.

The slope of the curve of emissivity obtained at this temperature is considerably steeper than that relating to experiments with the enclosure at a lower temperature.

The above-mentioned curves, which are given in fig. 7, all refer to the emissivity in dry air at atmospheric pressure.



The usual definition of the emissivity of a body, and the one I have adhered to throughout, may be expressed as follows:—The emissivity is numerically equal to the number of gramme-degrees, or therms, dissipated per second, per square centimetre of surface, per degree above the enclosure. The “true emissivity” at any temperature,  $t$ , might be defined as the slope at the temperature  $t$  of the tangent to the curve representing the total heat dissipated plotted in terms of the temperature. In other words, if  $h$  is the loss of heat per square centimetre per second at any temperature,  $t$ , the “true emissivity” at this temperature is  $dh/dt$ .

The Curves II. and III. (fig. 7), if plotted in terms of the true emissivity, would almost coincide, showing that in this case, at any rate, the value obtained depends to a very limited extent on the temperature of the enclosure.

A comparison of Curves I. and V. will show the effect of using a wire .6 millim. instead of 1.12 millims. in diameter.

Finally, in fig. 7, the points marked AA' refer to a platinum wire, 2.005 millims. in diameter, that is to say, nearly twice the diameter used in all previous experiments. The wire was fused in the gun-metal enclosure, and the results must therefore be compared with the point B, obtained under identical conditions, but with a wire 1.12 millims. in diameter.

It will be seen that the decrease in emissivity due to the use of the larger wire is, at this temperature, less than 7 per cent.

The emissivity at the points of fusion of palladium and platinum were deduced from the readings of instruments of the Weston type, which were afterwards calibrated by aid of the potentiometer. The wire was raised to within about  $100^\circ$  or  $200^\circ$  of the melting point. The current was then increased by equal amounts of  $\frac{1}{2}$  or  $\frac{1}{4}$  per cent., and current and electromotive force readings were taken at each successive rise. The current was noted at the instant the wire fused, the corresponding electromotive force being easily found by extrapolating the last few readings. Over 100 amperes were needed to fuse the 2 millims. wire, and for accurate results to be obtained it was necessary for this large current to remain constant to within a tenth of a per cent. This experiment, as many of those previously recorded, was only rendered possible by the exceptional facilities available at the Davy-Faraday Laboratory.

A large proportion of the work done on the subject of emissivity refers to the cooling of comparatively large bodies. This method was used by DULONG and PETIT, J. C. NICHOL,\* MCFARLANE,† C. F. BRUSH,‡ Dr. J. T. BOTTOMLEY, and many others. These results cannot serve as points of comparison. Added to the effect of the absolute size of the bodies used, the question of the relative surfaces of the enclosure and radiator comes here into play. Whenever the ratio of these two surfaces is not very

\* ‘Proc. Roy. Soc.,’ Edin., 1869, vol. 7, p. 206.

† ‘Proc. Roy. Soc.,’ 1872, vol. 15, p. 90.

‡ ‘Phil. Mag.,’ vol. 45, p. 31, Jan., 1898.



small, its absolute value has a considerable influence on the rate of cooling of the enclosed body.

SCHLEIERMACHER'S\* results only apply to radiation in a high vacuum, but BOTTOMLEY, in his researches on the same subject, incidentally gives one value at atmospheric pressure; this is for a temperature of 408° C. It works out at nearly one and a-half times the emissivity for the same temperature as shown in Curve I., fig. 7.

From the resistance given in Dr. BOTTOMLEY'S paper, it is probable that the wire used was about .3 millim. in diameter. If this be the case, the difference is what might have been predicted from the results obtained by Professors AYRTON and KILGOUR.†

The formulæ given by them for the connection between the emissivity of the wire and its diameter are

$$\text{At } 200^{\circ} \text{ C. } e = .00111 + .014303d^{-1}.$$

$$\text{At } 300^{\circ} \text{ C. } e = .001135 + .016084d^{-1}.$$

The wires were heated in a horizontal position and the diameters measured in millimetres.

The diameter of the wire used was 44 millims. Inserting this value for  $d$  in the above formulæ gives for the value of the emissivity at 200° and 300° C. .00144 and .00150 C.G.S. units. The values given in Curve I., fig. 7, are .00107 and .00121. The largest wires used in establishing the above formulæ were 14 millims. It is therefore not surprising if, after extrapolating for a wire over three times this diameter, the calculated and experimental values differ considerably. This divergence is also in some degree accounted for by the fact that the enclosures used in the two cases were of an entirely different shape and size.

In a paper recently published, Mr. C. F. BRUSH gives a number of observations "on the transmission of heat by gases," at temperatures of from 0° to 15° C. Unfortunately it is impossible to reduce these to absolute units. The relative value of the emissivity in air and in hydrogen is much the same as that which may be deduced at some hundred degrees higher from fig. 2. My measurements show that the emissivity in carbon dioxide is greater than in air at high temperatures, converging to the same value at 150° C. At about 10° C., according to Mr. C. F. BRUSH, the relative value of the emissivity in these two gases is reversed.

Many formulæ have been given, principally with regard to radiation in vacuo; but I am aware of none that will apply at atmospheric pressure and high temperatures.

Until more is known as to the physical constants of gases at temperatures ranging from the melting point of silver to the melting point of platinum, it is doubtful whether any general law can be obtained.

\* 'Wied. Ann.,' vol. 26, p. 287, 1885.

† 'Phil. Trans.,' A, vol. 183, p. 371, 1892.

## Part II.—A BOLOMETRIC STUDY OF THE LAW OF THERMAL RADIATION.

The relative merits of the laws of DULONG and PETIT, and STEFAN, have formed the subject of a number of researches. It is sufficient to recall the experimental work of SCHLEIERMACHER\* and SCHNEBELI,† which appeared in 1884, and the theoretical discussion of the subject by FERREL‡ some five years later. In 1888 Professor WEBER§ proposed a new expression connecting the thermal radiation with the absolute temperature of the radiator and the enclosure. This expression, as he showed, accounted satisfactorily for the greater part of the experimental results then available. It was with the hope that some additional data with regard to the total radiation at higher temperatures might help the elucidation of the problem that the present experiments were carried out.

Most of the apparatus employed has been already described in Part I. For the experiment we are now about to describe the radiating wire was placed in a metal enclosure kept at a constant temperature by a water circulation. A number of diaphragms, also provided with water circulations, shielded the bolometer from all external radiation.

The bolometer used was of a somewhat special construction. Many weeks were spent in preliminary experiments before the design of the instrument was finally fixed. It is with a desire to spare this loss of time to others that a somewhat full description of the bolometer is given here.||

The instrument is symmetrical throughout, one side containing the active platinum film which is exposed to the radiation, the other side an exactly similar film forming the second arm of the bridge. For the platinum silver foil from which these films were made I am indebted to the kindness of Mr. J. S. SELLON, of Messrs. JOHNSON, MATTHEY and Co.

A thin sheet of platinum was welded to a thick sheet of silver, the two sheets, protected on each side by a copper plate, were then rolled together and a leaf was thus obtained, of sufficient thickness to be quite easily handled. This I cut on the dividing engine into the shape of a grid (as shown in fig. 8, F). It was then placed in the holder (fig. 8, A), and the silver dissolved in dilute nitric acid. The film of platinum left was found to have a thickness of .00011 centim., or about twice the wave-length of sodium light. The active surface has the appearance of a grating, consisting of fifteen bars, 3 millims. wide, with a space of 1 millim. between each bar. The span from the upper to the lower edge of the carrier is 56 millims. The

\* 'Wied. Ann.,' vol. 26, p. 287.

† 'Wied. Ann.,' vol. 22, p. 430, 1884.

‡ 'American Journal of Science,' vol. 38, p. 3, 1889.

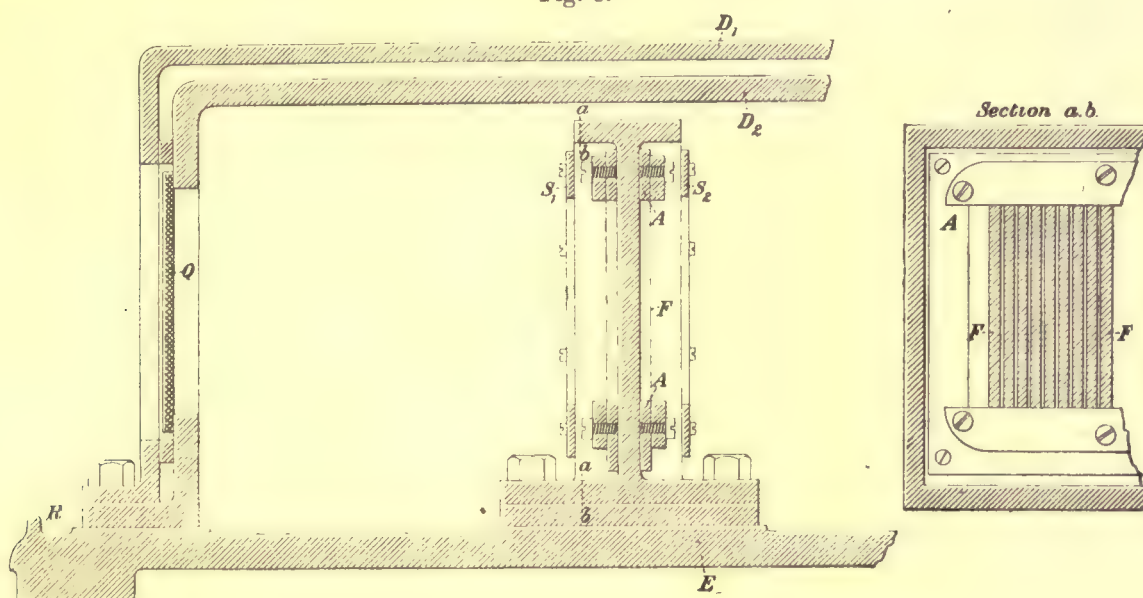
§ 'Berichte der Preussischen Akademie,' 1888, p. 933.

|| Those interested in this subject will do well to refer to an article by LUMMER and KURLBAUM ('Wied. Ann.,' vol. 46, p. 204, 1892; or, 'The Electrician,' vol. 34, pp. 168, 192, 1894).



practical difficulties in the construction of so thin a film are considerable, and in designing a new instrument it would be advisable either to reduce the size of the film or to increase its thickness. The very thin platinum, obtained in the way we have just described, though capable of withstanding a large amount of vibration, cannot resist the slightest mechanical strain, a touch with a single hair being sufficient to break it. Great care is needed in the manipulation of the film from the moment the silver is dissolved off it until it is screwed up in position in the instrument. During this time it has to receive an electrolytically deposited coating of platinum black, be washed several times, and carefully dried. When once in position in the instrument there is, however, comparatively little chance of an accident.

Fig. 8.



The desiderata of a good bolometer may be enumerated as follows:—

1. Great sensitiveness.
2. Constancy of zero.
3. Rapidity of action.

The first and third conditions are fulfilled, thanks to the extreme thinness of the platinum and to its high temperature coefficient. The sensitiveness was found to be increased by coating the films on one side only with platinum black, and by placing behind them a nickel mirror. With regard to rapidity of action it was found in the course of some subsequent experiments that the platinum rises in half a second to over 90 per cent. of its maximum temperature.

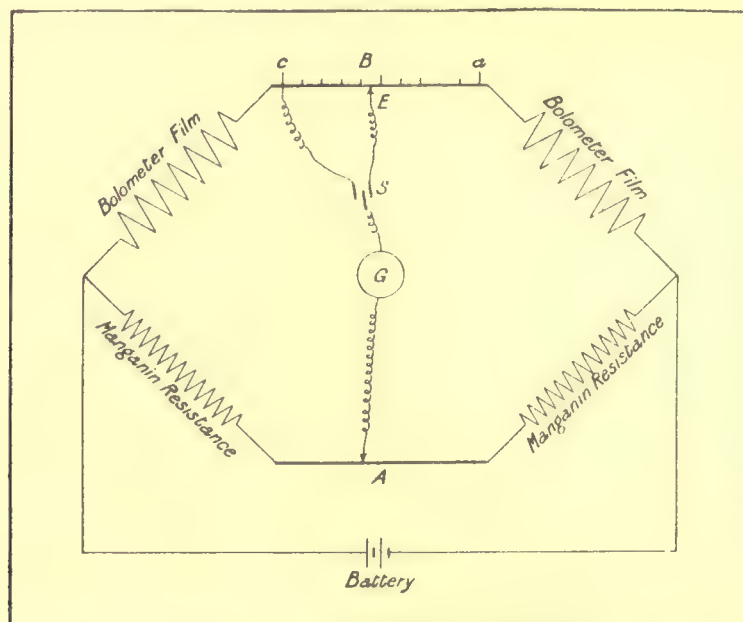
The second condition, namely the constancy of the zero point, is the most difficult to fulfil. It has already been pointed out that the four arms of the bridge must be in pairs as nearly as possible identical, not only with regard to resistance, but also with regard to shape, size, position, and temperature coefficient. But these pre-



cautions are not sufficient. To avoid thermo-electric disturbances it is necessary to keep all the working parts of the bolometer at the same temperature. To attain this object the entire instrument was made of metal. The insulation of the films at their upper and lower extremities where they are secured to the carriers is provided for by very thin strips of mica. The working parts of the bolometer were protected by two massive metal covers,  $D_1$  and  $D_2$ , fig. 8, fitting one over the other, and between which room is left for a water circulation.

The variations due to convection currents formed one of the most serious difficulties. Following the suggestion of Professor DEWAR, the instrument was constructed to allow of the films being used in a vacuum. As shown in fig. 8, the double cover is bolted down on to the bedplate,  $E$ . The joint is kept covered with oil which is poured into the rim,  $R$ .

Fig. 9.



A vacuum can only be used for experiments in which the absorption of the quartz plate,  $Q$ , is not a serious objection. The screens,  $S_1$  and  $S_2$ , are intended to regulate the path of the convection currents when the instrument is used under atmospheric pressure; they also screen the edges of the film from direct radiation. The two other arms of the bridge are made of manganin wire, the coils being placed in an earthenware vessel filled with oil.

The resistance of each of these two coils is 20 ohms, and the resistance of each of the bolometer films 23 ohms. The connections of the instrument are diagrammatically shown in fig. 9. The bar,  $A$ , is of German silver, and serves to adjust the zero of the galvanometer. The bar,  $B$ , between the potential contacts,  $c$  and  $d$ , is divided into 1000 parts of equal resistance, the length between the potential contacts is 80 centims., the resistance '007730. This bar serves to calibrate the galvano-

meter. By the switch, S, one terminal of the galvanometer can be thrown over from the zero point of the bar to the sliding contact, E, which has been previously set at any desired division.

If  $n$  be the number of divisions at which the sliding contact is set,  $n \times 2 \times \cdot 00000773$  is the change in resistance of the bolometer film,\* which would produce an equal galvanometer deflection. Where the total radiation received is large, a zero method can be used, and the change in resistance measured directly by the position of the slider, E, on the bar, B.

As regards sensitiveness, the deflection obtained with a candle at 1 metre from the instrument was 360 millims. The total electromotive force on the terminals of the bolometer was 2 volts, the time period of the galvanometer 10 seconds, and distance of the scale 1.3 metres. Each film before being mounted was tested at a pressure of 30 volts, so that any electromotive force up to 60 volts could be used without danger, but no advantage would be gained by increasing the pressure above 10 or 15 volts. The main object in view has not been to obtain excessive sensitiveness, but to construct an instrument which would give thoroughly reliable results. In almost all cases a much smaller degree of sensitiveness than that given above has been found amply sufficient.

The law of thermal radiation may be studied by two entirely distinct methods. We can either measure the heat lost by the radiating surface when in a nearly perfect vacuum, or the heat received by the sensitive surface of some type of radiometer.

The determinations by the first method will exceed the true value by some quantity,  $m.R$ , representing the heat carried away by any gas or vapour remaining in the enclosure.

The numbers obtained by the second method will fall below the actual value of the radiation by some quantity  $n.R$ , where  $n.R$  represents the part of the radiation not absorbed by the irradiated surface.

If  $R$  be, at any given temperature, the true value of the radiation, and  $O_1$  and  $O_2$  the observed values obtained by the two methods described above, we have :

$$\text{I. } O_1 = c_1 (R + m.R),$$

$$\text{II. } O_2 = c_2 (R - nR),$$

where  $c_1$  and  $c_2$  are constants depending merely on the experimental conditions and on system of units chosen.

In Equation I.,  $m$  represents the ratio between the heat dissipated by convection and conduction and the heat radiated. Now it is well known that this ratio decreases as the temperature increases. The slope of the curve of radiation obtained by this method will therefore always be somewhat less steep than the true curve of radiation.

\* The change in the resistance of the bolometer film produced by the radiation being thus known, it is an easy matter to express the heat received in gramme-degrees. In the present case, however, as we are only dealing with relative values, little advantage would be gained by the use of absolute units.



Considering now Equation II., we see that here also the second term decreases as the temperature rises; for it has been shown that whereas a surface coated with platinum-black or lamp-black almost totally absorbs all radiation of short wave-length, it reflects a comparatively large proportion of the infra-red rays. The curve obtained by any form of radiometer will therefore be steeper than that representing the actual relation between total radiation and temperature.

The foregoing conclusions may be summed up as follows :—

The rate of change of the total radiation with temperature will be too small when measured by the heat lost by the radiating body, whereas it will be too large when measured by the heat gained by an irradiated surface coated with lamp-black or platinum-black.

Finally we may note that, as both  $m$  and  $n$  decrease at higher temperatures, the accuracy of both methods increases as the temperature rises.

Let us now see to what extent the actual experimental work verifies the conclusions we have just arrived at.

In fig. 10 (Plate 21) the values obtained by the first method are represented by the results of Dr. J. T. BOTTOMLEY and of SCHLEIERMACHER, whereas the second method was used by F. PASCHEN and by myself.

To facilitate the comparison of the slopes of the curves, all the results are reduced to the same arbitrary value at a temperature of  $800^{\circ}$  C.

We see that Curve No. II., obtained by BOTTOMLEY, lies between the Curves No. I. and III. obtained respectively by SCHLEIERMACHER and by PASCHEN, or precisely in what is for theoretical reasons the most probable position of the true curve of radiation.

I have recently made a series of determinations by aid of the bolometer described above; the results within the limits of the probable experimental errors fall on this same curve.

It seems very probable that Curve II. represents the actual law of thermal radiation between  $400^{\circ}$  and  $800^{\circ}$  C., but even were this not so, we might safely admit that the true curve must fall somewhere between Curves I. and III., fig. 10. All expressions therefore giving values outside these limits may be discarded without further study. The formulæ of STEFAN, of DULONG and PETIT, and of ROSETTI, fail when tested by this criterion. WEBER's formula, on the other hand, agrees closely with Curve II. from  $400^{\circ}$  to  $800^{\circ}$  C. At all higher temperatures the rate of change of radiation with temperature calculated by this formula, is certainly too great.

Taking the value of WEBER's constant at  $800^{\circ}$  C. as unity and comparing the calculated results with Curve II., we obtain 1.01, .96 and .94 for the values of the constant for 700, 600 and 500 degrees centigrade, showing that the expression remains nearly correct over this range of temperature.

Table XVI. and fig. 11 give the results of a series of observations on the radiation from platinum at temperatures ranging from  $500^{\circ}$  C. to the point of fusion of the metal.



In Table XVII. a few results obtained from the readings of a thermopile, the surface of which was coated with lamp black, are recorded. These results, though much less accurate than those obtained by use of the bolometer, serve as a confirmation of the previous values and show that these values are not affected by the type of instrument used.

### PART III.—ON THE VARIATION OF THE INTRINSIC BRILLIANCY OF PLATINUM WITH TEMPERATURE.

The flux of light emitted by one square centimetre of platinum at a fixed temperature has many times been proposed as a standard of light. By VIOLLE\* this temperature is defined as the point of solidification of the metal, whereas LUMMER and KURLBAUM† advocate a lower temperature not far from the melting point of palladium. It was essentially as a preliminary to the study of these two standards that the present work was undertaken. The method employed for the temperature measurements was in every particular the same as used in the first part of this paper. The experiments were made with wires 1.12 millims. in diameter enclosed in a gun-metal box  $8 \times 8.5 \times 3$  centims., which could be fitted with diaphragms of any desired shape. The wire under observation was horizontal, the points of contact of the potential leads being a little more than one centimetre to the right and left of the edge of the diaphragm, the opening of which was 38 millims. in length. The walls of the gun-metal casting forming the enclosure were hollow, and a water circulation maintained the temperature at about  $15^{\circ}\text{C}$ .

As standards of comparison, incandescent lamps were used. These were first left burning for from fifty to one hundred hours to avoid the initial rapid change in candle-power and then compared with a number of standard candles; the usual corrections being in all cases applied where the candles burnt more or less than the prescribed forty grains of sperm in ten minutes. The candle-power of these lamps was finally established from the mean of some hundreds of photometric readings. While any one of these lamps was in use, the electromotive force at its terminals was kept constant by the aid of a potentiometer. No photometric reading during which the electromotive force had varied more than .05 per cent. from its correct value, was recorded.

Where the lights differ largely in colour, as was the case in most of the observations recorded in the table, the measurements become influenced mainly by two physiological phenomena—

- (1.) The relative value of two lights of different colour depends on the absolute illumination of the two surfaces used in comparing them.

\* 'Comptes Rendus,' 1884, p. 1032, vol. 98.

† 'Elektrotechnische Zeitschrift,' 1894, p. 475.

- (2.) LEPINAY and NICATI\* have shown that the intensity of a light is not the same when measured by the distinctness with which the details of an object can be seen as when measured by the apparent brightness of a surface. The form of photometer used is sufficient indication that it is the latter method which has been adopted for the experiments recorded here.

In 1879 VIOLLE† published a research on much the same subject as is here dealt with. His results and mine though in substantial agreement as to the intrinsic brilliancy of platinum at its melting point, differ considerably at lower temperatures. In the paper alluded to no information is given as to the methods or instruments used, it is therefore impossible to suggest to what the divergence may be due. The author sums up his results by giving for the relation between the temperature,  $t$ , and the light emitted,  $I$ , the formula

$$\text{Log } I = 8.244929 + 0.011475t - 0.00000297t^2.$$

This expression by a modification of the constants could doubtless be made to agree with the present results; but the formula given below, owing to its simplicity, will be found of more practical value.

If  $t$  denote the temperature in degrees centigrade, and  $b$  the intrinsic brilliancy of the surface in candle-power per square centimetre, the relation may be expressed as

$$(t - 400) = 889.6 \sqrt[6.9]{b}.$$

The temperature calculated by this formula‡ will be found in column 3 of Table XVIII. The crosses on fig. 13 represent the intrinsic brilliancy as calculated. The circles show the values obtained experimentally.§ Where the formula is used over so wide a range of temperature the agreement for the lower values is but approximate; but this expression will be found to give very correct values of  $b$  when used for a limited interval above and below any temperature for which the constant has been determined.

Finally, before bringing this paper to a close, I desire to express my deep indebted-

\* 'Annales de Chimie et de Physique,' 1883, vol. 30, p. 145.

† 'Comptes Rendus,' vol. 88, 1879, p. 171.

‡ When the platinum is observed in a dark room, the first grey light becomes visible at a temperature of 417° C. It is as the limiting temperature, so far as visible radiation is concerned, that 400° C. is used in this formula.

§ I have recently made a number of observations on the intrinsic brilliancy of the crater of the electric arc. As these results refer to a study at present far from complete, it will be sufficient to state here that the intrinsic brilliancy is about 11,000 candle-power per square centimetre. Inserting this value for  $b$  in the formula given above, we obtain 3830 as the temperature of the crater. This fact may be taken as a confirmation of the formula for temperatures above the melting point of platinum, though I do not think that it would be wise to attach much importance to a coincidence of this kind.



ness to Professors DEWAR and FLEMING, and to Dr. A. SCOTT, who throughout this work have most kindly helped me with their advice on the more difficult theoretical points, and with their long experience of all the details of experimental work.

NOTE ADDED 14TH SEPTEMBER, 1898.

It has been suggested that the standard resistance referred to on p. 502, owing to the large currents which were used, might in reality rise to a temperature much in excess of that indicated by the thermometer, and thus by the consequent change in resistance materially affect the accuracy of the observations. The following experiment was carried out to ascertain to what extent this actually took place.

A platinum resistance was constructed similar to the standard manganin resistance referred to above. This new resistance was calibrated as a platinum thermometer. It was then placed in the same oil bath as used for the manganin standard. An electric current of increasing intensity was passed through it, and after each successive increase of current the temperature of the oil was read on an ordinary mercury thermometer, and the temperature of the platinum strip was calculated from its resistance. The stirrer was, of course, kept in motion during the entire experiment.

The results, reduced so as to apply to the manganin standard, are given in the following table :—

Current through the standard.	Probable rise of the manganin strip above the temperature indicated by the mercury thermometer.
amperes.	° C.
18·6	0·3
39·6	1·4
57·0	3·1
74·3	4·9
102·0	9·3

This standard resistance was rarely used for currents above 50 amperes, and never for currents above 70 amperes. For larger currents it was replaced by a ·001 ohm resistance. However, taking an extreme case, and supposing 74·3 amperes to be passing (the temperature coefficient being ·0003 per cent.), it is clear that the rise of temperature would not involve an error of more than ·0015 per cent.



TABLE I.—On the Volume Specific Resistance of Platinum and Palladium at their Melting Points.

Platinum.			Palladium.		
Number of radiator.	Volume specific resistance.		Number of radiator.	Volume specific resistance.	
R <sub>2</sub>	C.G.S. units. 62,720	Mean volume specific resistance of platinum at its melting-point = 62,600 C.G.S. units.	R <sub>10</sub>	C.G.S. units. 53,960	Mean volume specific resistance of palladium at its melting-point = 54,700 C.G.S. units.
R <sub>3</sub>	63,140		R <sub>11</sub>	55,400	
R <sub>7</sub>	61,760				
R <sub>8</sub>	62,370				
R <sub>9</sub>	62,830				

TABLE II.—Emissivity in Dry Air at Three Atmospheres Pressure (See fig. 2.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.	24th February, 1898. Temperature of the enclosure 9° C.	° C.	C.G.S. units.	26th February, 1898. Temperature of the enclosure 10° C.
50	·0008234		765	·002006	
322	·001337		996	·002545	
800	·002082		1130	·002758	
1087	·002658		1370	·003520	

TABLE IIIA.—Emissivity in Dry Air at Atmospheric Pressure. (See fig. 2.)  
2nd March, 1898.

Current, first reading.	Electro- motive force, first reading.	Current, second reading.	Electro- motive force, second reading.	Mean current.	Mean electro- motive force.	Resist- ance.	$R - R_0$ $R_0 = 0.0066$ .	$R - R_0$ $R_0 = 0.0002345$ .	Tem- perature. ° C.	Energy dissipated in Watts.	$t_r - t_e$ . Tempera- ture of the radiator above the enclosure.	$\frac{W}{t_r - t_e}$ .	$\frac{W}{J \times S (t_r - t_e)}$ = emissivity
amperes.	volts.	amperes.	volts.	amperes.	volts.	ohms.		° pt.					
6.392	.07830	6.429	.07845	6.411	.078375	.01223	.00217	92.55	93	.5025	81	.006204	.0006640
9.125	.1280	9.130	.1280	9.1275	.1280	.01402	.00396	168.9	171	1.1685	159	.007348	.0007866
16.91	.3835	16.89	.3833	16.90	.3834	.02269	.01263	538.7	574	6.480	562	.01153	.001234
20.34	.5470	20.34	.5470	20.34	.5470	.02689	.01683	717.8	789	11.13	777	.01432	.001533
24.175	.7452	24.175	.7450	24.175	.7451	.03082	.02076	885.5	1003	18.01	991	.01817	.001945
28.75	.9910	28.73	.9915	28.74	.99125	.03449	.02443	1042	1219	28.49	1207	.02361	.002529

Temperature of the enclosure, 12° C.

" " standard resistance, 18.5° C.

" " Clarke's cell, 16° C.

Height of the barometer corrected to 0° C., 751.8 millims.

S = superficial area of wire between the potential terminals  
= 2.224 sq. centims.

$t_r$  = temperature of radiator in degrees centigrade.

$t_e$  = " of enclosure "

Radiator,  $R_2$ ,

$$pt = \frac{R - 0.01006}{.00002345},$$

$$t - pt = 1.3035 \left[ \left( \frac{t}{100} \right)^2 - \frac{t}{100} \right],$$

where  $pt$  is the temperature in platinum degrees, and  $t$  the temperature in degrees centigrade.

TABLE IIIB.—Emissivity in Dry Air at Atmospheric Pressure. (See fig. 2.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
1281	·002717	3rd February, 1898. Temperature of the enclosure 20° C.	528	·001141	3rd February, 1898. Temperature of the enclosure 20° C.
1172	·002378		387	·000992	
1044	·002031		293	·000878	
877	·001760				
700	·001380				
			Radiator. ° C.		
			R <sub>3</sub> 1779	·004960	

TABLE IV.—Emissivity in Air at Atmospheric Pressure saturated with moisture. (See fig. 2.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
1331	·002977	3rd March, 1898. Temperature of the enclosure 11° C.	142	·000833	28th February, 1898. Temperature of the enclosure 10° C.
1196	·002550		236	·000966	
1036	·002115		377	·001106	
888	·001800		534	·001290	
751	·001553				
533	·001257		Radiator. ° C.		
203	·000845		R <sub>10</sub> 1500	·003820	
74	·000666		R <sub>11</sub> 1500	·003945	
			R <sub>1</sub> 1779	·005228	

TABLE V.—Emissivity in Dry Air at 1 centim. Pressure. (See fig. 2.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
922	·001137	11th February, 1898. Temperature of the enclosure 13° C.	1291	·001145	22nd February, 1898. Temperature of the enclosure 9° C.
1002	·001365		715	·000870	
1137	·001590		268	·000490	
1244	·001995				

TABLE VI.—Emissivity in Dry Hydrogen at Three Atmospheres. (See fig. 3.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
119	·003336	3rd March, 1898. Temperature of the enclosure, 10° C.	971	·006720	2nd March, 1898. Temperature of the enclosure, 10° C.
183	·003686		188	·003679	
271	·004029				
408	·004535		141	·003499	8th March, 1898. Temperature of the enclosure, 9° C.
770	·005916		680	·005721	
			1281	·008270	



TABLE VII.—Emissivity in Dry Hydrogen at Atmospheric Pressure. (See fig. 3.)

Temperature.	Emissivity.	7th March, 1898. Temperature of the enclosure, 9° C.	Temperature.	Emissivity.	7th March, 1898. Temperature of the enclosure, 9° C.
° C.	C.G.S. units.		° C.	C.G.S. units.	
123	·002577		1020	·005386	
222	·002970		1324	·006459	
291	·003209		1305	·006435	
384	·003509				
526	·003955				
692	·004385		Radiator. ° C.		
690	·004373		R <sub>6</sub> 1779	·00942	
886	·004962		R <sub>7</sub> 1779	·00940	

TABLE VIII.—Emissivity in Dry Hydrogen at 6 centims. Pressure. (See fig. 3.)

Temperature.	Emissivity.	8th March, 1898. Temperature of the enclosure, 9° C.	Temperature.	Emissivity.	8th March, 1898. Temperature of the enclosure, 9° C.
° C.	C.G.S. units.		° C.	C.G.S. units.	
195	·002213		1120	·004520	
487	·002853		427	·002668	
725	·003413				
741	·003350		1265	·004988	
954	·003954		288	·002345	
					7th March, 1898. Temperature of the enclosure, 9° C.

TABLE IX.—Emissivity in Dry Carbon Dioxide at Three Atmospheres Pressure.  
(See fig. 4.)

Temperature.	Emissivity.	28th February, 1898. Temperature of the enclosure, 9° C.	Temperature.	Emissivity.	1st March, 1898. Temperature of the enclosure, 9° C.
° C.	C.G.S. units.		° C.	C.G.S. units.	
371	·001428		821	·002322	
277	·001271		1020	·002808	
157	·001031		1160	·003230	
61	·000783		1269	·003594	
			1331	·004098	
554	·001776	1st March, 1898. Temperature of the enclosure, 9° C.			
685	·002028				

TABLE X.—Emissivity in Dry Carbon Dioxide at Atmospheric Pressure. (See fig. 4.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
1278	·002892	17th February, 1898. Temperature of the enclosure, 11° C.	1249	·002809	18th February, 1898. Temperature of the enclosure, 11° C.
1267	·002816		1361	·003197	
1167	·002541		1419	·003407	
1026	·002157				
762	·001582		90	·000655	28th February, 1898. Temperature of the enclosure, 9° C.
			241	·000888	
			483	·001155	

TABLE XI.—Emissivity in Dry Carbon Dioxide at 6 centims. Pressure. (See fig. 4.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
188	·000422	21st February, 1898. Temperature of the enclosure, 10° C.	1097	·001712	21st February, 1898. Temperature of the enclosure, 10° C.
312	·000530		1200	·001971	
988	·001447		1374	·002540	
708	·000975				
			1334	·002385	18th February, 1898. Temperature of the enclosure, 10° C.
			521	·000715	

TABLE XII.—Emissivity in Oxygen at Three Atmospheres. (See fig. 4.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
815	·002092	28th February, 1898. Temperature of the enclosure, 9° C.	1135	·002882	28th February, 1898. Temperature of the enclosure, 9° C.
413	·001429		1291	·003463	
723	·001914		61	·000848	

TABLE XIII.—Emissivity in Steam at Atmospheric Pressure. (See fig. 5.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
962	·002392	3rd March, 1898. Temperature of the enclosure, 107° C.	925	·002174	3rd March, 1898. Temperature of the enclosure, 107° C.
1293	·003489		754	·001714	
764	·001742		534	·001268	
1122	·002790				

TABLE XIV.—Emissivity in Air at Atmospheric Pressure. Wire horizontal in cylindrical iron enclosure. (See fig. 7, Curves II. and III.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
858	·00216	29th January, 1898. Temperature of the enclosure, 455° C.	61	·000676	29th January, 1898. Temperature of the enclosure, 22° C.
919	·00234		106	·000848	
1007	·00260		188	·000991	
1142	·00309		381	·001225	
629	·00155				
778	·00189				
912	·00239				

TABLE XV.—Emissivity in Air at Atmospheric Pressure. Gun-metal Enclosure. Radiating Wire Horizontal. Diameter of Wire ·112 centim. (See fig. 7, Curve I.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
64	·000814	Temperature of the enclosure 17° C.	771	·001952	Temperature of the enclosure 17° C.
89	·000899		785	·001972	
118	·000962		804	·002034	
133	·000976		861	·002147	
219	·001133		942	·002312	
276	·001159		945	·002344	
351	·001272		1016	·002532	
385	·001309		1115	·002832	
420	·001393		1150	·002915	
481	·001477		1171	·003018	
510	·001540		1274	·003387	
615	·001693		1371	·003754	

TABLE XVI.—Emissivity in Air at Atmospheric Pressure. Gun-metal Enclosure. Radiating Wire Horizontal. Diameter of Wire ·0600 centim. Radiator No. R<sub>24</sub>. (See fig. 7, Curve V.)

Temperature.	Emissivity.		Temperature.	Emissivity.	
° C.	C.G.S. units.		° C.	C.G.S. units.	
1012	·002782	Temperature of the enclosure 16° C.	522	·001990	Temperature of the enclosure 16° C.
890	·002546		489	·001938	
775	·002379		383	·001779	
664	·002203		194	·001462	
564	·002056				



TABLE XVII.—On the Law of Thermal Radiation.

Heat received by the Bolometer; Sensitive Surface coated with Platinum Black;  
Temperature of the Bolometer, 17° C. (See figs. 10 and 11.)

4th May, 1898.		9th May, 1898.	
Temperature.	Mean corrected readings.	Temperature.	Mean corrected readings.
° C.		° C.	
511	4.9	1197	115.2
665	12.9	1121	91.1
716	16.6	1014	62.4
768	22.6	939	46.8
830	29.8	765	22.3
899	40.6	657	11.9
968	53.4		
1050	74.3		
1117	96.4		
1224	144.0		

Radiation received from palladium at its melting point . . .	{ Radiator No. R <sub>27</sub> . . .	407
	„ No. R <sub>28</sub> . . .	396
Radiation received from melting platinum . . . . .	{ Radiator No. R <sub>29</sub> . . .	830
	„ No. R <sub>21</sub> . . .	804

Heat received by a Thermopile, the Sensitive Surface of which was coated with  
Lamp Black.

Temperature.	Corrected galvanometer deflection.
° C.	
1166	97.8
1059	72.7
900	41.8
635	11.2

TABLE XVIII. (See figs. 12 and 13.)

Temperature.	Intrinsic brilliancy of surface in candle- power per sq. centim. of projected area.	Calculated temperature $(t-400) = 889.6 \sqrt[6]{b}$ .	
° C.			
1036	.0677	1002	November, 1897. Enclosure at about 15° C. Projected area of radiator .4256 sq. centim.
1076	.131	1063	
1140	.250	1128	
1189	.437	1189	
1132	.231	1119	
1240	.735	1251	
1286	1.14	1307	
1360	1.96	1381	

	Temperature.	Intrinsic brilliancy of surface in candle- power per sq. centim. of projected area.	Calculated temperature $(t-400) = 889.6 \sqrt[6]{b}$ .
	° C.		
Radiator No. R <sub>23</sub> . . . . .	1500	4.15	} 1509
„ No. R <sub>24</sub> . . . . .	1500	4.95	
„ No. R <sub>21</sub> . . . . .	1779	20.53	} 1766
„ No. R <sub>22</sub> . . . . .	1779	18.00	

Fig. 2.

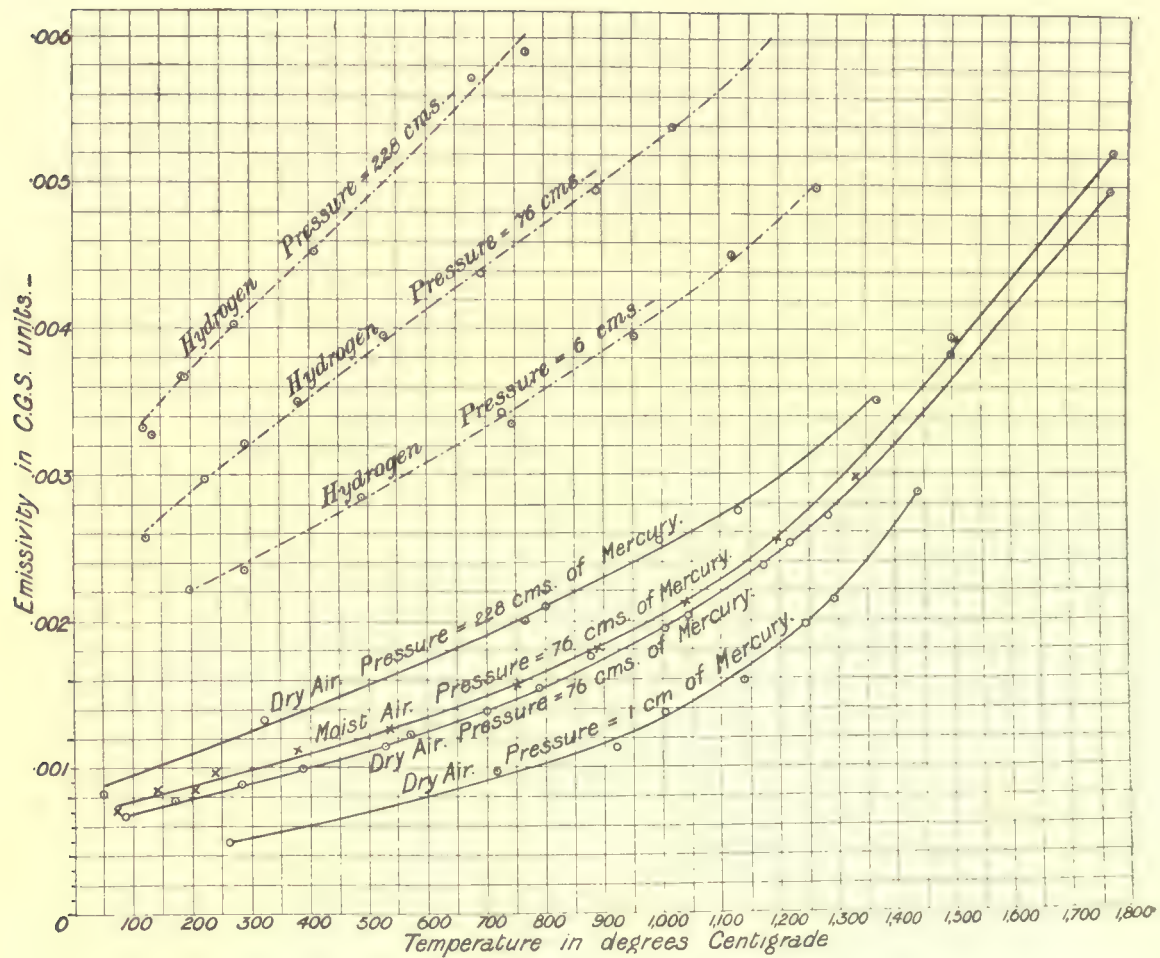


Fig. 3.

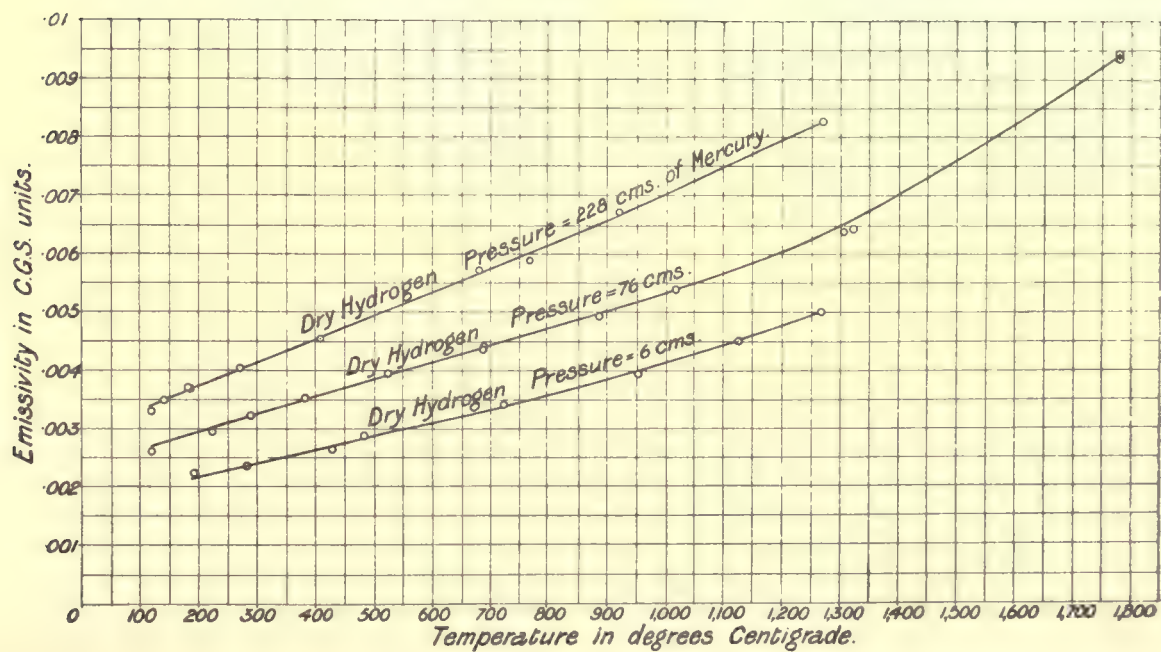






Fig. 4.

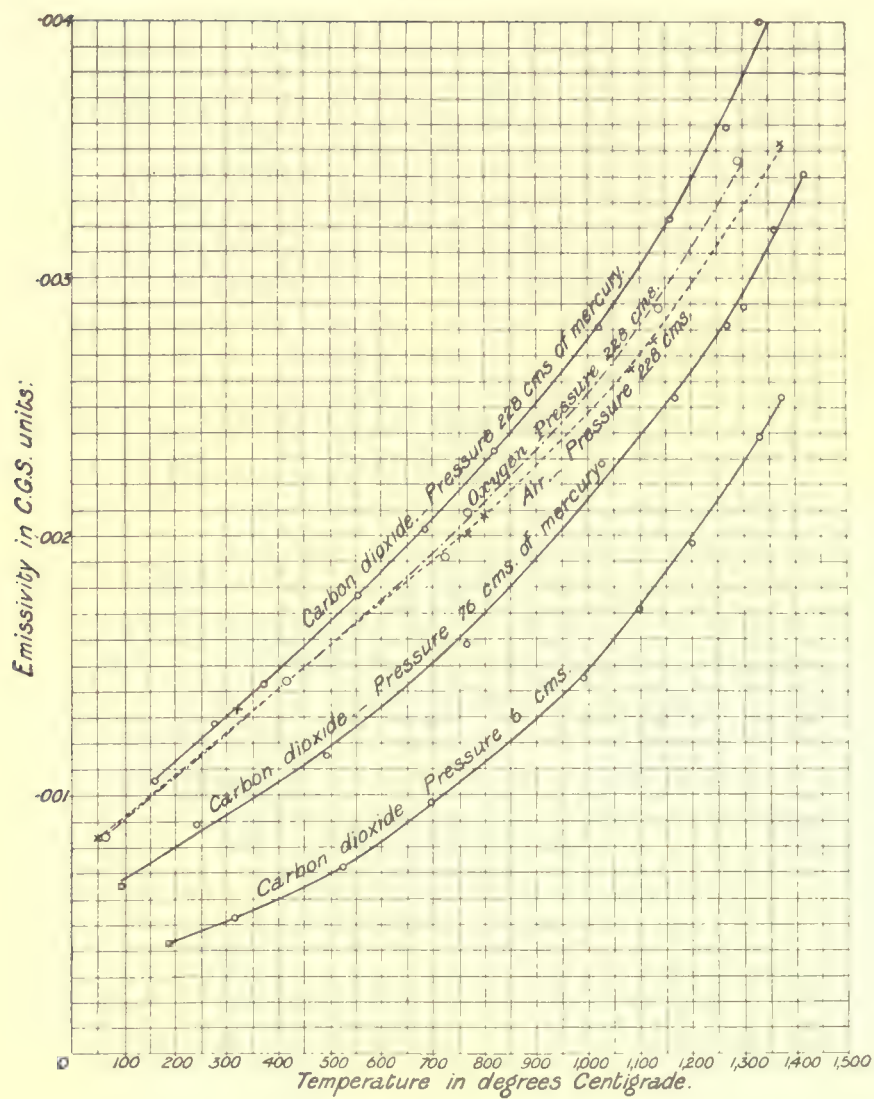


Fig. 6.

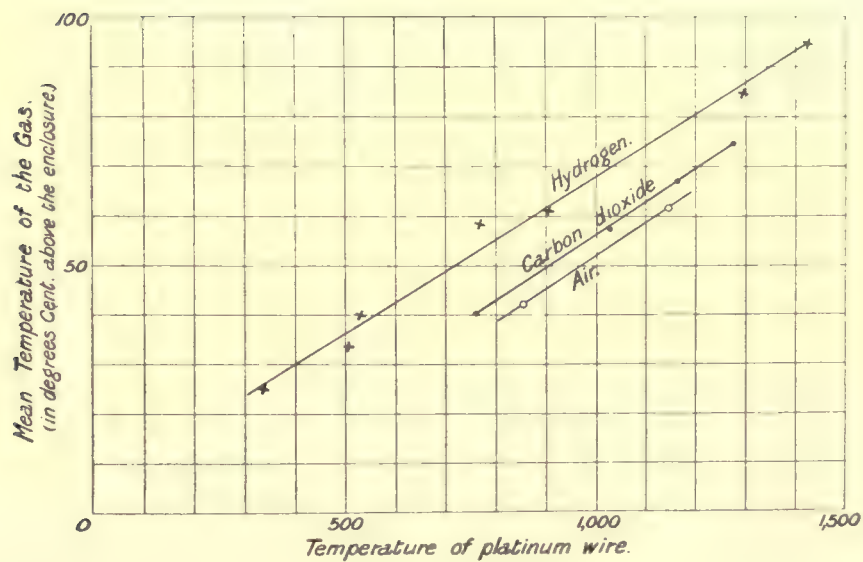






Fig. 7

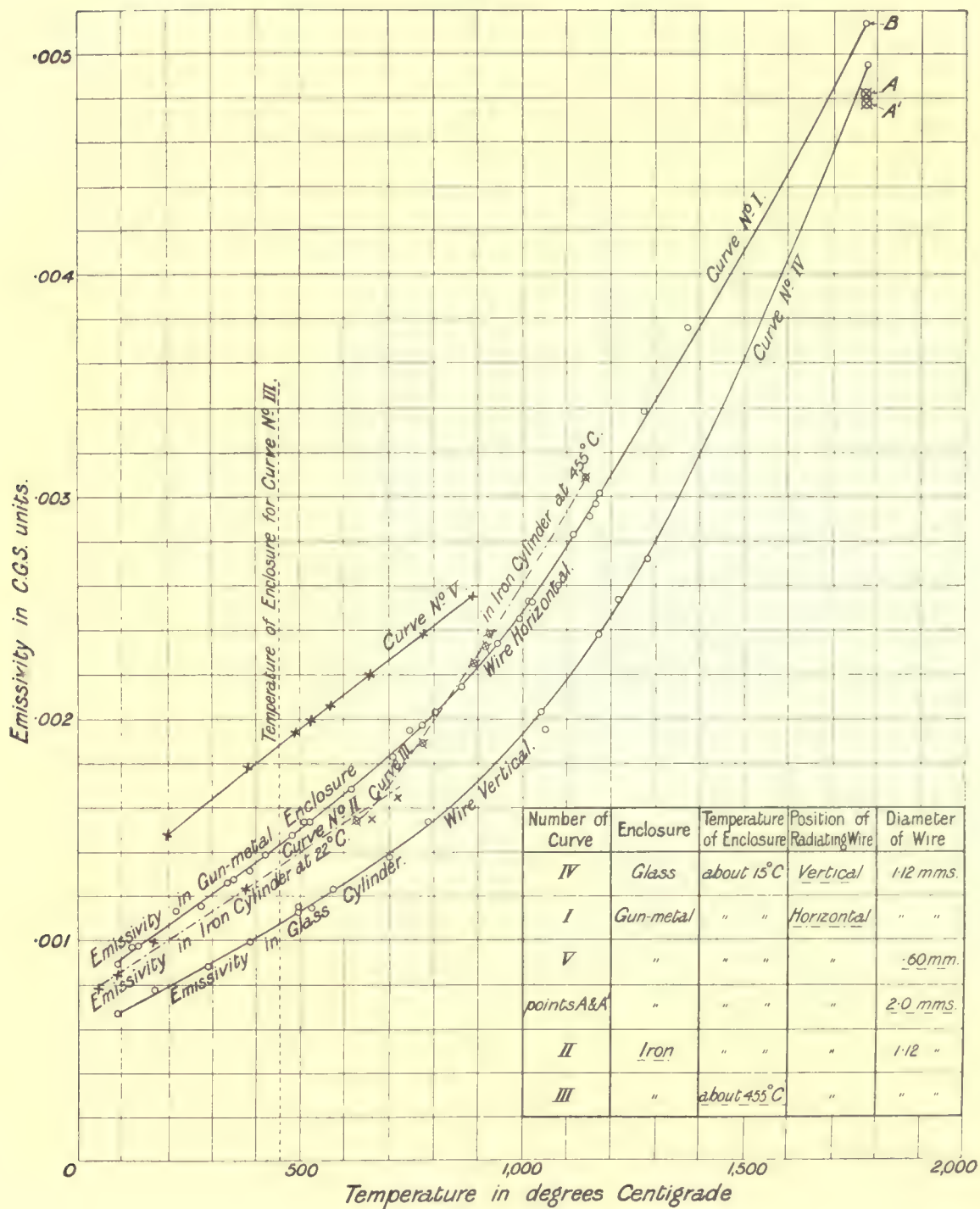




Fig. 5.

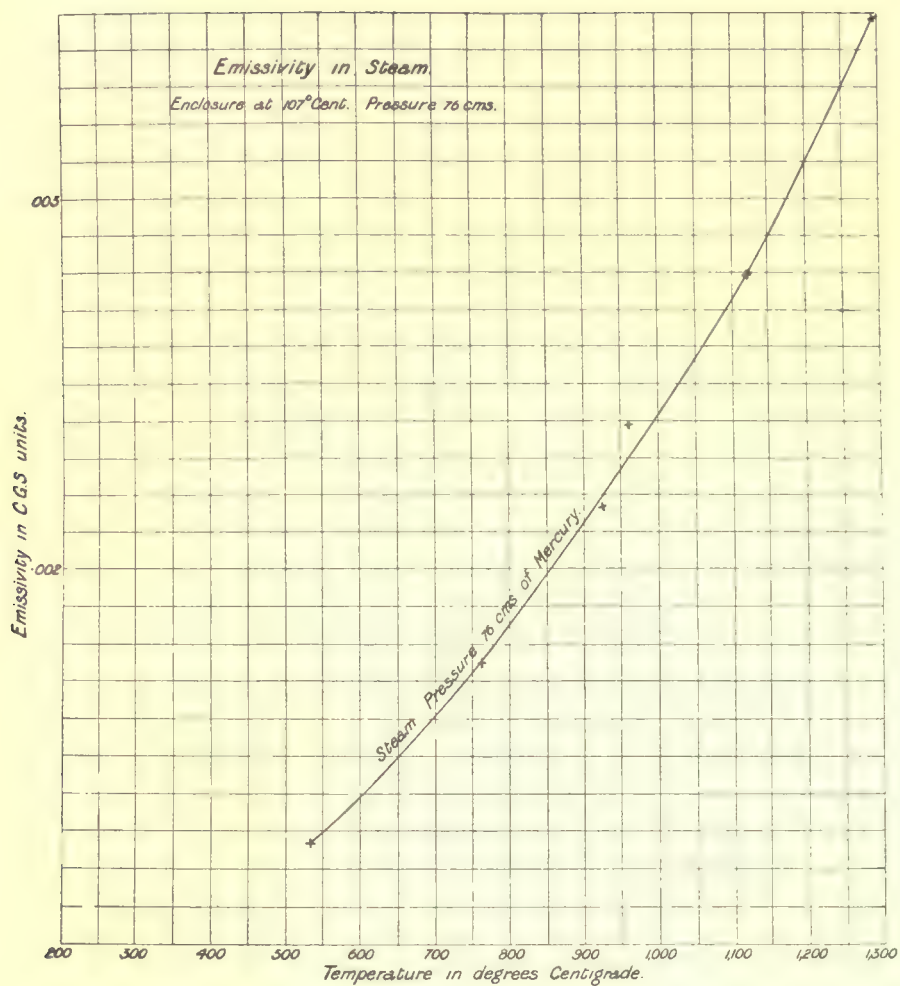


Fig. 10.

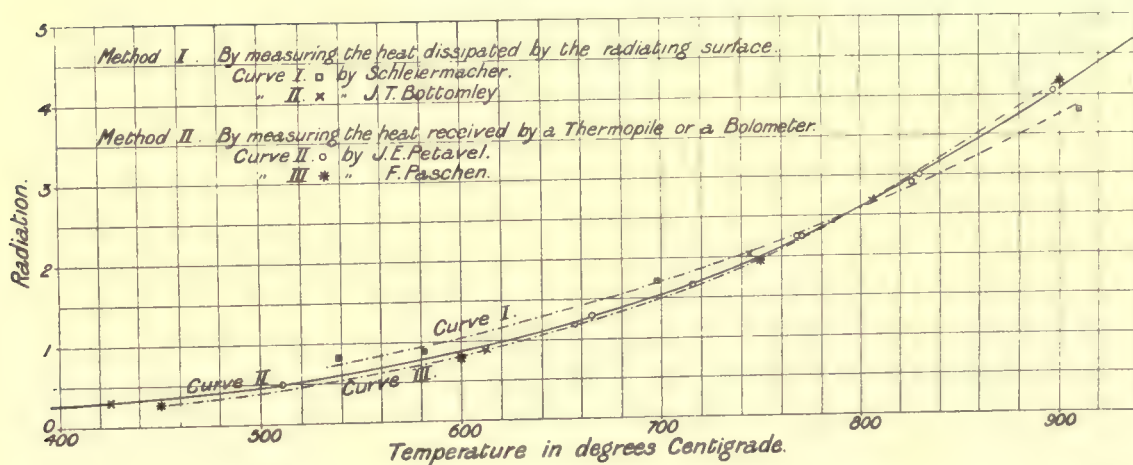






Fig. 11.

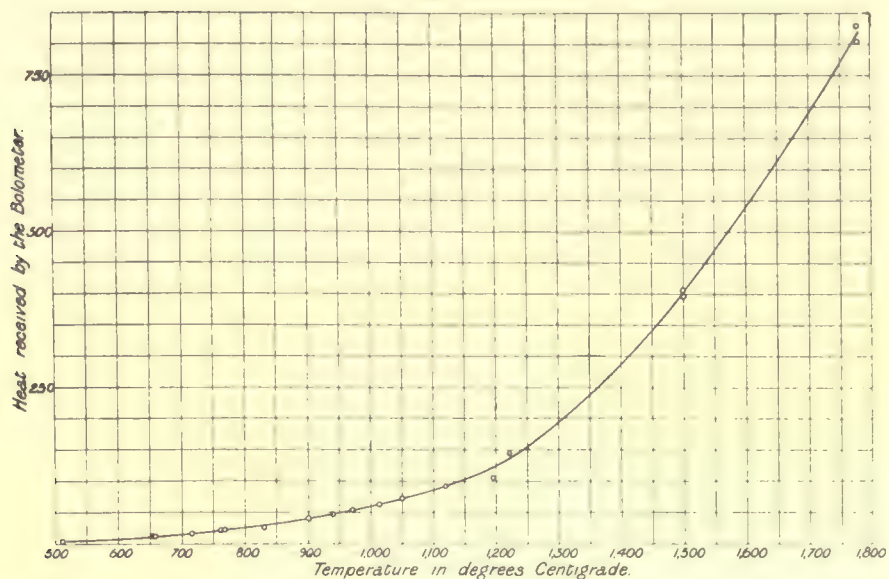


Fig. 12.

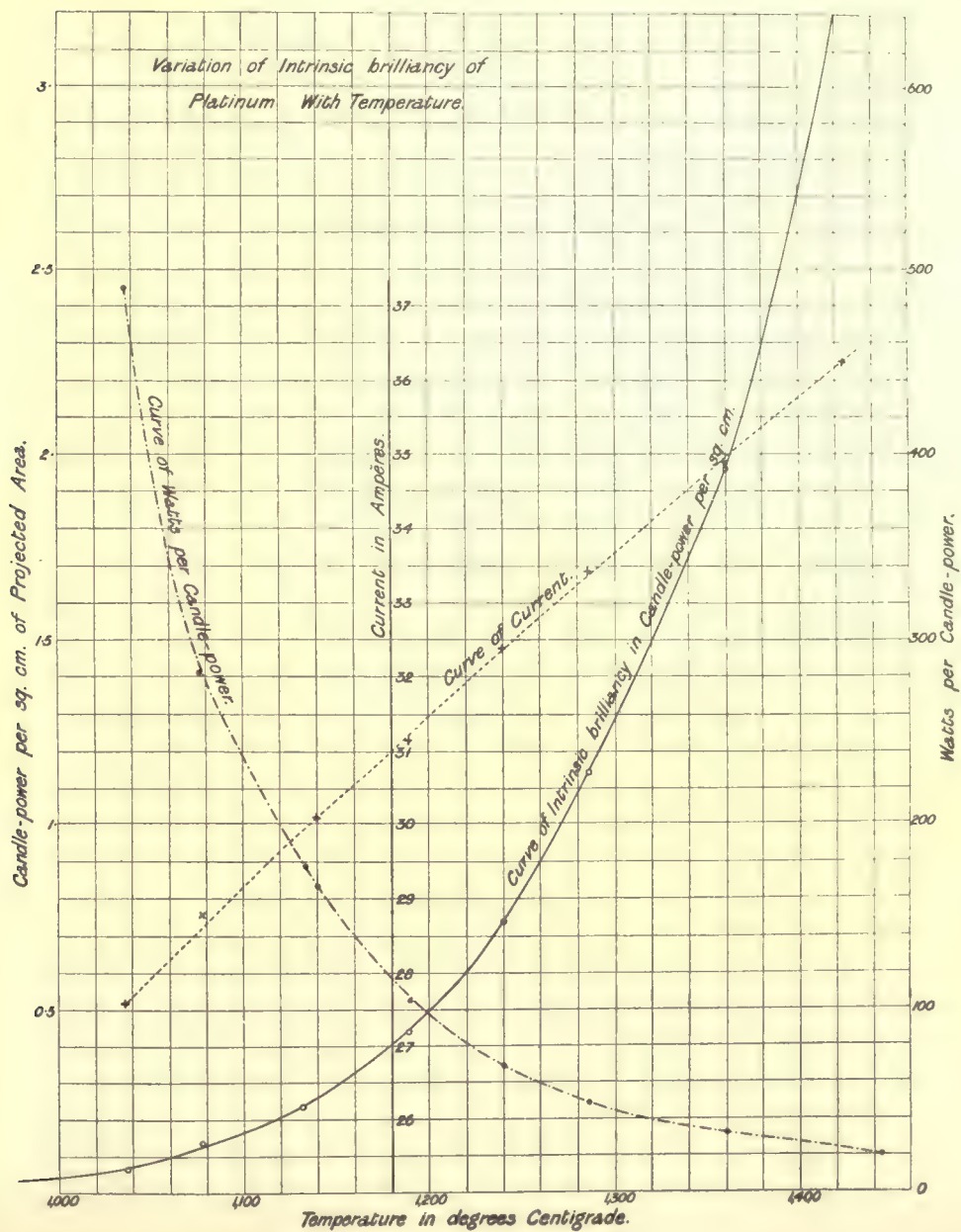
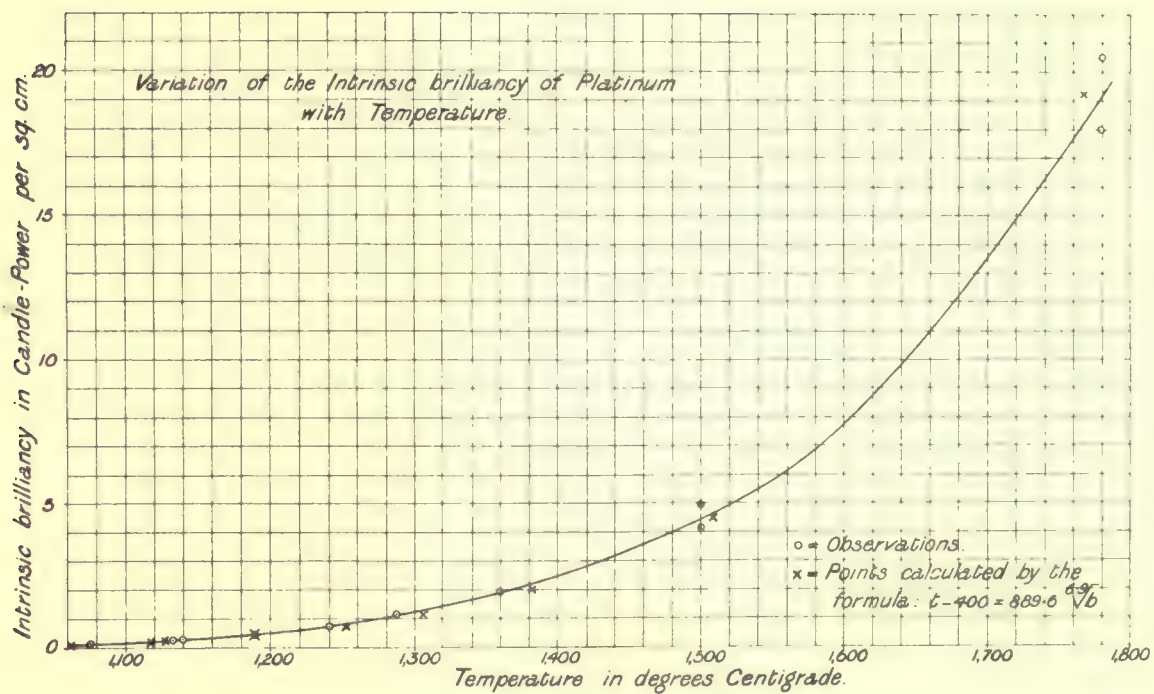






Fig. 13.





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